MA162: Finite mathematics

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University of Kentucky

November 2, 2011

SCHEDULE:

- HW 6A is due Friday, Nov 4th, 2011.
- HW 6B is due Wednesday, Nov 9th, 2011.
- HW 6C is due Friday, Nov 11th, 2011.
- Exam 3 is Monday, Nov 14th, 5:00pm-7:00pm in CB106.

Today we will cover 6.2: Counting

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
  - Compound interest
  - Sinking funds
  - Amortized loans

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations

6.1: Equality drill

- Two sets are **equal** if they have the same elements.
- $\{1, 2, 3\} \stackrel{?}{=} \{1, 2, 3\}$
- $\{1, 2, 3\} \stackrel{?}{=} \{1, 2\}$
- $\{1, 2, 3\} \stackrel{?}{=} \{3, 1, 2\}$
- $\{1, 2, 3\} \stackrel{?}{=} \{1, 2, 2, 3, 3, 3\}$
- $\{1, 2, 3\} \stackrel{?}{=} \{$ positive integers whose square has one digit $\}$
- $\{1, 2, 3\} \stackrel{?}{=} \{$ odd numbers less than 4 $\}$

6.1: Equality drill

- Two sets are **equal** if they have the same elements.
- $\{1,2,3\} = \{1,2,3\}$
  Yes! Exactly the same.
- $\{1,2,3\} \neq \{1,2\}$
  No! Right hand set is missing "3"
- $\{1,2,3\} = \{3,1,2\}$
  Yes! Order does not matter.
- $\{1,2,3\} = \{1,2,2,3,3,3\}$
  Yes! Repeats don't matter.
- $\{1,2,3\} = \{$ positive integers whose square has one digit $\}$
  Yes! Long-winded doesn't matter.
- $\{1,2,3\} \neq \{$ odd numbers less than 4 $\}$
  No! Right hand set is missing "2"

6.1: Union and intersection drill

- $\bigcup$ The **union** includes anything in either, and is big. $\bigcup$
- $\bigcap$ The **intersection** includes only those in both, and is small. $\bigcap$
- $\{1, 2, 3\} \cup \{3, 4, 5\} =$
- $\{1, 2, 3\} \cap \{3, 4, 5\} =$
- $\{1, 2, 3\} \cup \{1\} =$
- $\{1, 2, 3\} \cap \{1\} =$

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- $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
- $\{1, 2, 3\} \cup \{1\} = \{1, 2, 3\}$
- $\{1, 2, 3\} \cap \{1\} = \{1\}$

6.1: Difference drill

- $-$ The **difference** keeps the first, but not in the second. $-$
- $\{1, 2, 3\} - \{1\} =$
- $\{1, 2, 3\} - \{2, 3\} =$
- $\{1, 2, 3\} - \{3, 4, 5\} =$
- $\{1, 2, 3\} - \{4, 5, 6\} =$
- $\{1, 2, 3\} - \{1, 2, 3\} =$

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- $\{1, 2, 3\} - \{4, 5, 6\} = \{1, 2, 3\}$
- $\{1, 2, 3\} - \{1, 2, 3\} = \{\}$ The **empty set** containing nothing.

6.1: Laws of sets

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- $A = \{1,2,3\}$, and $B = \{3,4,5\}$. Compare $A \cap B$ and $A - B$.

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- $A \cap B = \{3\}$ and $A - B = \{1,2\}$
- $A = (A \cap B) \cup (A - B)$

6.2: Counting the missing piece

- Out of 100 coffee drinkers surveyed, 70 take cream, and 60 take sugar. How many take it black (with neither cream nor sugar)?
- Well, it is hard to say, right?
  30 don't use cream, 40 don't use sugar, but...

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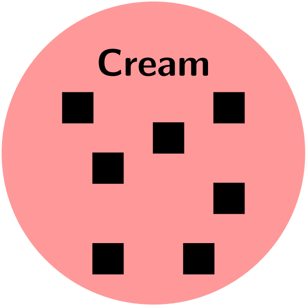

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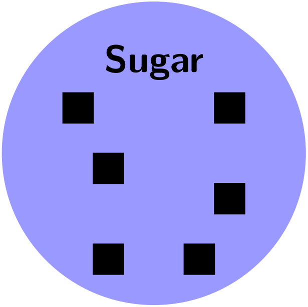

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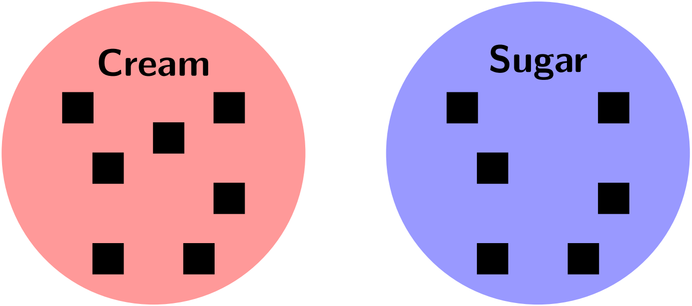

- $60 + 70 = 130$ is way too big. What happened?
  Try it yourself!

6.2: The overlap

- In order to figure out how many take it black, we need to know how many take it with cream or sugar or both.

$$\#\text{Black} = 100 - n(C \cup S)$$

- However, in order to find out how many take either, we kind of need to know how many take both:

$$n(C \cup S) = n(C) + n(S) - n(C \cap S) = 70 + 60 - n(C \cap S)$$

- So what if 50 people took both?

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- Then $n(C \cup S) = 130 - 50 = 80$ and so $100 - 80 = 20$ took neither.

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- Suppose we know that there were 200 people in the testing pool. About how many were drug users?
- Assuming exactly 5% of non-users returned positive, there is a unique answer. Let me know when you've found it.

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- All 10 are false positives; 100% wrong, but 95% accurate?
  Be careful what you are counting.

6.2: More overlaps

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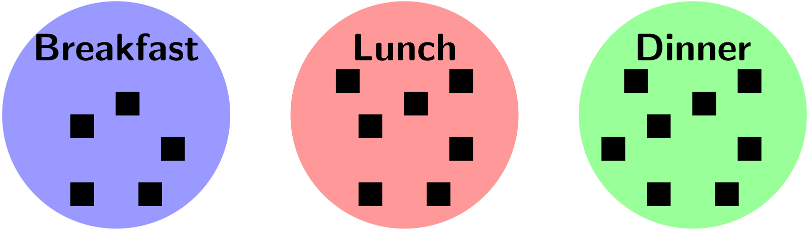
- No more than 50, right? What is the bare minimum?
- At least 20 ate both breakfast and lunch, right?
- What if those were exactly the 20 people that didn't eat dinner?
- Could be 0%, could be 50%. We need to know more!

6.2: More information and a picture

- If we let B, L, D be the sets of people, then we are given

$$n(B) = 50, n(L) = 70, n(D) = 80,$$

and we want to know $n(B \cap L \cap D)$.



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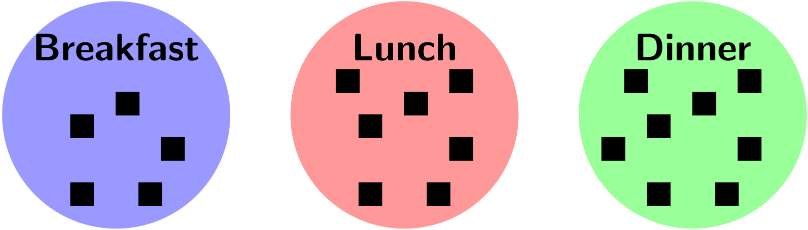

- What if we find out:

$$n(B \cap L) = 30, n(B \cap D) = 40, n(L \cap D) = 40$$

We can find the overlaps!

6.2: More information and a formula

- Just like before, there is a formula relating all of these things:

$$n(B)+n(L)+n(D)+n(B\cap L\cap D) = n(B\cup L\cup D)+n(B\cap L)+n(L\cap D)+n(D\cap B)$$

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- We plugin to get:

$$55 + 65 + 80 + n(B \cap L \cap D) = 100 + 34 + 46 + 40$$

$$n(B \cap L \cap D) = 100 + 34 + 46 + 40 - 55 - 65 - 80 = 20$$

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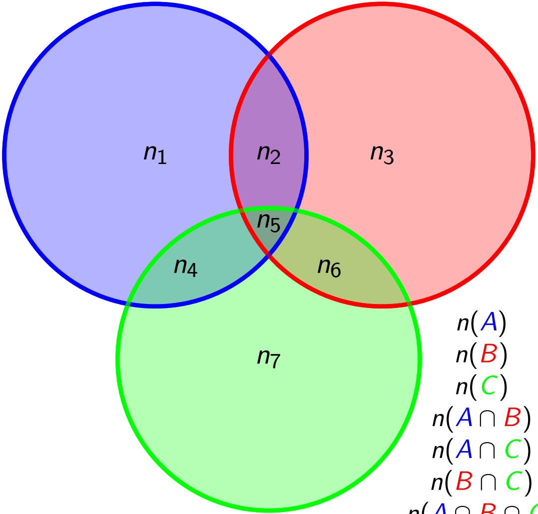
- **Inclusion-exclusion formula** will be given on the exam,
  but make sure you know how to use it!

6.2: Picture and formula



$$
\begin{aligned}
n(A) &= n_1 + n_2 + n_4 + n_5 \\
n(B) &= n_2 + n_3 + n_5 + n_6 \\
n(C) &= n_4 + n_5 + n_6 + n_7 \\
n(A \cap B) &= n_2 + n_5 \\
n(A \cap C) &= n_4 + n_5 \\
n(B \cap C) &= n_5 + n_6 \\
n(A \cap B \cap C) &= n_5 \\
n(A \cup B \cup C) &= n_1 + n_2 + n_3 + n_4 \\
&\quad n_5 + n_6 + n_7
\end{aligned}
$$

6.2: Summary

- We learned the notation $n(A) =$ the number of things in the set A
- We learned the basic inclusion-exclusion formulas:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

- Make sure to complete HW 6A and 6B