Symplectic centralizers of certain unipotent elements

Jack Schmidt

The book (Malle & Testerman, 2011) has been a wonderful help, but I worry there is an error in an exercise 20.10, page 174, asking one to show that a specific unipotent element does not lie in the connected component of the identity of its centralizer in Sp_4. I calculated the centralizer explicitly, and I believe its coordinate ring is an integral domain.

I noticed there is a result of (Springer, 1966, Theorem 4.12, p. 134) that $u \notin C_G(u)^o$ when $p = 2$, G is of type B, and u is a regular element. However, in the exercise, u is not a regular element.

1 Exercise as stated

I quote the exercise verbatim to ensure I have not misconstrued its meaning:

Exercise 20.10 Show that the unipotent element

$$u := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}_4$$

of $G = \mathrm{Sp}_4$ over a field of characteristic 2 is not contained in the connected component $C_G(u)^o$ of its centralizer.

This is false. $C_G(u)$ is connected.

Proof. Let $L = \left(\begin{smallmatrix} 1 & 1 \\ . & 1 \end{smallmatrix}\right)$ and

$$u = I \otimes L = \left(\begin{smallmatrix} 1 & 1 & . & . \\ . & 1 & . & . \\ . & . & 1 & 1 \\ . & . & . & 1 \end{smallmatrix}\right).$$

We first calculate the centralizer in M_4 of u, and then restrict to $\mathrm{Sp}(4, K)$. A direct calculation or principles of Kronecker products yields

$$\begin{aligned} C_{M_4}(u) &= \langle A \otimes B : A \in C_{M_2}(I), B \in C_{M_2}(L) \rangle \\ &= \left\{ \left(\begin{smallmatrix} a & b & c & d \\ . & a & . & c \\ e & f & g & h \\ . & e & . & g \end{smallmatrix}\right) : a, b, c, d, e, f, g, h \in K \right\} \end{aligned}$$

where $C_{M_2}(I) = M_2$ is clear, and $C_{M_2}(L) = K[L]$ is easy to verify directly and well known for Jordan blocks.

Such a matrix A lies in Sp_4 if and only if $ae - cg = 1$ and $ah + bg = cf + de$, simply by writing down the coordinate-wise equations $A^T JA = J$ where J is defined on page 7 as

$$J = \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & -1 & . & . \\ -1 & . & . & . \end{pmatrix}.$$

My commutative algebra is not terribly strong, but I believe $P = (ae-cg-1, ah+bg-cf-de)$ is a prime ideal in $R = K[a,b,c,d,e,f,g,h]$ for any field K. For instance, by (Popov, 1974), $D = K[a,c,e,g]/(ae-cg-1)$ is a UFD, and then $ah+bg-cf-de$ is a linear polynomial of content 1 in $R = D[b,d,f,h]$, so prime. □

2 Revised exercise

I believe the regular elements of Sp_4 are the elements of order 4 when K has characteristic 2, such as

$$u = \begin{pmatrix} 1 & 1 & . & . \\ . & 1 & 1 & -1 \\ . & . & 1 & -1 \\ . & . & . & 1 \end{pmatrix}.$$

This element has centralizer

$$C_{M_4}(u) = \left\{ \begin{pmatrix} a & b & c & d \\ . & a & b & -b-c \\ . & . & a & -b \\ . & . & . & a \end{pmatrix} : a,b,c,d \in K \right\}$$

in the algebra of matrices, and these are in Sp_4 iff $a^2 = 1$ and $b^2 = ab + 2ac$.

When the characteristic of K is not 2, then $C_G(u)$ has two connected components, since $(a^2-1, b^2-ab-2ac) = (a-1, b^2-b-2c)(a+1, b^2+b+2c)$. In simpler terms, the two components are distinguished by $a = 1$ versus $a = -1$. Clearly both u and the identity lie in the first component.

When the characteristic of K is 2, then the equations simplify to $a = 1$, $b^2 = b$, and so $C_G(u)$ has two components since $(a-1, b^2-b) = (a-1, b)(a-1, b-1)$. In simpler terms, the two components are distinguished by $b = 0$ versus $b = 1$. The identity lies in the former, and u in the latter, providing an example of the indicated behavior.

References

Malle, G., & Testerman, D. (2011).
Linear algebraic groups and finite groups of Lie type (Vol. 133).
Cambridge: Cambridge University Press.
MR2850737

Popov, V. L. (1974).
Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fiberings.
Izv. Akad. Nauk SSSR Ser. Mat., *38*, 294–322.
MR0357399

Springer, T. A. (1966).
Some arithmetical results on semi-simple Lie algebras.
Inst. Hautes Études Sci. Publ. Math.(30), 115–141.
MR0206171URL:http://www.numdam.org/item?id=PMIHES_1966__30__115_0

Jack Schmidt
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
USA
jack.schmidt@uky.edu
http://www.ms.uky.edu/~jack/

Contents