# DEPARTMENT OF MATHEMATICS

Ma 162 Final Exam December 14, 2011

**Instructions:** No cell phones or network-capable devices are allowed during the exam. You may use calculators, but you must show your work to receive credit. If your answer is not in the box or if you have no work to support your answer, you will receive no credit. The test has been carefully checked and its notation is consistent with the homework problems. No additional details will be provided during the exam.

| **Problem** | **Maximum Score** | **Actual Score** |
|---|---|---|
| 1 | ? | |
| 2 | ? | |
| 3 | ? | |
| 4 | ? | |
| 5 | ? | |
| 6 | ? | |
| 7 | ? | |
| 8 | ? | |
| Total | 100 | |

**NAME:** ______________________________ **Section:** ____________

**Last four digits of Student ID:** __________________

Answers with no work receive no credit.

1. **Setup the following linear equation.**

   Mr. Marjoram runs a stuffed animal factory, and is very worried about paying taxes on his rather large inventory of plush fabric, cloud-like stuffing, and whimsical trim. He decides he is going to use every last bit of his inventory to make the 2011 Marjoram Menagerie! His menagerie only includes Pandas, Saint Bernards, and Onery Ostriches. Each Panda requires 1.5 square yards of plush, 30 cubic feet of stuffing, and 12 pieces of trim. Each Saint Bernard requires 2 square yards of plush, 35 cubic feet of stuffing, and 8 pieces of trim. Each Onery Ostrich requires 2.5 square yards of plush, 25 cubic feet of stuffing, and 5 pieces of trim. Marjoram's storage room has 110 square yards of velvety plush, 1400 cubic feet of fluffy stuffing, and 350 pieces of tremendous trim. How many stuffed animals of each type should he make in order to use up all of his inventory?

| | Plush | Stuffing | Trim |
|---|---|---|---|
| Panda | 1.5 | 30 | 12 |
| Bernard | 2 | 35 | 8 |
| Ostrich | 2.5 | 25 | 5 |
| Inventory | 110 | 1400 | 350 |

The variables describing the decision are:

The equations to be solved are:

The augmented matrix describing the equations is:

Answers with no work receive no credit.

2. **Interpret the final matrix for this solution to a word problem.**

Vincent is trying to optimize his profit by solving a system of linear equations. He sets $X$ to be the number of Sunshine paintings to produce, $Y$ to be the number of Lollipop paintings to produce, $A$ to be the tubes of Amarillo paint left over, $B$ to be the tubes of Berry Red paint left over, $C$ to be the number of canvasses left over, $D$ to be tubes of Dark Blue paint left over, and $P$ to be the profit. His decision is governed by the equations on the right.

$$\begin{aligned} 3X + Y + A &= 25 \\ 3X + 2Y + B &= 26 \\ X + Y + C &= 10 \\ X + 3Y + D &= 24 \\ 10X + 12Y &= P \end{aligned}$$

Converting this to a matrix, he quickly reduced this to something very similar to RREF:

$$\left(\begin{array}{rr|rrrr|r|r} X & Y & A & B & C & D & P & rhs \\ \hline 3 & 1 & 1 & 0 & 0 & 0 & 0 & 25 \\ 3 & 2 & 0 & 1 & 0 & 0 & 0 & 26 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 10 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 24 \\ \hline -10 & -12 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_4-(1/3)R_1,\ R_5+(10/3)R_1]{R_1/(3),\ R_2-R_1,\ R_3-(1/3)R_1} \left(\begin{array}{rr|rrrr|r|r} X & Y & A & B & C & D & P & rhs \\ \hline 1 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 25/3 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2/3 & -1/3 & 0 & 1 & 0 & 0 & 5/3 \\ 0 & 8/3 & -1/3 & 0 & 0 & 1 & 0 & 47/3 \\ \hline 0 & -26/3 & 10/3 & 0 & 0 & 0 & 1 & 250/3 \end{array}\right) \xrightarrow[R_4-(8/3)R_2,\ R_5+(26/3)R_2]{R_1-(1/3)R_2,\ R_3-(2/3)R_2}$$

$$\left(\begin{array}{rr|rrrr|r|r} X & Y & A & B & C & D & P & rhs \\ \hline 1 & 0 & 2/3 & -1/3 & 0 & 0 & 0 & 8 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & -2/3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 7/3 & -8/3 & 0 & 1 & 0 & 13 \\ \hline 0 & 0 & -16/3 & 26/3 & 0 & 0 & 1 & 92 \end{array}\right) \xrightarrow[R_4-(7)R_3,\ R_5+(16)R_3]{R_1-(2)R_3,\ R_2+(3)R_3,\ (3)R_3} \left(\begin{array}{rr|rrrr|r|r} X & Y & A & B & C & D & P & rhs \\ \hline 1 & 0 & 0 & 1 & -2 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & -2 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & -7 & 1 & 0 & 6 \\ \hline 0 & 0 & 0 & -2 & 16 & 0 & 1 & 108 \end{array}\right) \xrightarrow[R_4/(2),\ R_5+R_4]{R_1-(1/2)R_4,\ R_2+(1/2)R_4,\ R_3+R_4}$$

$$\left(\begin{array}{rr|rrrr|r|r} X & Y & A & B & C & D & P & rhs \\ \hline 1 & 0 & 0 & 0 & 3/2 & -1/2 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 & 0 & 7 \\ 0 & 0 & 1 & 0 & -4 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -7/2 & 1/2 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 9 & 1 & 1 & 114 \end{array}\right)$$

(a) Which variables are free?

(b) Convert the last row to an equation, and solve it for the non-free variable.

(c) What value should the free variables have to maximize $P$? (assuming they cannot be negative)

(d) Solve the third row for a non-free variable, and replace the free variables by their values from part (c).

Answers with no work receive no credit.

3. During Winter Vacation your pal Vincent decides to start his own roadside art business to fund a action-packed road trip to the Bahamas. He may have drifted off in art class most days, but he did learn to draw a pretty awesome Sunshine! and some sweet Lollipops. Like his artistic repertoire, his art supplies are limited. He has 25 tubes of Amarillo, 26 tubes of Berry Red, 10 Canvasses, and 24 tubes of Dark Blue. Despite his limited artistry, his marketing skills are unparalleled and he can sell every painting he paints. The requirements and profits of his two painting styles are given in the following table:

| | Amarillo | Berry Red | Canvasses | Dark Blue | Profit |
|---|---|---|---|---|---|
| Sunshine! | 3 | 3 | 1 | 1 | 10 |
| Lollipops | 1 | 2 | 1 | 3 | 12 |
| Inventory | 25 | 26 | 10 | 24 | |

How many Sunshine! paintings and Lollipop paintings should Vincent produce in order to maximize his profit?

Vincent should paint ______ Sunshine! paintings, and ______ Lollipop paintings, to achieve the maximum profit of ______ .

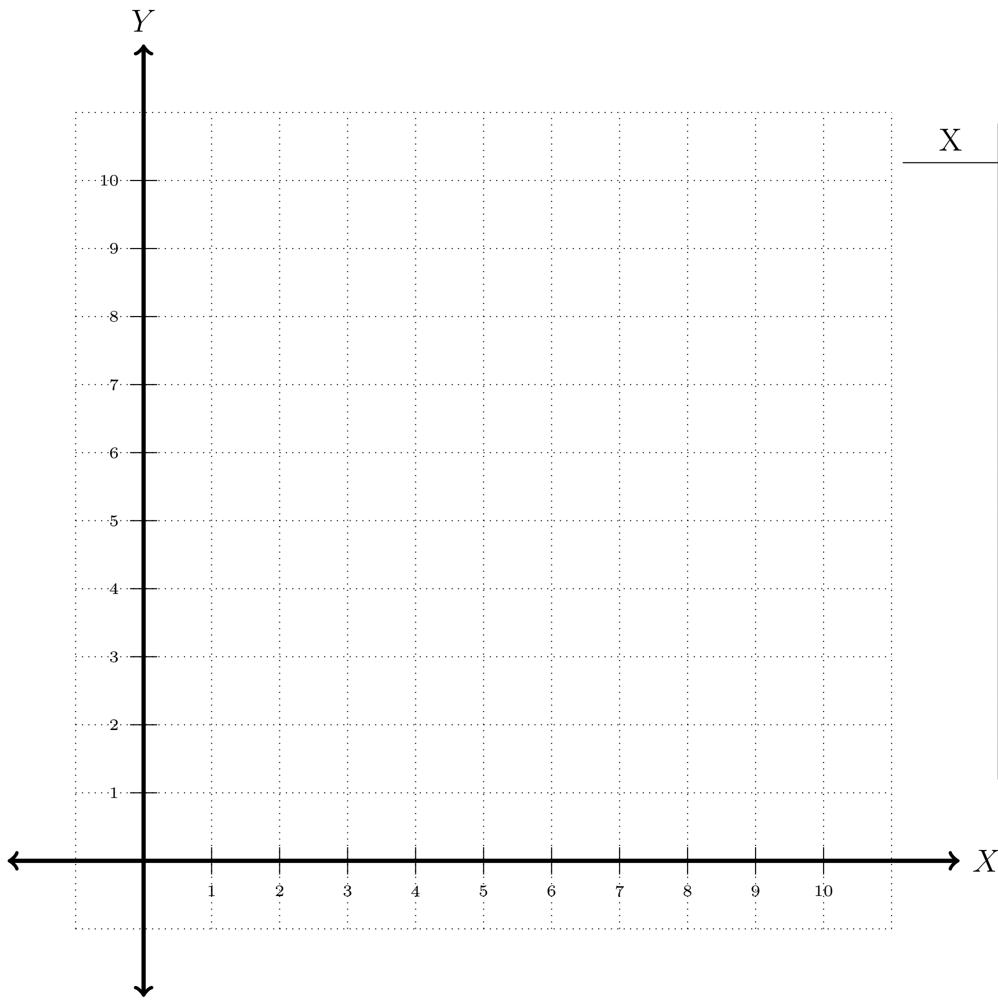
This will leave him with:

- ______ tubes of Amarillo, and
- ______ tubes of Berry Red, and
- ______ blank canvasses, and
- ______ tubes of Dark Blue.

Answers with no work receive no credit.

4. **Completely solve the following LPP using the graphical method.** Graph the feasible region for the following LPP. You will be graded on three aspects: correctly drawn edges, correctly shaded region, and correctly labelled corners. List the corners, determine if the region is bounded or unbounded, and find the maximum value of $P$.

$$\text{Maximize } P = 10x + 12y \text{ subject to } \left\{ \begin{array}{r} 3x+\ \ y \le 25 \\ 3x+2y \le 26 \\ x+\ \ y \le 10 \\ x+3y \le 24 \end{array} \right\} \text{ and } x \ge 0,\ y \ge 0.$$




| X | Y | P |
|---|---|---|
| | | |

The maximum value of $P$ is ______ and it occurs at ($x =$ ______, $y =$ ______).

Answers with no work receive no credit.

5. Two fair dice are rolled.

   (a) What is the probability that the first die is odd?

   (b) What is the probability that the total roll is 9 or larger?

   (c) What is the probability that both the total roll is 9 or larger and the first die rolled was odd?

   (d) What is the probability that the total roll is 9 or larger given that the first die rolled was odd?

   (e) Are the events “total roll is 9 or larger” and “the first die is odd” independent, mutually exclusive, both, or neither?

Answers with no work receive no credit.

6. A survey of 100 College students were asked for their opinions about pizza. They were specifically whether they liked pepperoni, mushrooms, and garlic.

   - 44 students liked pepperoni.
   - 40 students liked mushrooms.
   - 38 students liked garlic.
   - 14 students liked both pepperoni and mushrooms.
   - 13 students liked both pepperoni and garlic.
   - 12 students liked both mushrooms and garlic.
   - 9 students liked all three toppings.

   Based on the above information, answer the following questions. You must show your calculations to receive credit.

   (a) What is the probability that a random student did not like any of the toppings?

   **Answer:** [ ]

   (b) What is the probability that a random student liked at least two of the toppings?

   **Answer:** [ ]

Answers with no work receive no credit.

7. You are examining a budget cut proposal. In the cut, 85 out of 340 managers will be laid off. A total of 230 out of 940 employees will be laid off, including the managers.

   (a) What is the probability a random employee will be laid off?

   (b) What is the probability a random non-manager will be laid off?

   (c) What is the probability an employee that gets laid off is a manager?

   (d) Are the events "getting laid off" and "being a manager" independent?

Answers with no work receive no credit.

8. A drug test is 98% accurate: out of 100 drug users, 98 will get a positive result, and 2 a negative; out of 100 non-users 98 will get a negative result, and 2 a positive. A company (somehow) knows that exactly 1 of its 100 employees is a drug user, but (somehow) does not know which employees are which.

   (a) An employee is picked at random to be tested, and tests positive. What is the probability that an employee is the drug user, given that they tested positive? Hint: It is NOT 98%.

   (b) The company wants to be sure, and so tested the employee again. Positive. again. What is the probability that an employee is the drug user, given that they tested positive twice?

   (c) What is the probability that the drug test would correctly report on all 100 employees, assuming each drug test is run independently?

   (d) An employee is picked at random to be tested twice, and tests positive once and negative once. What is the probability an employee is the drug user, given that they tested positive once and negative once?

# Answers:

**1** Identify the variables as the decision to be made. What do you have (limited but direct) control over? Identify the constraints as the extra requirements and relations between the variables. How much flexibility do you have with those decisions?

**2** *In general* Identify columns as either free or basic. The basic columns have all 0s except one 1, and are said to "own" the row containing the 1. If more than one column looks basic but owns the same row, then you just choose one of those columns to be basic, and the other is still free. Write out each row of the matrix as an equation, and then solve that equation for the basic variable in terms of the free variables.

**2a** C and D are free. Every other column (besides the right hand side, rhs) consists only of 0s and one 1.

**2b** The last row means $0X + 0Y + 0A + 0B + 9C + 1D + 1P = 114$, and so solving for the non-free variable $P$ one gets:

$$P = 114 - 9C - D$$

Notice the number involved is the maximum from #4. Look at the other matrices. Do you recognize their bottom right corners from #4?

**2c** Everytime we increase $C$ by 1, we decrease profit by 9. Everytime we increase $D$ by 1, we decrease profit by 1. Obviously, we want $C$ and $D$ to be as small as possible. Since they cannot be negative, we set them both to 0. $C = D = 0$.

Notice how the previous matrices had $P$ equal to a number *plus* some of the free variables, so setting those free variables to 0 would not have been optimal, only feasible.

**2d** $A - 4C + D = 9$ becomes $A = 9 + 4C - D$, but we already set $C = D = 0$, so this simplifies to just $A = 9$, meaning there are 9 tubes of Amarillo paint left unused.

**3** We begin by determining the decision variables:

- Let $x$ be the number of Sunshine! paintings to produce, and
- Let $y$ be the number of Lollipop paintings to produce

Now we list the constraints:

$$\begin{cases} 3x + \ \ y \leq 25 & \text{Amarillo supply} \\ 3x + 2y \leq 26 & \text{Berry red supply} \\ x + \ \ y \leq 10 & \text{Canvas supply} \\ x + 3y \leq 24 & \text{Dark blue supply} \end{cases}$$

and of course $x \geq 0$, $y \geq 0$.

Now we determine the objective is to maximize the profit:

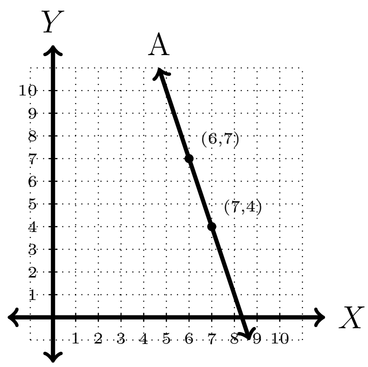
$$P = 10x + 12y$$

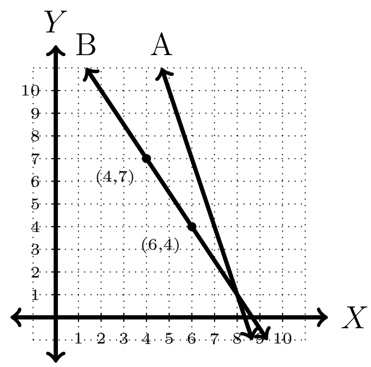
We consult problem #4 for the maximum which is achieved at $(x = 3, y = 7)$, giving $P = 30 + 84 = 114$. Hence we answer the question:

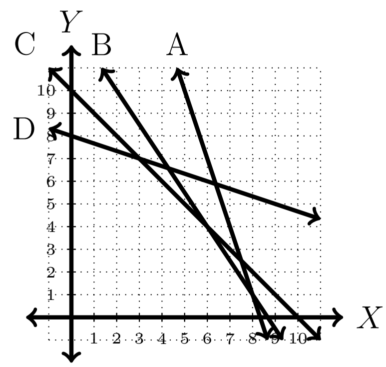
- Vincent should make $x = 3$ Sunshine! paintings,
- and $y = 7$ Lollipop paintings,
- in order to get a profit of $P = \$114$, while leaving
- $25 - 3x - y = 9$ tubes of Amarillo,
- $26 - 3x - 2y = 3$ tubes of Berry red,
- $10 - x - y = 0$ Canvassses, and
- $24 - x - 3y = 0$ tubes of Dark blue.

**4** This is a long problem with many parts. Begin by drawing the lines corresponding to the constraints. Most of the lines can be drawn using the cover up method (plug in $x = 0$ to get an $(x = 0, y)$ point, and then plug in $y = 0$ to get an $(x, y = 0)$ point).

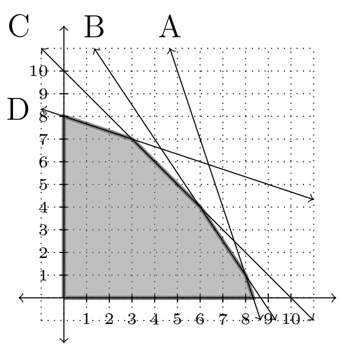
Draw $A : 3x + y = 25$ by plugging in $x = 6$ to get $y = 7$, and plugging in $x = 7$ to get $y = 4$. That is two points $(6, 7)$ and $(7, 4)$, and there is only one line that goes through both. Draw $B : 3x + 2y = 26$ by plugging in $x = 6$ to get $y = 4$, and then plugging in $x = 4$ to get $y = 7$; that is two points $(6, 4)$ and $(4, 7)$ and $B$ is the line between them.

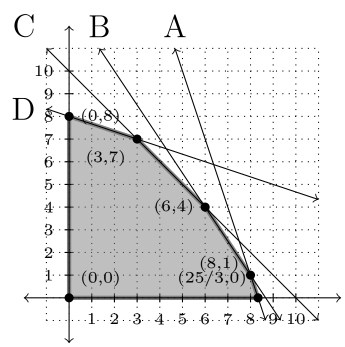



Now test which region is correct. I recommend the guess-and-check method. Choose a point that is not on any of the lines, but which you think might be a reasonable answer. For instance 1 of each kind of painting will not use up his supplies. In other words plugging $(x = 1, y = 1)$ into all of the inequalities always results in true statements. For example $A : 3(1) + (1) \leq 25$ since $3 \leq 25$, etc. Hence the correct region to shade is the region with $(1, 1)$ inside.




| X | Y | P |
|---|---|---|
| 0 | 0 | (10)(0) + (12)(0) = 0 |
| 0 | 8 | (10)(0) + (12)(8) = 96 |
| 3 | 7 | (10)(3) + (12)(7) = 114 |
| 6 | 4 | (10)(6) + (12)(4) = 108 |
| 8 | 1 | (10)(8) + (12)(1) = 92 |
| 8.3 | 0 | (10)(8.3) + (12)(0)= 83 |

Next we find the corners. Since we drew a nice picture we only need to intersect the lines that actually cross at corners. The bottom left is the easiest intersection: $x = 0$ with $y = 0$ intersects exactly at the point $(x = 0, y = 0)$. The corner above it is the intersection of $x = 0$ with $D : x + 3y = 24$, which simplifies to $0 + 3y = 24$, that is, the point $(x = 0, y = 8)$. The corner to the right of it is the intersection of $D : x + 3y = 24$ with $C : x + y = 10$; subtracting the two equations gives $2y = 14$, so $y = 7$ and $x + 7 = 10$, so the intersection is $(x = 4, y = 7)$. Continuing in this way we get all the corners.

Finally we plug the corners into the objective function to calculate the profit at all of the extreme strategies (where we use up at least two of the supplies). The maximum profit occurs at the $(x = 3, y = 7)$ strategy, with 114 as the profit.

**5a** I suggest just listing the possibilities: #{ 11 12 13 14 15 16 31 32 33 34 35 36 51 52 53 54 55 56} = 18 ways, out of 36 total, so 18/36 = 50%. Of course, it is clear that the second die is irrelevant, so one could just list #{ 1 3 5 } = 3 out of 6, so 3/6 = 50%, but in some sense this is being too smart.

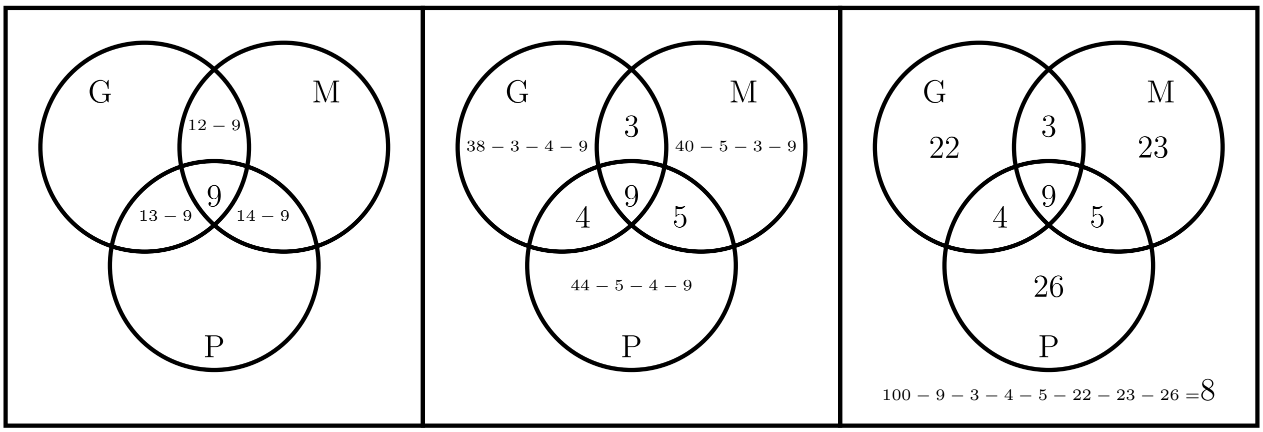
**5b** I suggest just listing the possibilities: #{ 36 45 54 63 46 55 64 56 65 66 } = 10 ways, out of 36 total, so 10/36 = 27.78%.

**5c** I suggest just listing the possibilities: #{ 36 54 55 56 } = 4 ways out of 36 ways, so 4/36 = 11.11%.

**5d** I suggest just listing the possibilities: #{ 36 54 55 56 } = 4 ways to get both out of 18 ways to get the first die odd, so 4/18 = 22.22%. Notice how the universe has shrunk from 36 total possibilities, to only the 18 possibilities from part (a).

**5e** They are neither. They are not mutually exclusive because 36 is a sample point contained in both events. In other words, it is possible to both get a total roll 9 or above and to have the first die roll an odd number. They are not independent, because the probability of both happening (4/36) is not the product of each individual event happening (10/36 and 18/36).

**6** For this problem it may be easiest just to figure out the entire Venn diagram. 12 liked at least mushroom and garlic, but 9 liked all three, so only $12 - 9 = 3$ liked exactly mushroom and garlic, but not pepperoni. Similarly we can fill in the inner flower. 44 liked pepperoni, but 5, 4, and 9 also liked some other topping, so only $44 - 5 - 4 - 9 = 26$ liked exactly pepperoni. One could also say 44 liked at least pepperoni, but 14 also liked mushroom, and 13 also liked garlic, while 9 liked all three, giving $44 - 14 - 13 + 9 = 26$ that liked exactly pepperoni.




**6a** One can either simply look at the outer portion of the Venn diagram to get

$$100 - 9 - 3 - 4 - 5 - 22 - 23 - 26 = 8$$

or one can use the inclusion exclusion formula:

$$100 - 44 - 40 - 38 + 14 + 12 + 13 - 9 = 8$$

**6b** The easiest way is just to consult the Venn diagram to get:

$$9 + 5 + 3 + 4 = 21$$

However, one can also use inclusion exclusion to get:

$$14 + 12 + 13 - 9 - 9 - 9 + 9 = 21$$

or the slightly more intelligent version which notices that if we add up all three "petals" then we added up the center of the flower 3 times instead of just once, so we just subtract it out twice to fix things.

$$14 + 12 + 13 - 9 - 9 = 21$$

**7** This problem is easier once you make a table. First fill in the numbers form the problem.

| | Mngr | Nonm | Employee |
|---|---|---|---|
| Laid Off | 85 | | 230 |
| Kept | | | |
| Total | 340 | | 940 |

Then you know the margins are just the totals, so you can solve for most of the missing entries.

| | Mngr | Nonm | Employee |
|---|---|---|---|
| Laid Off | 85 | 230-85 | 230 |
| Kept | 340-85 | | 940-230 |
| Total | 340 | 940-340 | 940 |

The middle entry is both $710 - 255$ and $600 - 145$.

| | Mngr | Nonm | Employee |
|---|---|---|---|
| Laid Off | 85 | 145 | 230 |
| Kept | 255 | 455 | 710 |
| Total | 340 | 600 | 940 |

**7a** There are 940 total employees, 230 of which got laid off. This is $230/940 = 24.47\%$.

**7b** There are 600 non-managers, 145 of which will be laid off, so the probability is $145/600 = 24.17\%$. Nearly the same as the over-all probability of any employee being laid-off.

**7c** There are 230 employees that are being laid off, and 85 of them are managers, so the probability is $85/230 = 36.96\%$. We can see that most people being laid off are not managers. Is this an indication of bias?

**7d** The probability of a manager getting laid off is $85/340 = 25\%$, which is a bit higher than the probability, 24.47% of a random employee getting laid off, so these events are not perfectly independent. However, the difference in probability is only 0.53%. If just two extra managers had been kept, then the probability would have been $83/340 = 24.41\%$, so the bias is certainly not strong. One could suggest two more managers be kept to minimize the dependence, but certainly the dependence is very weak.

**8a** The question asks for the probability that an employee is a drug user, given that they tested positive. In other words, our population is only employees that test positive. Two kinds of employees test positive: the drug-users where the test works correctly, and the non-users where the test works incorrectly. First we calculate the probability of a random employee testing positive:

$$\begin{aligned}
P(\text{positive}) &= P(\text{true positive}) + P(\text{false positive}) \\
&= P(\text{user})P(\text{correct}|\text{user}) + P(\text{non user})P(\text{incorrect}|\text{non user}) \\
&= \left(\tfrac{1}{100}\right)\cdot\left(\tfrac{98}{100}\right) + \left(\tfrac{100-1}{100}\right)\cdot\left(\tfrac{2}{100}\right) \\
&= 0.97809\% + 1.97906\% \\
&= 2.95715\%
\end{aligned}$$

Now we find the requested probability:

$$\begin{aligned}
P(\text{drug user}|\text{positive}) &= \tfrac{P(\text{true positive})}{P(\text{positive})} \\
&= \tfrac{0.97809\%}{2.95715\%} \\
&= 33.07718\%
\end{aligned}$$

The 98% figure comes from a population consisting entirely of drug-users, an entirely different population.

$$P(\text{positive}|\text{drug user}) = 98\%$$

**8b** This is very similar to the previous question, except to it was either a true positive twice in a row, or a false positive twice in a row.

$$\begin{aligned}
P(\text{2x positive}) &= P(\text{2x true positive}) + P(\text{2x false positive}) \\
&= P(\text{user})P(\text{correct}|\text{user})^2 + P(\text{non user})P(\text{incorrect}|\text{non user})^2 \\
&= \left(\tfrac{1}{100}\right)\cdot\left(\tfrac{98}{100}\right)^2 + \left(\tfrac{100-1}{100}\right)\cdot\left(\tfrac{2}{100}\right)^2 \\
&= 0.95978\% + 0.03815\% \\
&= 0.99792\%
\end{aligned}$$

Now we find the requested probability:

$$\begin{aligned}
P(\text{drug user}|\text{2x positive}) &= \tfrac{P(\text{2x true positive})}{P(\text{2x positive})} \\
&= \tfrac{0.95978\%}{0.99792\%} \\
&= 96.17015\%
\end{aligned}$$

**8c** We want all 100 tests to run correctly, and each test is independent, so we just multiply these probabilities together:

$$P(\text{100x correct}) = P(\text{correct})^{100} = \left(\frac{98}{100}\right)^{100} = 13.24463\%$$

**8d** There are two ways to test positive once and negative once: either be a drug user and get one correct and one incorrect, or be a non user and get one correct and one incorrect.

$$\begin{aligned}
P(\text{pos,neg}) &= P(\text{true pos, false neg}) + P(\text{false pos, true neg}) \\
&= P(\text{user})P(\text{correct}|\text{user})P(\text{wrong}|\text{user}) + P(\text{non})P(\text{wrong}|\text{non})P(\text{correct}|\text{non}) \\
&= \left(\tfrac{1}{100}\right)\cdot\left(\tfrac{98}{100}\right)\cdot\left(\tfrac{2}{100}\right) + \left(\tfrac{100-1}{100}\right)\cdot\left(\tfrac{2}{100}\right)\cdot\left(\tfrac{98}{100}\right) \\
&= 0.01831\% + 1.93939\% \\
&= 1.95923\%
\end{aligned}$$

Now we find the requested probability:

$$\begin{aligned}
P(\text{drug user}|\text{pos,neg}) &= \tfrac{P(\text{true pos, false neg})}{P(\text{pos,neg})} \\
&= \tfrac{0.01831\%}{1.95923\%} \\
&= 1.0\%
\end{aligned}$$

Notice that this is exactly the probability of a random employee being a drug-user, regardless of any test results.