MA162: Finite mathematics

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SCHEDULE:

- Exam 2 returned during recitation
- HW 5.1,5.2 are due Fri, March 23rd, 2012
- HW 5.3,6.1 are due Fri, March 30th, 2012
- HW 6.2,6.3 are due Fri, April 6th, 2012
- Exam 3 is Monday, Apr 9th, 5:00pm-7:00pm in CB106 and CB118.

Today we will cover 5.3: amortized loans. We will be using calculators today.

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
    short term, interest not reinvested
  - Compound interest
    one payment, interest reinvested
  - Sinking funds
    recurring payments, big money in the future
  - Amortized loans
    recurring payments, big money in the present

- Chapter 6, Counting
  - Inclusion exclusion
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  - Multiplication principle
  - Permutations and combinations

5.2: Summary

- Monday we learned about **annuities**, **present value**, **future value**, and **total payout**
  - Future value of annuity, paying out n times at per-period interest rate i

$$A = R\frac{(1+i)^n - 1}{i}$$

  - Present value of annuity is just future value divided by $(1+i)^n$
  - Total payout is just nR, n payments of R each
- You should be done with homework for 5.1 and 5.2.
- Today we handle 5.3.

5.3: Buying annuities

- How much would you pay today for an annuity paying you back \$100 per month for 12 months?
- No more than \$1200 for sure,
  if you had \$1200 you could just pay yourself
- If you have a 12% APR (1% per month) account,
  then you could invest the money each month,
  In one year you have \$1268.25.
- How much would you need right now (one payment)
  in order to have \$1268.25 in the account after one year?

5.3: Buying annuities

- We solve a 5.1 problem:

$$\begin{aligned} P &= ? \\ i &= 0.12/12 = 0.01 \text{ per month} \\ n &= 12 \text{ months} \\ A &= \$1268.25 \\ A &= P(1+i)^n \\ \$1268.25 &= P(1.01)^{12} \\ P &= \$1268.25/(1.01)^{12} = \$1125.50 \end{aligned}$$

- If we had \$1125.50 right now, we could invest it to end up with \$1268.25

- If we got \$100 every month, we could invest it to end up with \$1268.25

- So the cash flow is worth \$1125.50 now

5.3: Pricing annuities again

- What if we don't want to invest it?
  What if we want to spend \$100 every month?
- Well, put \$1125.50 in the bank and remove \$100 every month
- How much is left at the end of the year?

Date	Old Balance	Interest on Old	Withdrawal	New Balance
Jan	\$1125.50	\$11.26	\$100.00	\$1036.76
Feb	\$1036.76	\$10.37	\$100.00	\$ 947.12
Mar	\$ 947.12	\$ 9.47	\$100.00	\$ 856.59
Apr	\$ 856.59	\$ 8.57	\$100.00	\$ 765.16
May	\$ 765.16	\$ 7.65	\$100.00	\$ 672.81
Jun	\$ 672.81	\$ 6.73	\$100.00	\$ 579.54
Jul	\$ 579.54	\$ 5.80	\$100.00	\$ 485.33
Aug	\$ 485.33	\$ 4.85	\$100.00	\$ 390.19
Sep	\$ 390.19	\$ 3.90	\$100.00	\$ 294.09
Oct	\$ 294.09	\$ 2.94	\$100.00	\$ 197.03
Nov	\$ 197.03	\$ 1.97	\$100.00	\$ 99.00
Dec	\$ 99.00	\$ 0.99	\$100.00	\$ -0.01

5.3: Pricing an annuity

- To price an annuity using our old formulas:
- Find the future value $A = R((1+i)^n - 1)/(i)$
- Find the present value by solving $A = P(1+i)^n$

$$P = A/((1+i)^n)$$

- If you like new formulas, the book divides the $(1+i)^n$ using algebra:

$$P = R\left(1 - (1+i)^{(-n)}\right)/(i)$$

5.3: Perspective

- Alex borrows \$100 per month from Bart at 1% per month interest, compounded monthly
- Bart thinks of Alex as a savings account
- Bart expects \$1268.25 in his account at the end of the year
- Alex owes Bart \$1268.25 at the end of the year

- What if the bank called you up and wanted to buy an annuity?
- What if Bart wants Alex to pay in advance?
  How much does Alex owe Bart up front?

5.3: Amortized loan

- Most people don't say "the bank purchased an annuity from me"
- "I owe the bank money every month, because they gave me a loan"
- So the bank gives you \$1125.50 and expects 1% interest per month
- You give the bank \$100 back at the end of the month
- You owe:
  \$1125.50 + (1% of it) - \$100
  = \$1125.50 + \$11.26 - \$100
  = \$1036.76

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- Amortized loans are just present values of annuities

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  After one month of paying, we estimate two months fewer
  How many is it really?

5.3: Finding the time

- The debt is paid once the future value of the annuity is equal to the future value of the debt
- Annuity:

$$\begin{aligned} A &= R((1+i)^n - 1)/(i) \\ \\ R &= \$20 \\ i &= 0.12/12 = 0.01 \\ n &= ? \\ A &= \ldots \end{aligned}$$

  Debt:

$$\begin{aligned} A &= P(1+i)^n \\ \\ P &= \$1000 \\ i &= 0.01 \\ n &= ? \\ A &= \$1000(1.01)^n \end{aligned}$$

- So solve:

$$\$20(1.01^n - 1)/0.01 = \$1000(1.01)^n$$

5.3: Algebra

Need to solve:

$$\$20(1.01^n - 1)/0.01 = \$1000(1.01)^n$$

divide both sides by \$1000 and notice $\$20/0.01/\$1000 = 2$:

$$2(1.01^n - 1) = 1.01^n$$

distribute:

$$2(1.01^n) - 2 = 1.01^n$$

subtract 1.01^n from both sides, add 2 to both sides:

$$1.01^n = 2$$

Now what?

5.3: Logarithms

- To solve:

$$1.01^n = 2$$

- Take **logarithms** of both sides:

$$(n)(\log(1.01)) = \log(2)$$

- $\log(1.01)$ is just a number (some might say 0.004321373783)
- Divide both sides by $\log(1.01)$ to get:

$$n = \log(2)/\log(1.01) \approx 69.66 \approx 70$$

- $n = 70$ months
- Monthly payments are worth the same as the debt after 70 months