MA162: Finite mathematics

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September 5, 2012

Schedule:

- HW 2.1-2.2 are due Friday, Sep 7th, 2012.
- HW 2.3-2.4 are due Friday, Sep 14th, 2012.
- Exam 1 is Monday, Sep 24th, 5:00pm-7:00pm in BS107 and BS116.

Today we will cover 2.2, augmented matrices, and the elimination algorithm
Start the ZESTY problem on the worksheet now!

2.2: Do we already know this?

- You and the crew have lunch at Fried-ees most days
- Day 1: you got the Zesty meal for $5
- Day 2: You and a pal got the Yummy bunch, and your apprentice got the Zesty; one check for $17
- Day 3: Your pal got the Xtra crispy, your apprentice got the Yummy, and you got the Zesty for $18
- How much does the Xtra, the Yummy, and the Zesty each cost?

2.2: Do it with equations

$$\begin{aligned} X + Y + Z &= 18 \\ 2Y + Z &= 17 \\ Z &= 5 \end{aligned}$$

- If $Z = 5$, then we know Z:

$$\begin{aligned} X + Y + 5 &= 18 \\ 2Y + 5 &= 17 \end{aligned}$$

- If $2y + 5 = 17$, then $2y = 12$ and $y = 6$

$$X + 6 + 5 = 18$$

- If $x + 6 + 5 = 18$, then $x = 7$.

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2.2: Efficiently solving systems

- We solved systems last time with two variables
- Real decisions involve balancing half a dozen variables
- Two main changes to handle this:
  - **Write down less** to see the important parts clearly
  - Use a **systematic** method to solve

2.2: Efficient notation

- We worked some equations with the variables x, y
- We could have used M and T
- The letters we used did not matter; just placeholders
- Why do we even write them down?
- The plus signs and equals are pretty boring too.
- The only part we need are the numbers
  (and where the numbers are)

2.2: Augmented matrices

$$\begin{aligned} x + 2y &= 4 \\ y + 5z &= 7 \\ 8x + 17y - z &= 9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{array}\right]$$

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$$\begin{aligned} x + 2y &= 4 \\ y + 5z &= 7 \\ 8x + 17y - z &= 9 \end{aligned}$$

$$\begin{matrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{array}\right]$$

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$$\begin{aligned} x + 2y &= 4 \\ y + 5z &= 7 \\ 8x + 17y - z &= 9 \end{aligned}$$

1	2	0	4
0	1	5	7
8	17	-1	9

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{array}\right]$$

2.2: More examples

$$\begin{aligned} 2x + 3z &= 4 \\ 6z + 5y &= 7 \\ 8x + 9y &= 1 \end{aligned} \qquad \begin{aligned} 2x &+ 0y &+ 3z &= 4 \\ 0x &+ 5y &+ 6z &= 7 \\ 8x &+ 9y &+ 0z &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 8 & 9 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} 4x + 3z &= 2 \\ 8z - y &= 7 \\ 5x - 9y &= 6 \end{aligned} \qquad \begin{aligned} 4x &+ 0y &+ 3z &= 2 \\ 0x &- 1y &+ 8z &= 7 \\ 5x &- 9y &+ 0z &= 6 \end{aligned} \qquad \left[\begin{array}{rrr|r} 4 & 0 & 3 & 2 \\ 0 & -1 & 8 & 7 \\ 5 & -9 & 0 & 6 \end{array}\right]$$

$$\begin{aligned} y &= 3 - 2x \\ z &= 7 + 4y \\ x &= 6 + 5z \end{aligned} \qquad \begin{aligned} 2x &+ 1y &+ 0z &= 3 \\ 0x &- 4y &+ 1z &= 7 \\ x &+ 0y &- 5z &= 6 \end{aligned} \qquad \left[\begin{array}{rrr|r} 2 & 1 & 0 & 3 \\ 0 & -4 & 1 & 7 \\ 1 & 0 & -5 & 6 \end{array}\right]$$

2.2: Efficient notation

- We now have a very clean way to write down systems of equations
- Make sure you can convert from a system of equations to the **augmented matrix**
- Make sure you can convert from an augmented matrix to a system of equations

2.2: A systematic procedure

- Now we will learn a method of solving systems
- We will transform the equations until they look like (REF):

$$\begin{aligned} x + 2y + 3z &= 4 \\ 5y + 6z &= 7 \\ 8z &= 9 \end{aligned}$$

- Next time, we will transform them until they look like (RREF):

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

- We will do this by following a set of rules
- Your work on the exam is graded **strictly**

2.2: First step: Find pivots

- The 0th step is to make sure you have got an augmented matrix
- Once you do we look for **pivots**
- Each row should have a pivot;
  it is the **first nonzero** number in the row

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{array}\right]$$

- We want **one pivot per column**
- We are usually **disappointed**

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2.2: Second step: Choose target

- If there are two pivots in one column, we **eliminate** one of them
- The **active pivot** is the first pivot in the first bad column
- The **target pivot** is the next pivot in the first bad column

Active → 1

Target → 8

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{array}\right]$$

- We want to ZERO out the target
  by subtracting a multiple of the active

2.2: Third step: Eliminate the target

- We are now going to subtract a multiple of the active row from the target row

- We choose the multiple: $\dfrac{\text{target pivot}}{\text{active pivot}}$

- In our example, we choose $\dfrac{8}{1} = 8$

$$\begin{array}{rrrrrrcrrrrr}
R_3 & & 8 & 17 & -1 & 9 & & & 8 & 17 & -1 & 9 \\
-8R_1 & -8\cdot(& 1 & 2 & 0 & 4) & & + & -8 & -16 & 0 & -32 \\
\hline
\text{New } R_3 & & & & & & & & 0 & 1 & -1 & -23
\end{array}$$

- We changed the old 8 to a zero!

- This new row will replace our old target row

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2.2: Fourth step: regroup

- Now we rewrite our new matrix and start over with an **easier** system

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 8 & 17 & -1 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & -1 & -23 \end{array}\right]$$

- We also need to **show our work** in a **very specific way**

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2.2: First step again: find pivots

- Now we begin again with our new **simpler** system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & -1 & -23 \end{array}\right]$$

- We find the pivots
- Second column has pivots in rows 2 and 3, so need to clear again!
- This time you do it! (On worksheet)

2.2: First step again: find pivots

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Brief version of 2-4 (again)

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & -1 & -23 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -6 & -30 \end{array}\right]$$

$$\xrightarrow[\text{(optional)}]{R_3/(-6)} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

- The pivots are in a nice diagonal
- No column has more than one pivot, so **REF**
- We can solve this using algebra, first for z, then for y, then for x

2.2: Final step – Back substitution

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -6 & -30 \end{array}\right] \quad \begin{cases} 1X + 2Y + \ \ 0Z = \ \ 4 \\ 0X + 1Y + \ \ 5Z = \ \ 7 \\ 0X + 0Y + -6Z = -30 \end{cases}$$

- $-6Z = -30$ is better known as $Z = 5$
- $Y + 5Z = 7$ is better known as $Y + 25 = 7$, or $Y = -18$ to its friends
- $X + 2Y = 4$ is better known as $X - 36 = 4$, so $X = 40$.
- $(X = 40, Y = -18, Z = 5)$

2.2: Real question

- You have three types of workers: packers, sewers, cutters.
- You have three types of products: short-sleeve, sleeveless, long-sleeve.
- It takes the following amount of time to make them:

	Short	**Less**	**Long**
Pack	4	3	4
Sew	24	22	28
Cut	12	9	15

- You have 24 hours of packers, 80 hours of cutters, and 160 hours of sewers
- How many of each should you make to keep everyone working?

2.2: As system, as matrix

- As a system of equations:
  Make x short-sleeve, y sleeveless, z long-sleeve

$$\begin{cases} 4x + 3y + 4z = 1440 \\ 24x + 22y + 28z = 9600 \\ 12x + 9y + 15z = 4800 \end{cases}$$

- As a matrix:

$$\left(\begin{array}{ccc|c} 4 & 3 & 4 & 1440 \\ 24 & 22 & 28 & 9600 \\ 12 & 9 & 15 & 4800 \end{array}\right)$$

2.2: REF it

$$\left(\begin{array}{ccc|c} 4 & 3 & 4 & 1440 \\ 24 & 22 & 28 & 9600 \\ 12 & 9 & 15 & 4800 \end{array}\right) \xrightarrow[R_3-6R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 4 & 3 & 4 & 1440 \\ 0 & 4 & 4 & 960 \\ 0 & 0 & 3 & 480 \end{array}\right) \qquad \text{REF}$$

- As equations: $\begin{cases} 4x + 3y + 4z = 1440 \\ \phantom{4x + {}} 4y + 4z = 960 \\ \phantom{4x + 3y + {}} 3z = 480 \end{cases}$
- $z = 480/3 = 160$, then $4y + 4(160) = 960$ and $y = 80$, then . . . and $x = 140$
- So make 140 short-sleeve, 80 sleeveless, and 160 long-sleeves