MA162: Finite mathematics

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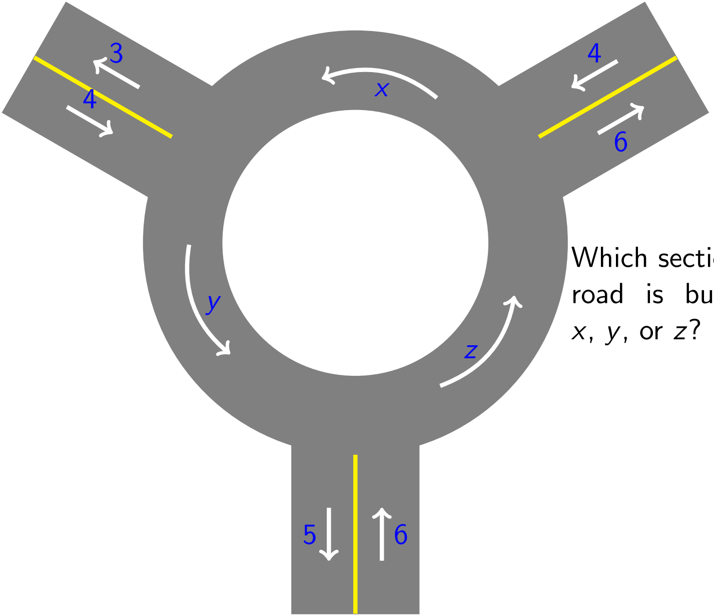
January 30, 2012

Schedule:

- HW 2.3-2.4 are due Friday, Feb 3rd, 2012.
- Exam 1 is Monday, Feb 6th, 5:00pm-7:00pm in CB106 and CB118.

Today we will cover 2.3 and pages 7-8 of the appendix: degeneracy and RREF

Roundabout: An applied problem



Which section of road is busiest, x, y, or z?

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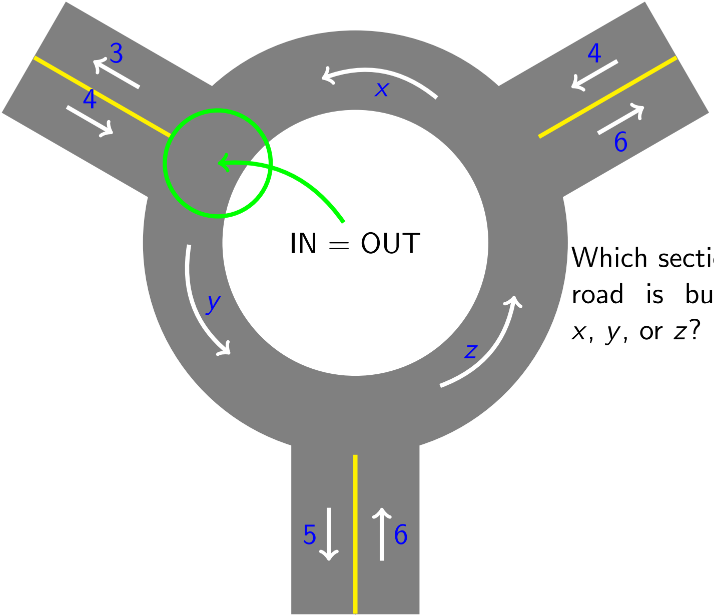

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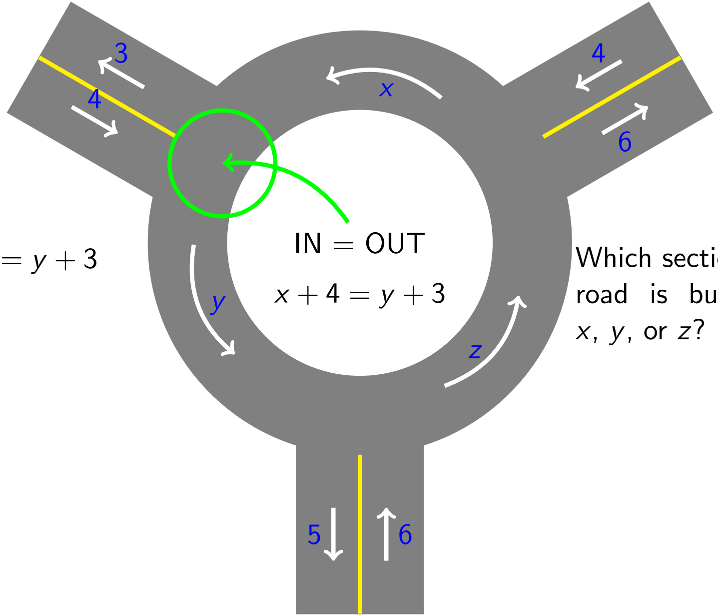

$x + 4 = y + 3$

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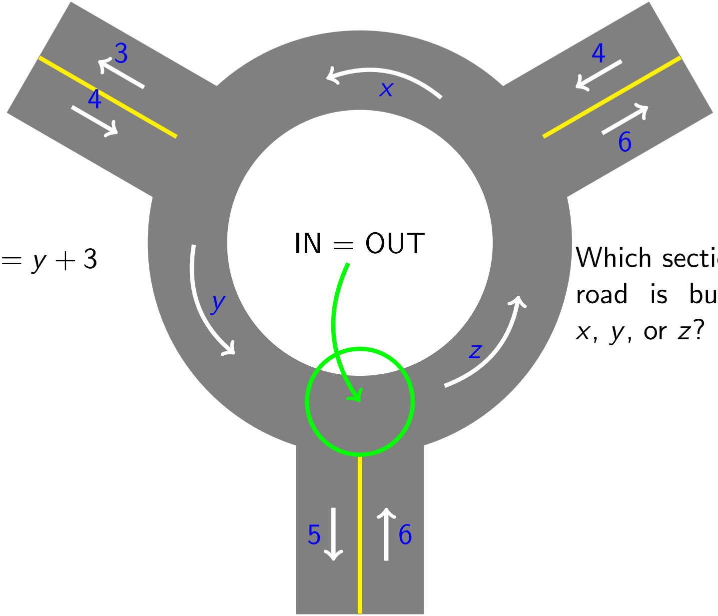

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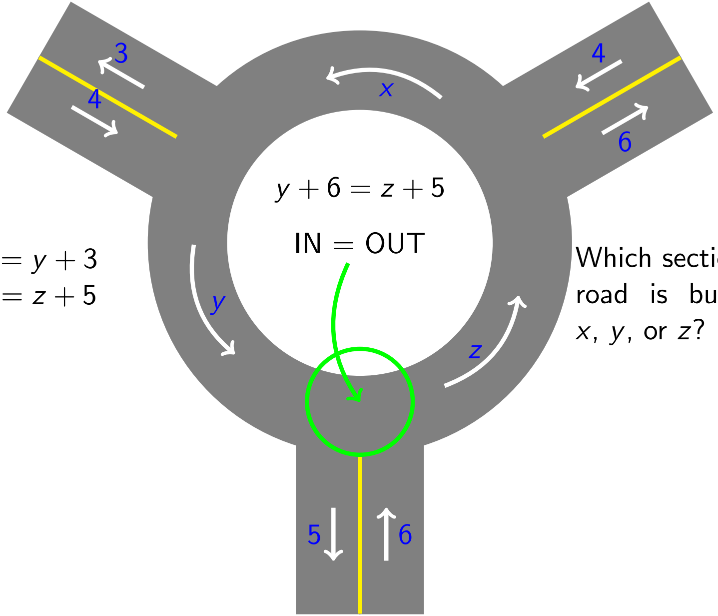

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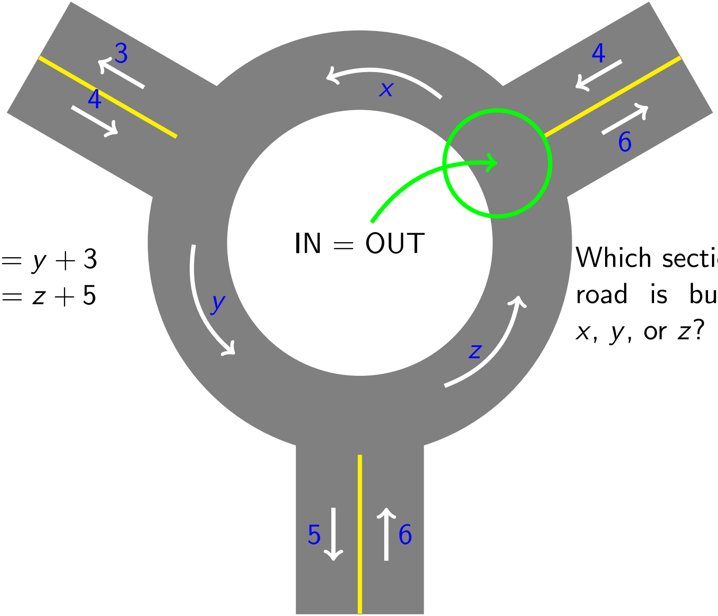

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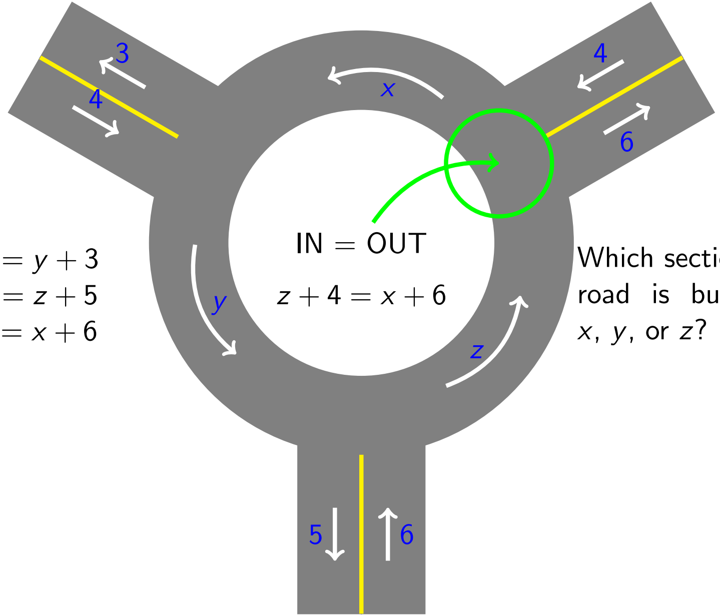

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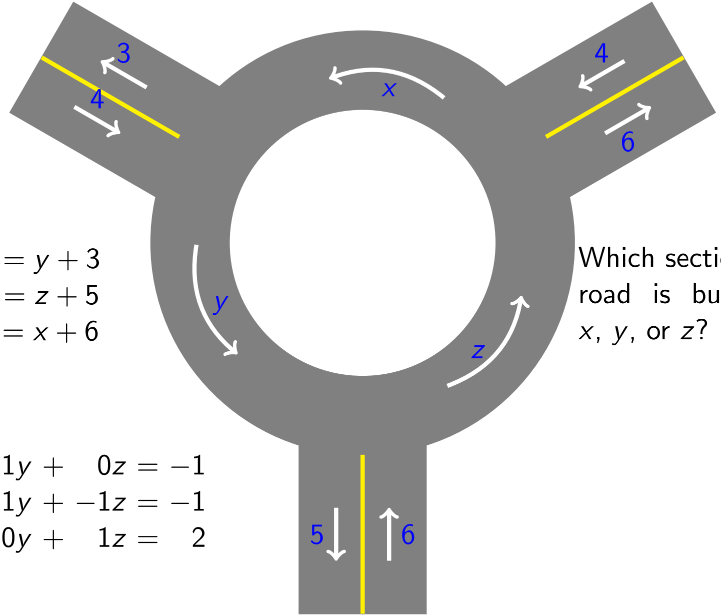
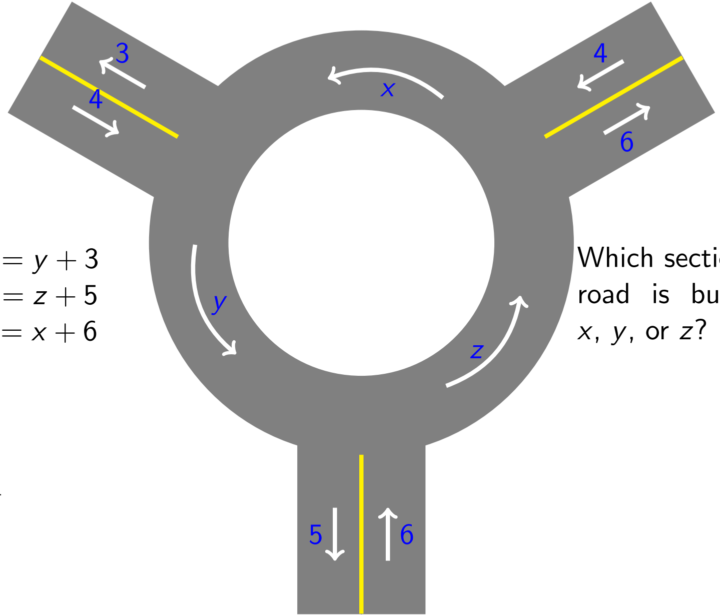


$$x + 4 = y + 3$$
$$y + 6 = z + 5$$
$$z + 4 = x + 6$$

Which section of road is busiest, x, y, or z?

$$\begin{aligned} 1x + -1y + \;\; 0z &= -1 \\ 0x + \;\; 1y + -1z &= -1 \\ -1x + \;\; 0y + \;\; 1z &= \;\; 2 \end{aligned}$$

Roundabout: An applied problem



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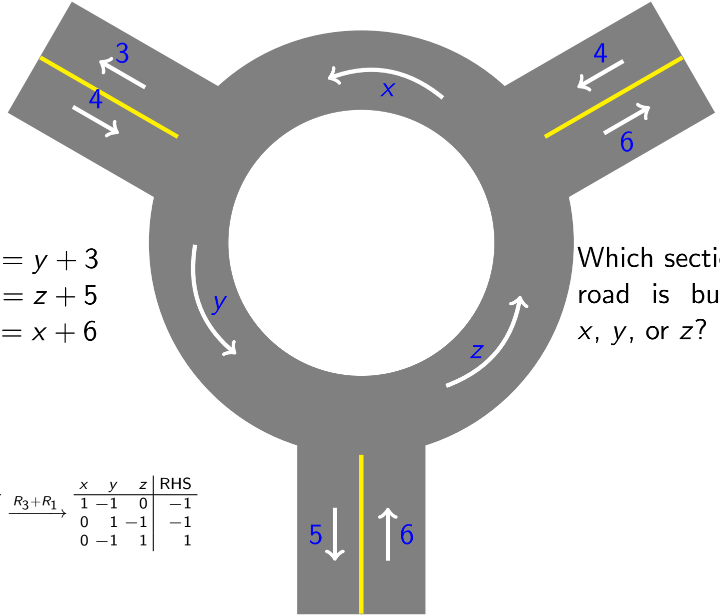

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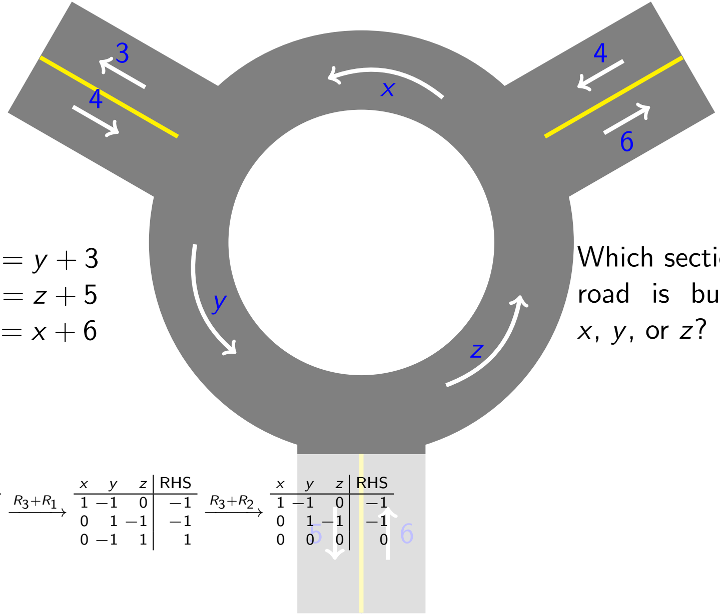
$\xrightarrow{R_3+R_1}$

x	y	z	RHS
1	−1	0	−1
0	1	−1	−1
0	−1	1	1

Roundabout: An applied problem



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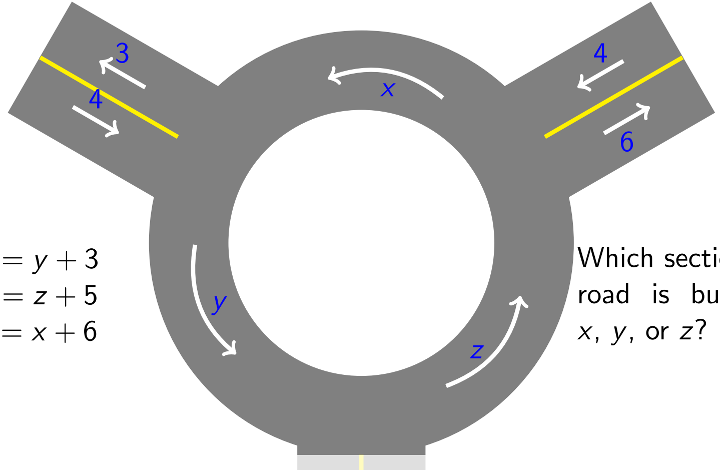

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\begin{aligned}
x + 4 &= y + 3 \\
y + 6 &= z + 5 \\
z + 4 &= x + 6
\end{aligned}
$$

Which section of road is busiest, x, y, or z?

$$
\begin{array}{rrr|r}
x & y & z & \text{RHS} \\ \hline
1 & -1 & 0 & -1 \\
0 & 1 & -1 & -1 \\
-1 & 0 & 1 & 2
\end{array}
\xrightarrow{R_3+R_1}
\begin{array}{rrr|r}
x & y & z & \text{RHS} \\ \hline
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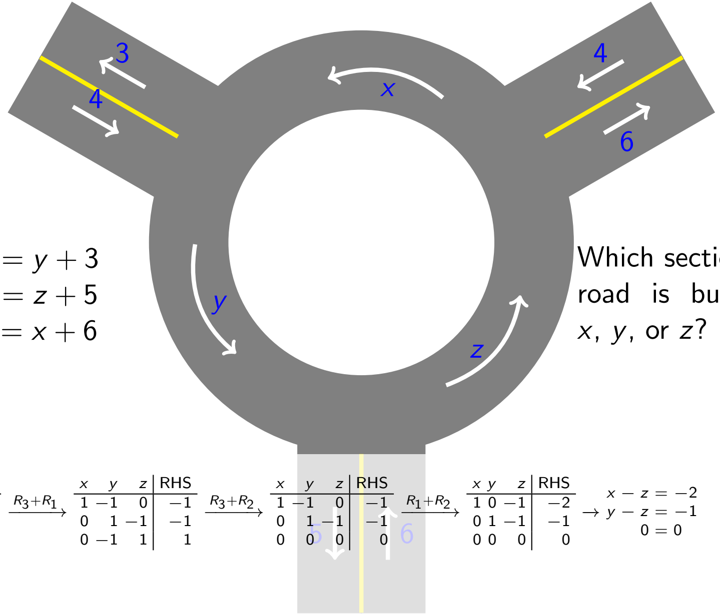

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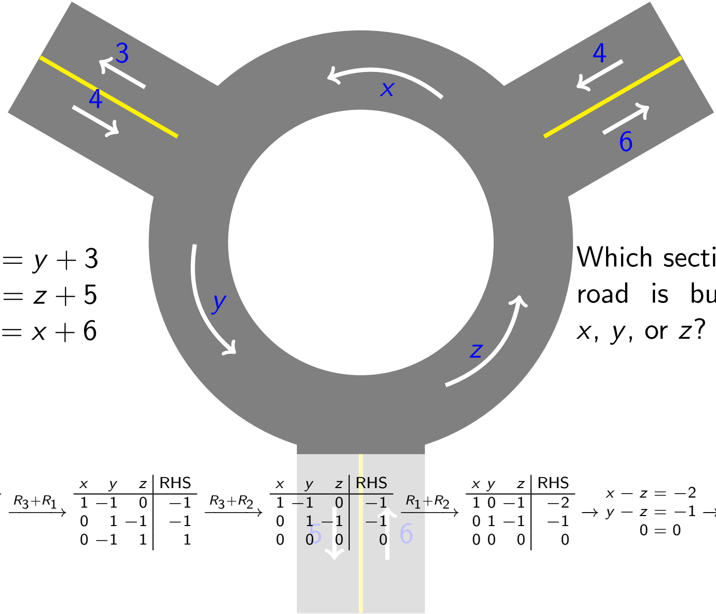

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$\xrightarrow{R_1+R_2}$

x	y	z	RHS
1	0	−1	−2
0	1	−1	−1
0	0	0	0

$$\rightarrow \begin{aligned} x - z &= -2 \\ y - z &= -1 \\ 0 &= 0 \end{aligned} \rightarrow \begin{aligned} x &= z - 2 \\ y &= z - 1 \\ 0 &= 0 \end{aligned}$$

Appendix: Very efficiently solving systems

- We managed to solve some fairly big systems last time using our **new** number crunching skills.
- Mostly it was repetitive, routine, soothing.
- But near the end we stopped the number pushing and revived the variables, which totally harshed my zen.
- Today we learn to finish the easy way

Appendix: Cleaning above as well as below

- A matrix is in **REF** if no column (left of the bar) has two pivots
- This means that below and to the left of each pivot are zeros
- A matrix is in **RREF** if
  - it is in REF,
  - there are only zeros above pivots, and
  - pivots are equal to 1

Appendix: How to clean

- If a matrix is in REF, then a **possible target** is a non-zero number above a pivot
- We choose the right-most column with a possible target, and then choose the bottom-most possible target in that column
- The row operation is the same as before:

$$R_{target} - \frac{target}{active} \cdot R_{active}$$

- If a pivot is not equal to one, then we can divide the whole row by the pivot

Appendix: Example

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow$$

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$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

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$$\xrightarrow{\frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Appendix: Example

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RREF

2.3: More practice

- Row reduce these matrices:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 59 \\ 0 & 1 & 5 & 47 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 16 \\ 0 & 1 & -4 & -25 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

2.3: More practice

- Row reduce these matrices:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 59 \\ 0 & 1 & 5 & 47 \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow[R_2-5R_3]{R_1-4R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 27 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 16 \\ 0 & 1 & -4 & -25 \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow[R_2+4R_3]{R_1-3R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -8 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

- Notice how two different REFs have the same RREF

2.3: What if things go wrong?

- Is this matrix in REF? RREF?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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- Is this matrix in REF? RREF?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- What could we do to fix it?

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  - Row 2 can only make row 1 worse and vice versa!

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  - Row 3 cannot do anything at all!

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- What could we do to fix it?
  - Row 2 can only make row 1 worse and vice versa!
  - Row 3 cannot do anything at all!
- Let's write it out in variables, and see what is going on:

$$x + 2y = 3 \qquad z = 4 \qquad 0 = 0$$

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$$x + 2y = 3 \qquad z = 4 \qquad 0 = 0$$

- Well that is not too bad? $x = 3 - 2y$, y is free, $z = 4$.

2.3: What if things go wrong?

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- We can read this right from the matrix

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- Is this matrix in REF? RREF?

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- Let's write it out in variables, and see what is going on:

$$x + 2y = 3 \qquad z = 4 \qquad 0 = 0$$

- Well that is not too bad? $x = 3 - 2y$, y is free, $z = 4$.
- We can read this right from the matrix
- We do say this matrix is in REF and RREF

2.3: Free variables

- If a column (for a variable) has no pivot, then that variable is **free**
- Be careful when reading the answer off the matrix
  120|3 means $x + 2y = 3$, so $x = 3 - 2y$
- If a variable is free, then (assuming there are any solutions) there are **infinitely many solutions**
- What does "no solution" look like in matrix format?

2.3: What if things go wrong?

- Is this matrix in REF? RREF?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

2.3: What if things go wrong?

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$$x + 2y = 3 \qquad z = 4 \qquad 0 = 5$$

- Well that is not too bad? $x = 3 - 2y$, y is free, $z = 4$, and **0=5? What?!**

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- **No solution**, **inconsistent**

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- Well that is not too bad? $x = 3 - 2y$, y is free, $z = 4$, and **0=5? What?!**
- **No solution**, **inconsistent**
- We can read this right from the matrix
- We do say this matrix is in REF and RREF