

MA162: Finite mathematics

Jack Schmidt

University of Kentucky

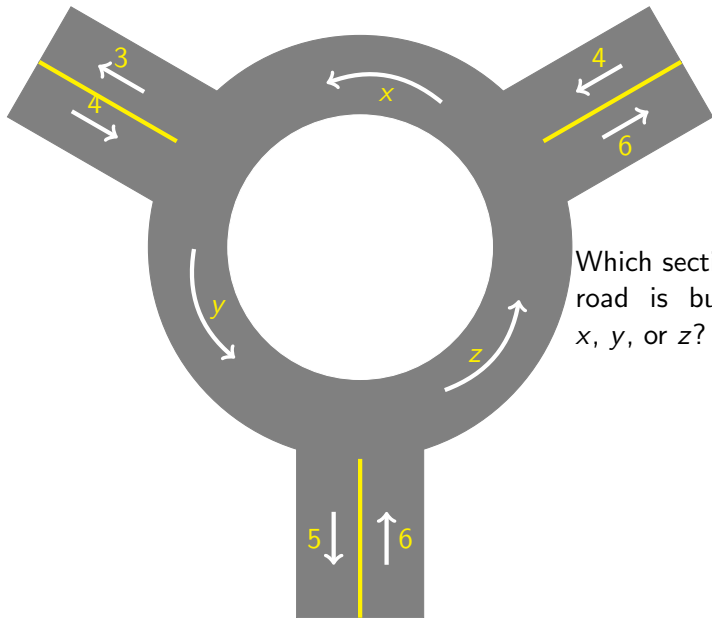
January 28, 2013

SCHEDULE:

- HW 1.1-1.4 due Friday, Jan 18, 2013 (Late)
- HW 2.1-2.2 due Friday, Jan 25, 2013 (Late)
- HW 2.3-2.4 due Friday, Feb 01, 2013
- Exam 1, Monday, Feb 04, 2013, from 5pm to 7pm
- HW 3.1 due Friday, Feb 08, 2013

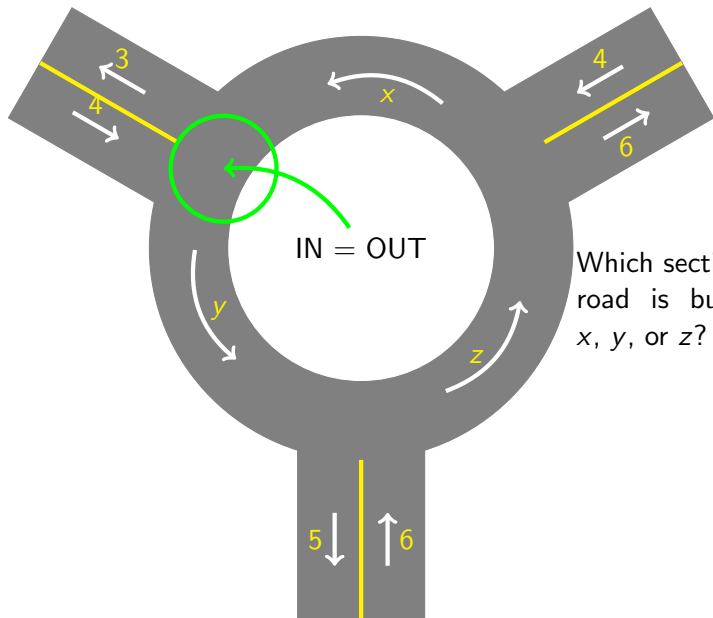
Today we cover 2.3: free variables

Roundabout: An applied problem



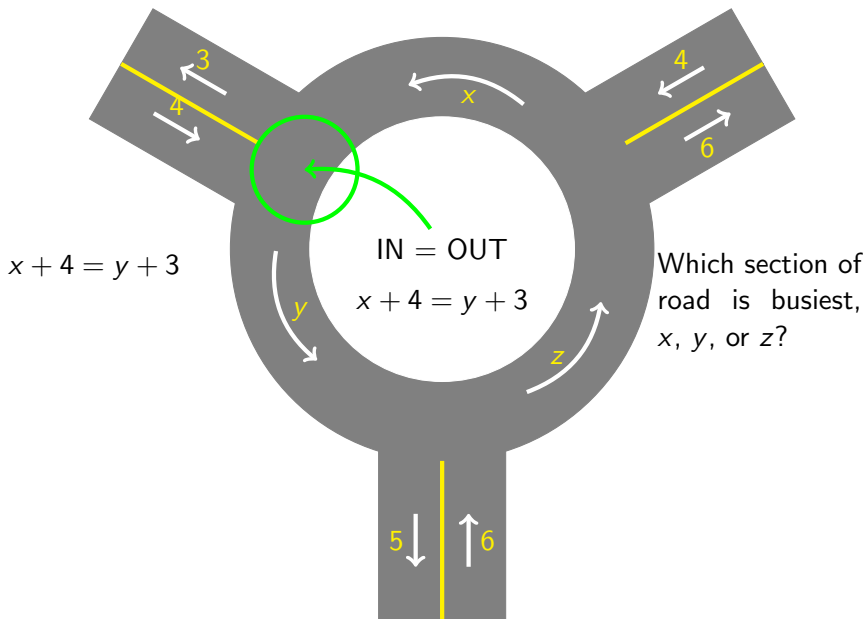
Which section of road is busiest, x , y , or z ?

Roundabout: An applied problem

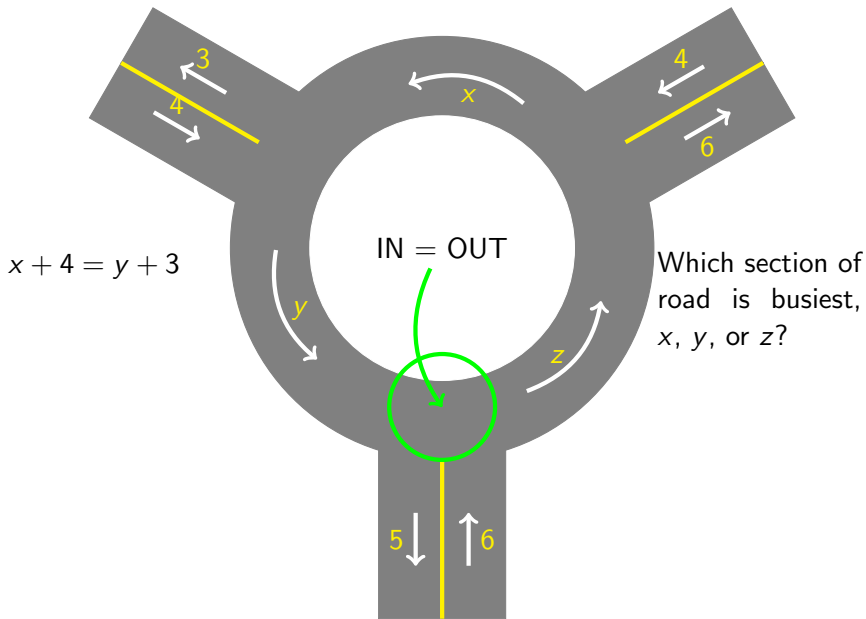


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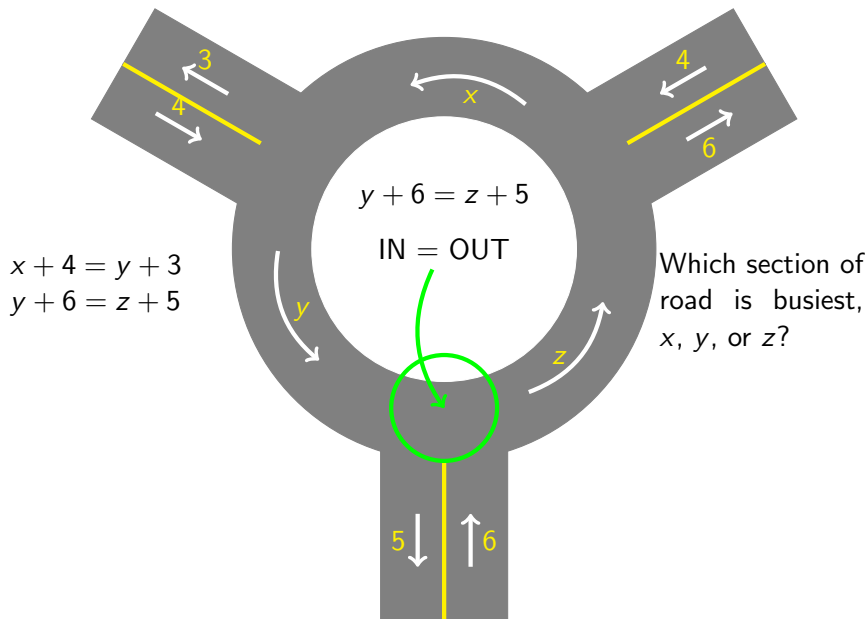
Roundabout: An applied problem



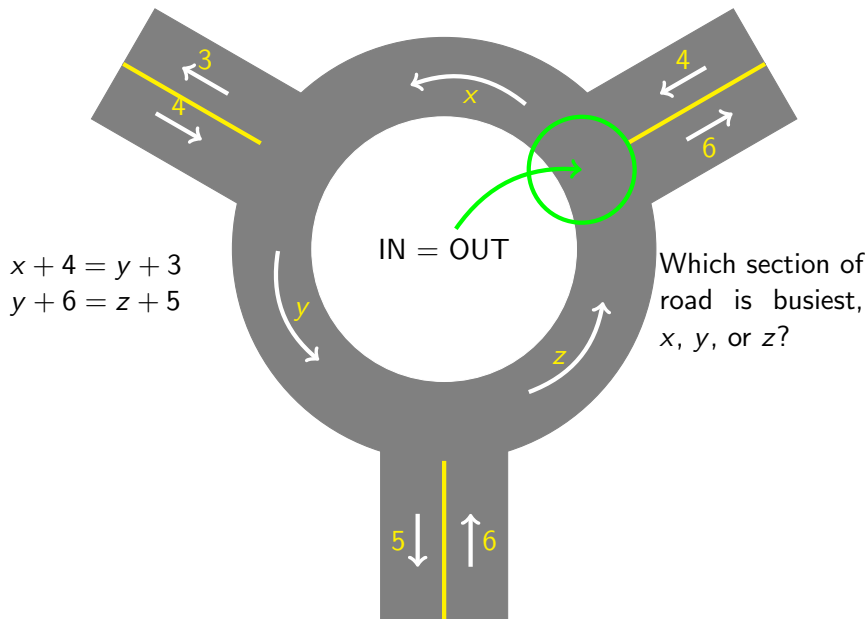
Roundabout: An applied problem



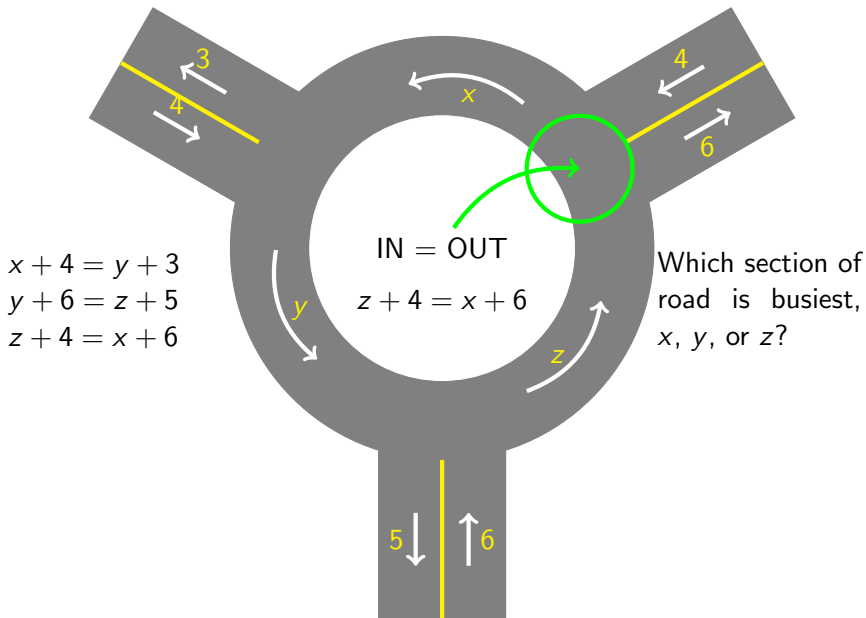
Roundabout: An applied problem



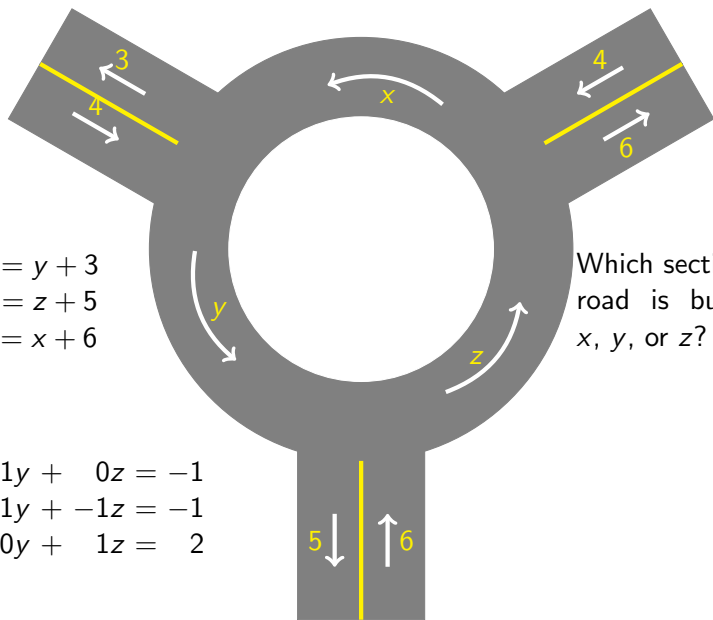
Roundabout: An applied problem



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Roundabout: An applied problem



$$x + 4 = y + 3$$

$$y + 6 = z + 5$$

$$z + 4 = x + 6$$

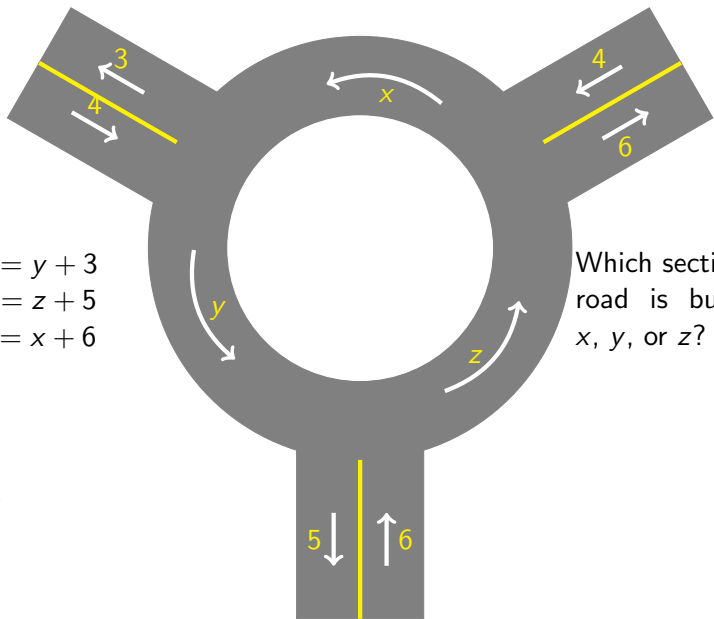
Which section of road is busiest, x , y , or z ?

$$1x + -1y + 0z = -1$$

$$0x + 1y + -1z = -1$$

$$-1x + 0y + 1z = 2$$

Roundabout: An applied problem



$$x + 4 = y + 3$$

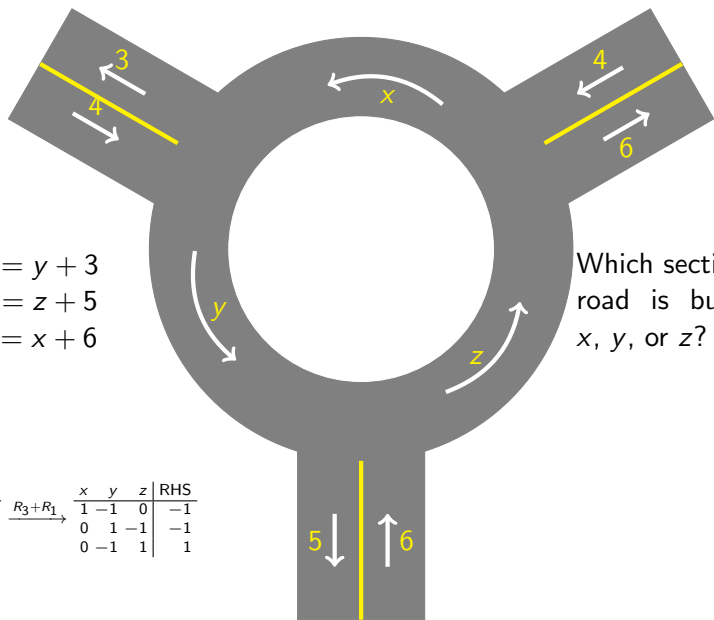
$$y + 6 = z + 5$$

$$z + 4 = x + 6$$

Which section of road is busiest, x , y , or z ?

x	y	z	RHS
1	-1	0	-1
0	1	-1	-1
-1	0	1	2

Roundabout: An applied problem

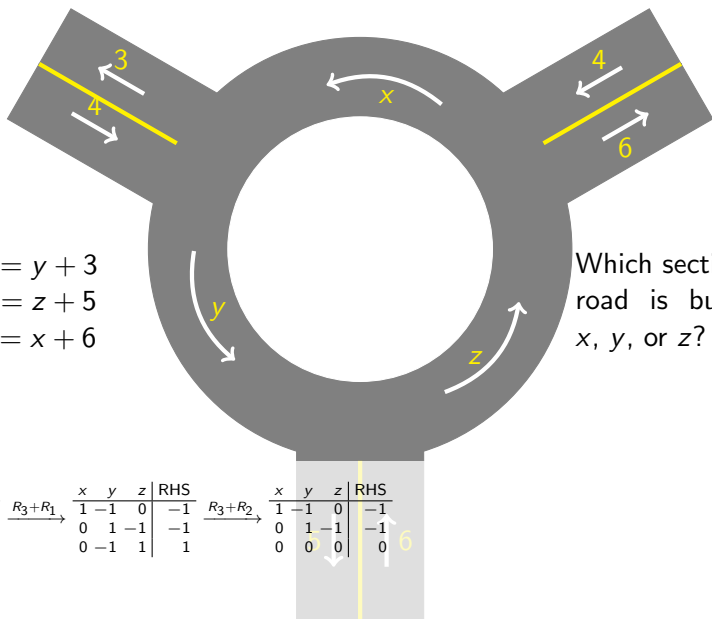


$$\begin{aligned}x + 4 &= y + 3 \\y + 6 &= z + 5 \\z + 4 &= x + 6\end{aligned}$$

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x	y	z	RHS		x	y	z	RHS
1	-1	0	-1	$\xrightarrow{R_3+R_1}$	1	-1	0	-1
0	1	-1	-1		0	1	-1	-1
-1	0	1	2		0	-1	1	1

Roundabout: An applied problem

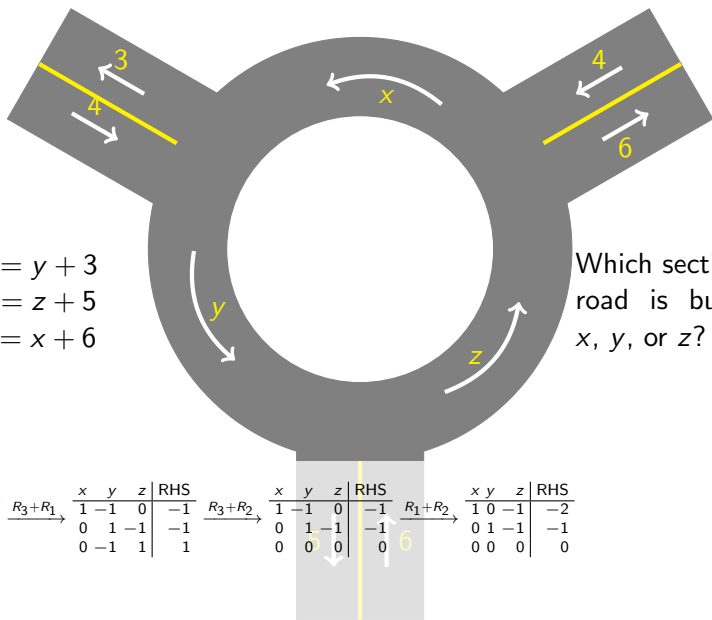


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Which section of road is busiest, x , y , or z ?

x	y	z	RHS		x	y	z	RHS		x	y	z	RHS
1	-1	0	-1	$R_3 + R_1 \rightarrow$	1	-1	0	-1	$R_3 + R_2 \rightarrow$	1	-1	0	-1
0	1	-1	-1		0	1	-1	-1		0	1	-1	-1
-1	0	1	2		0	-1	1	1		0	0	0	0

Roundabout: An applied problem

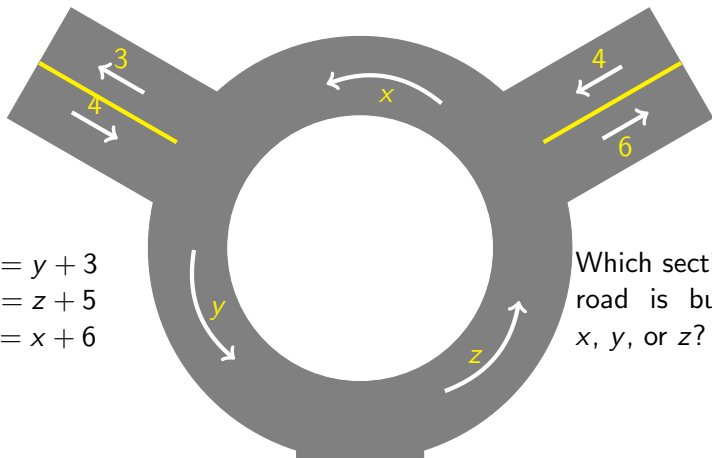


$$\begin{aligned} x + 4 &= y + 3 \\ y + 6 &= z + 5 \\ z + 4 &= x + 6 \end{aligned}$$

Which section of road is busiest, x , y , or z ?

$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 2 \end{array}$	$\xrightarrow{R_3+R_1}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}$	$\xrightarrow{R_3+R_2}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$	$\xrightarrow{R_1+R_2}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$
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Roundabout: An applied problem

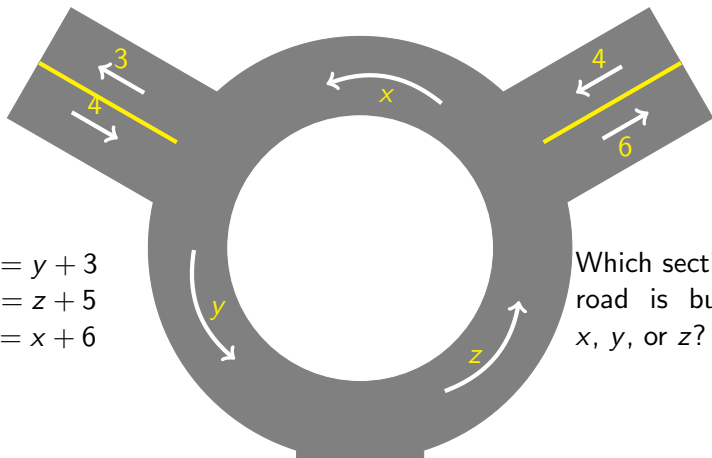


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$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 2 \end{array}$	$\xrightarrow{R_3+R_1}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}$	$\xrightarrow{R_3+R_2}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$	$\xrightarrow{R_1+R_2}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$	\rightarrow	$\begin{aligned}x - z &= -2 \\y - z &= -1 \\0 &= 0\end{aligned}$
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Roundabout: An applied problem



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Which section of road is busiest, x , y , or z ?

$\begin{array}{ccc c} x & y & z & \text{RHS} \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 2 \end{array}$	$\xrightarrow{R_3+R_1}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}$	$\xrightarrow{R_3+R_2}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$	$\xrightarrow{R_1+R_2}$	$\begin{array}{ccc c} x & y & z & \text{RHS} \\ \hline 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$	\rightarrow	$\begin{array}{l} x - z = -2 \\ y - z = -1 \\ 0 = 0 \end{array}$	\rightarrow	$\begin{array}{l} x = z - 2 \\ y = z - 1 \\ 0 = 0 \end{array}$
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Appendix: Very efficiently solving systems

- We managed to solve some fairly big systems last time using our **new** number crunching skills.
- Mostly it was repetitive, routine, soothing.
- But near the end we stopped the number pushing and revived the variables, which totally harshed my zen.
- Today we learn to finish the easy way

Appendix: Cleaning above as well as below

- A matrix is in **REF** if no column (left of the bar) has two pivots
- This means that below and to the left of each pivot are zeros
- A matrix is in **RREF** if
 - it is in REF,
 - there are only zeros above pivots, and
 - pivots are equal to 1

Appendix: How to clean

- If a matrix is in REF, then a **possible target** is a non-zero number above a pivot
- We choose the right-most column with a possible target, and then choose the bottom-most possible target in that column
- The row operation is the same as before:

$$R_{target} - \frac{target}{active} \cdot R_{active}$$

- If a pivot is not equal to one, then we can divide the whole row by the pivot

Appendix: Example

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array} \right] \longrightarrow$$

Appendix: Example

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array} \right] \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

Appendix: Example

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$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

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$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

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Appendix: Example

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Appendix: Example

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array} \right] & \xrightarrow{R_2 - R_3} & \left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right] \\ & \xrightarrow{R_1 - R_3} & \left[\begin{array}{ccc|c} 2 & 1 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right] \\ & \xrightarrow{R_1 - R_2} & \left[\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right] \\ & \xrightarrow{\frac{1}{2}R_1} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right] \end{aligned}$$

Appendix: Example

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array} \right] \xrightarrow{R_2 - R_3}$$

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$$\xrightarrow{\frac{1}{2}R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

RREF

2.3: More practice

- Row reduce these matrices:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 59 \\ 0 & 1 & 5 & 47 \\ 0 & 0 & 1 & 8 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 16 \\ 0 & 1 & -4 & -25 \\ 0 & 0 & 1 & 8 \end{array} \right]$$

2.3: More practice

- Row reduce these matrices:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 59 \\ 0 & 1 & 5 & 47 \\ 0 & 0 & 1 & 8 \end{array} \right] \xrightarrow{\substack{R_1 - 4R_3 \\ R_2 - 5R_3}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 27 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array} \right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 16 \\ 0 & 1 & -4 & -25 \\ 0 & 0 & 1 & 8 \end{array} \right] \xrightarrow{\substack{R_1 - 3R_3 \\ R_2 + 4R_3}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -8 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array} \right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array} \right]$$

- Notice how two different REFs have the same RREF

2.3: What if things go wrong?

- Is this matrix in REF? RREF?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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- Is this matrix in REF? RREF?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- What could we do to fix it?

2.3: What if things go wrong?

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 - Row 2 can only make row 1 worse and vice versa!

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 - Row 3 cannot do anything at all!

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 - Row 2 can only make row 1 worse and vice versa!
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- Let's write it out in variables, and see what is going on:

$$x + 2y = 3 \quad z = 4 \quad 0 = 0$$

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$$x + 2y = 3 \quad z = 4 \quad 0 = 0$$

- Well that is not too bad? $x = 3 - 2y$, y is free, $z = 4$.

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- Well that is not too bad? $x = 3 - 2y$, y is free, $z = 4$.
- We can read this right from the matrix
- We do say this matrix is in REF and RREF

2.3: Free variables

- If a column (for a variable) has no pivot, then that variable is **free**
- Be careful when reading the answer off the matrix
 $120|3$ means $x + 2y = 3$, so $x = 3 - 2y$
- If a variable is free, then (assuming there are any solutions) there are **infinitely many solutions**
- What does “no solution” look like in matrix format?

2.3: What if things go wrong?

- Is this matrix in REF? RREF?

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

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What?!

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- **No solution, inconsistent**

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What?!
- **No solution, inconsistent**
- We can read this right from the matrix
- We do say this matrix is in REF and RREF