MA162: Finite mathematics

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February 18, 2013

SCHEDULE:

- HW 3.1-3.3 (Late)
- HW 4.1 due Friday, Feb 22, 2013
- HW 2.5-2.6 due Friday, Mar 01, 2013
- Exam 2, Monday, Mar 04, 2013, from 5pm to 7pm

Today we will cover 4.1: solving larger linear programming problems

Exam 1 question

- On the first exam we tried to solve:

  "While using all resources, maximize profit"

Cost	**Output X**	**Output Y**	**Output Z**	**Output W**	**Available**
Resource 1	4 min	3 min	4 min	4 min	60 hours
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Resource 3	12 min	9 min	15 min	6 min	204 hours
Profit	\$1	\$1	\$1	\$1	

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$$\left[\begin{array}{cccc|r} X & Y & Z & W & RHS \\ 4 & 3 & 4 & 4 & (60)(60) \\ 24 & 22 & 28 & 0 & (416)(60) \\ 12 & 9 & 15 & 6 & (204)(60) \end{array}\right]$$

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- General way to use all resources is

$$(X = 150 - 6W, Y = 360 + 4W, Z = 480 + 2W, W = FREE)$$

Exam 1 question

- Part (b) addressed the "maximize profit" part:

$$\begin{aligned} P &= X + Y + Z + W \\ &= (150 - 6W) + (360 + 4W) + (480 + 2W) + W \\ &= 990 + W \end{aligned}$$

- Clearly making W large increases profit
- But how large? Too large and it'll be impossible!
- We can solve an (easy) algebra problem, but can't we do it without algebra?

Why we should keep pivotting after RREF

- W can be anything we want, but we don't know what we want!
- We look at the other variables, do any have a $-W$ in them?

  $(X = 150 - 6W, Y = 360 + 4W, Z = 480 + 2W, W = FREE)$
- Let's make X be free, and solve for W using X
- $X = 150 - 6W$ means $W = 25 - \frac{1}{6}X$
- $P = 990 + 25 - \frac{1}{6}X$, X is free
- How big should X be? Every X is costing us money. X=0, duh.
- So $W = 25$, $Y = 460$, $Z = 530$, and $X = 0$.

Key idea

- a '+W' in profit and '-W' in a variable is hard
- Switch which variables are free to fix it
- (A '+W' in profit with no '-W' in variables means unlimited profit!)

As matrices

- The pivot is which variable you solve for
- To solve for W in the X equation, we need to change pivots

$$\left[\begin{array}{cccc|c} X & Y & Z & W & RHS \\ \textcircled{1} & 0 & 0 & 6 & 150 \\ 0 & \textcircled{1} & 0 & -4 & 360 \\ 0 & 0 & \textcircled{1} & -2 & 480 \end{array}\right] \xrightarrow{R_1/6} \left[\begin{array}{cccc|c} X & Y & Z & W & RHS \\ \frac{1}{6} & 0 & 0 & \textcircled{1} & 25 \\ 0 & \textcircled{1} & 0 & -4 & 360 \\ 0 & 0 & \textcircled{1} & -2 & 480 \end{array}\right]$$

$$\xrightarrow[R_3+2R_1]{R_2+4R_1} \left[\begin{array}{cccc|c} X & Y & Z & W & RHS \\ 1/6 & 0 & 0 & \textcircled{1} & 25 \\ 4/6 & \textcircled{1} & 0 & 0 & 460 \\ 2/6 & 0 & \textcircled{1} & 0 & 530 \end{array}\right]$$

- $(X = \text{FREE}, Y = 460 - \frac{2}{3}X, Z = 530 - \frac{1}{3}X, W = 25 - \frac{1}{6}X)$
- What about P? Could use algebra, but. . .

As matrices, really

- $P = X + Y + Z + W$ is just another equation
- $-X - Y - Z - W + P = 0$ is more matrixy

$$\left[\begin{array}{ccccc|c} X & Y & Z & W & P & RHS \\ 1/6 & 0 & 0 & \textcircled{1} & 0 & 25 \\ 4/6 & \textcircled{1} & 0 & 0 & 0 & 460 \\ 2/6 & 0 & \textcircled{1} & 0 & 0 & 530 \\ \hline -1 & -1 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_4+(R_1+R_2+R_3)}$$

$$\left[\begin{array}{ccccc|c} X & Y & Z & W & P & RHS \\ 1/6 & 0 & 0 & \textcircled{1} & 0 & 25 \\ 4/6 & \textcircled{1} & 0 & 0 & 0 & 460 \\ 2/6 & 0 & \textcircled{1} & 0 & 0 & 530 \\ \hline \frac{1}{6} & 0 & 0 & 0 & \textcircled{1} & 1015 \end{array}\right]$$

- $(X = \text{FREE}, Y = 460 - \frac{2}{3}X, Z = 530 - \frac{1}{3}X, W = 25 - \frac{1}{6}X, P = 1015 - \frac{1}{6}X)$

As matrices, simple RREF

- Doing row ops ourselves can be tedious
- Ask the calculator to swap the columns so they are in the "right" order

$$\left[\begin{array}{rrrrr|r} X & Y & Z & W & P & RHS \\ 4 & 3 & 4 & 4 & 0 & (60)(60) \\ 24 & 22 & 28 & 0 & 0 & (416)(60) \\ 12 & 9 & 15 & 6 & 0 & (204)(60) \\ -1 & -1 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{Swap!}}$$

$$\left[\begin{array}{rrrrr|r} W & Y & Z & P & X & RHS \\ 4 & 3 & 4 & 0 & 4 & (60)(60) \\ 0 & 22 & 28 & 0 & 24 & (416)(60) \\ 6 & 9 & 15 & 0 & 12 & (204)(60) \\ -1 & -1 & -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF!}} \left[\begin{array}{ccccc|r} W & Y & Z & P & X & RHS \\ \hline \textcircled{1} & 0 & 0 & 0 & 1/6 & 25 \\ 0 & \textcircled{1} & 0 & 0 & 2/3 & 460 \\ 0 & 0 & \textcircled{1} & 0 & 1/3 & 530 \\ 0 & 0 & 0 & \textcircled{1} & 1/6 & 1015 \end{array}\right]$$

- $W = 25 - X/6$, $Y = 460 - 2X/3$, $Z = 530 - X/3$, $P = 1015 - X/6$, $X = \text{FREE}$
- $P = 1015 - X/6$? I guess $X = 0$! and $W = 25$, $Y = 460$, $Z = 530$

Exam 1 #9

Cost	Output X	Output Y	Output Z	Output W	Available
Resource 1	4 min	3 min	4 min	4 min	60 hours
Resource 2	24 min	22 min	28 min	0 min	416 hours
Resource 3	12 min	9 min	15 min	6 min	204 hours
Profit	\$1	\$1	\$1	\$1	

- Allow resource 2 to be wasted ("fire the sewers")
- Should we just make W as big as possible? ("just make scarves")
- Imagine a new product U that uses one minute of sewing time, but gives no profit
- Any leftover sewing time can be spent on U
- So $24X + 22Y + 28Z + 0W + 1U = (416)(60)$

Exam 1 #9

Cost	**Output X**	**Output Y**	**Output Z**	**Output W**	**Output U**	**Available**
Resource 1	4 min	3 min	4 min	4 min	0 min	60 hours
Resource 2	24 min	22 min	28 min	0 min	1 min	416 hours
Resource 3	12 min	9 min	15 min	6 min	0 min	204 hours
Profit	\$1	\$1	\$1	\$1	\$0	

- Equations are:

$$\left[\begin{array}{cccccc|c} X & Y & Z & W & U & P & RHS \\ 4 & 3 & 4 & 4 & 0 & 0 & 3600 \\ 24 & 22 & 28 & 0 & 1 & 0 & 24960 \\ 12 & 9 & 15 & 6 & 0 & 0 & 12240 \\ \hline -1 & -1 & -1 & -1 & 0 & 1 & 0 \end{array}\right]$$

- RREF is easy (just update the U column)

$$\left[\begin{array}{cccccc|c} X & Y & Z & W & U & P & RHS \\ 1/6 & 0 & 0 & \textcircled{1} & -1/32 & 0 & 25 \\ 2/3 & \textcircled{1} & 0 & 0 & 1/8 & 0 & 460 \\ 1/3 & 0 & \textcircled{1} & 0 & -1/16 & 0 & 530 \\ \hline 1/6 & 0 & 0 & 0 & 1/32 & \textcircled{1} & 1015 \end{array}\right]$$

- Writing this in terms of equations we get

$$\left(\begin{array}{lll} X = \text{FREE}, & Y = 460 - 2x/3 - u/8, & Z = 530 - x/3 + u/16, \\ W = 25 - x/6 + u/32, & U = \text{FREE}, & P = 1015 - x/6 - u/32 \end{array}\right)$$

Exam 1 #9

$$\left(\begin{array}{lll} X = \text{FREE}, & Y = 460 - 2x/3 - u/8, & Z = 530 - x/3 + u/16, \\ W = 25 - x/6 + u/32, & U = \text{FREE}, & P = 1015 - x/6 - u/32 \end{array} \right)$$

- X and U are free, what should set them to?
- $P = 1015 - x/6 - u/32$ and $x \geq 0$, $u \geq 0$
- Each unused sewing minute costs us around $0.03!
- If we want to do better, we also have to allow unused cutting and packing minutes.
- Each new unused resource is another column in the matrix, but we have seen that is not hard!

Exam 1 #10

Cost	Output X	Output Y	Output Z	Output W	Output T	Output U	Output V	Available
Resource 1	4 min	3 min	4 min	4 min	1 min	0 min	0 min	60 hours
Resource 2	24 min	22 min	28 min	0 min	0 min	1 min	0 min	416 hours
Resource 3	12 min	9 min	15 min	6 min	0 min	0 min	1 min	204 hours
Profit	\$1	\$1	\$1	\$1	\$0	\$0	\$0	

- Equations are:

$$\left[\begin{array}{rrrr|rrrr|r} X & Y & Z & W & T & U & V & P & RHS \\ 4 & 3 & 4 & 4 & \textcircled{1} & 0 & 0 & 0 & 3600 \\ 24 & 22 & 28 & 0 & 0 & \textcircled{1} & 0 & 0 & 24960 \\ 12 & 9 & 15 & 6 & 0 & 0 & \textcircled{1} & 0 & 12240 \\ \hline -1 & -1 & -1 & -1 & 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right]$$

- It is already in RREF, well, if the columns had been ordered T, U, V, P, X, Y, Z, RHS

Exam 1 #10

- Order the columns W, Y, V, P, X, Z, T, U, RHS

W	Y	V	P	X	Z	T	U	RHS
4	3	0	0	4	4	①	0	3600
0	22	0	0	24	28	0	①	24960
6	9	①	0	12	15	0	0	12240
-1	-1	0	①	-1	-1	0	0	0

$\xrightarrow{\text{RREF it!}}$

W	Y	V	P	X	Z	T	U	RHS
①	0	0	0	2/11	1/22	1/4	$-3/88$	540/11
0	①	0	0	12/11	14/11	0	1/22	12480/11
0	0	①	0	12/11	36/11	$-3/2$	$-9/44$	19080/11
0	0	0	①	3/11	7/22	1/4	1/88	13020/11

- $P = 13020/11 - 3X/11 - 7Z/22 - T/4 - U/88$
- Better make $X = Z = T = U = 0$! $P = 13020/11 \approx \$1183.64$
- Compare to "fire the sewers, and just make scarves" at \$900

Summary so far

- Get the system to RREF, so it is "easy" to find solutions
- In the equation for P, any '$+$' free variables mean we are not done
- Want those variables to be big, but ...
- Any ① variable that has a '-' of that free variable is a constraint
- We try to swap who is free and who is ①
- In our examples today, there was only one choice of swap
- Next class: which one do we swap if there is a choice?