MA162: Finite mathematics

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Schedule:

- HW 3.1-3.3 (Late)
- HW 4.1 due Friday, Feb 22, 2013
- HW 2.5-2.6 due Friday, Mar 01, 2013
- Exam 2, Monday, Mar 04, 2013, from 5pm to 7pm

Today we will cover 4.1: solving larger linear programming problems

4.1: Setup

- We have **choices** (how much of each product to make)
- We have linear **constraints** (resource usage $\leq$ resource budget)
- We have a linear **objective** (profit)

- By creating new (profitless, resource wasting) products,
we can change $\leq$ to $=$
- We now have a system of linear equations
- We want solutions where all production levels are non-negative (feasible)
- Best feasible solution is one with best level for the objective

4.1: Easy example

- We have three products and five resources.

	Prod X	Prod Y	Prod Z	Budget
Res A	1	1	1	100
Res B	5	4	8	500
Res C	3	3	3	1000
Res D	1	1	2	150
Res E	2	1	1	120
Profit	1	2	3	

- If we need to spend all the resources, then we are doomed.

  [3 products, 5 resources, usually cannot be done]

4.1: Easy example

- Let's invent 5 profitless products A, B, C, D, E that use up one resource of the specified type

- So solve:

$$
\left[\begin{array}{ccc|ccccc|c|r}
X & Y & Z & A & B & C & D & E & P & RHS \\
\hline
1 & 1 & 1 & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 100 \\
5 & 4 & 8 & 0 & \textcircled{1} & 0 & 0 & 0 & 0 & 500 \\
3 & 3 & 3 & 0 & 0 & \textcircled{1} & 0 & 0 & 0 & 1000 \\
1 & 1 & 2 & 0 & 0 & 0 & \textcircled{1} & 0 & 0 & 150 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 & 120 \\
\hline
-1 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0
\end{array}\right]
$$

- It is already in RREF, with X,Y,Z the FREE variables

- We can make them be anything. . . legal but it is hard to tell what is legal

- Only easy answer is the **basic solution**:

  $X = Y = Z = 0$ stays under budget

4.1: Strategy

- Bring the linear system to RREF (can list all solutions! hrm, even insane ones)
- We make FREE variables $= 0$ to get a "basic solution"
- Make the "bad products" be FREE, (so we can freely make none of them)
- If the RHS are all non-negative, then the basic solution is feasible
- If the bottom line is (also) all non-negative, then the basic solution is optimal
- So we just want to make the bottom positive,
  while keeing the right hand side positive
- Which products are bad but not FREE?
  Which products are FREE but not bad?

4.1: Easy example

X	Y	Z	A	B	C	D	E	P	RHS
1	1	1	①	0	0	0	0	0	100
5	4	8	0	①	0	0	0	0	500
3	3	3	0	0	①	0	0	0	1000
1	1	2	0	0	0	①	0	0	150
2	1	1	0	0	0	0	①	0	120
−1	−2	−3	0	0	0	0	0	①	0

- $X = Y = Z = 0$ stays under budget, but is not optimal
- Stays under budget:

  $X + Y + Z + A = 100$, but $X = 0$, so

  $0 + 0 + 0 + A = 100$, and $A = 100$

  A, B, C, D, E are equal to the RHS, and none are negative,

  so all good on the feasible front

4.1: Easy example

X	Y	Z	A	B	C	D	E	P	RHS
1	1	1	①	0	0	0	0	0	100
5	4	8	0	①	0	0	0	0	500
3	3	3	0	0	①	0	0	0	1000
1	1	2	0	0	0	①	0	0	150
2	1	1	0	0	0	0	①	0	120
−1	−2	−3	0	0	0	0	0	①	0

- $X = Y = Z = 0$ stays under budget, but is not optimal
- Not optimal:

  $-1X - 2Y - 3Z + 0A + 0B + 0C + 0D + 0E + P = 0$

  is usually written $P = X + 2Y + 3Z$

  $P = X$ PLUS $2Y$ PLUS $3Z$, adding 0 is legal,

  but wouldn't it be nice to add more?

4.1: Pivot rules: Step 1, find pivot column

- Step 1: Find a free variable that is still profitable.
- How? Look at the bottom line. A column with a negative number in the bottom line is the column of a free, profitable variable
- Why? $\left[\begin{array}{ccccc|c} X & Y & Z & W & P & RHS \\ \hline 2 & -3 & 0 & 0 & \textcircled{1} & 10 \end{array}\right]$

  means $2X - 3Y + 0Z + 0W + P = 10$

  solve for $P = 10 - 2X + 3Y$, the Y still adds to profit

4.1: Pivot rules: Step 2, find pivot row

- Step 2: Find a useless product that is not free
- How? Look at RHS compared to the pivot column.

  Smallest non-negative ratio is the winner.

  That row has a ① in some column; that column becomes free, and the pivot column gets the new ①
- Why? $\left[\begin{array}{ccc|c} Y & Z & W & RHS \\ \hline 1 & \textcircled{1} & 0 & 10 \\ 2 & 0 & \textcircled{1} & 10 \end{array}\right]$

  means $Z = 10 - 1Y$ and $W = 10 - 2Y$;

  the W equation will go negative before the Z one if we try to make Y big, so we'll want to make $W = 0$; might as well make $W =$ FREE

4.1: Easy example

- We have three products and five resources.

	Prod X	Prod Y	Prod Z	Budget
Res A	1	1	1	100
Res B	5	4	8	500
Res C	3	3	3	1000
Res D	1	1	2	150
Res E	2	1	1	120
Profit	1	2	3	

- If we need to spend all the resources, then we are doomed.

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4.1: Easy example

- Let's invent 5 profitless products A, B, C, D, E that use up one resource of the specified type
- So solve:

$$\left[\begin{array}{ccc|ccccc|c|c} X & Y & Z & A & B & C & D & E & P & RHS \\ \hline 1 & 1 & 1 & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 100 \\ 5 & 4 & 8 & 0 & \textcircled{1} & 0 & 0 & 0 & 0 & 500 \\ 3 & 3 & 3 & 0 & 0 & \textcircled{1} & 0 & 0 & 0 & 1000 \\ 1 & 1 & 2 & 0 & 0 & 0 & \textcircled{1} & 0 & 0 & 150 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 & 120 \\ \hline -1 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right]$$

- It is already in RREF, but the FREE variables are all profitable

$P = X + 2Y + 3Z$, $X = \text{FREE}$, $Y = \text{FREE}$, $Z = \text{FREE}$

4.1: First new pivot

X	Y	Z	A	B	C	D	E	P	RHS
1	1	1	①	0	0	0	0	0	100
5	4	8	0	①	0	0	0	0	500
3	3	3	0	0	①	0	0	0	1000
1	1	2	0	0	0	①	0	0	150
2	1	1	0	0	0	0	①	0	120
−1	−2	−3	0	0	0	0	0	①	0

- I have a good feeling about Y: moderate price, moderate profit
- Which resource is it going to use up first?

  A: $100/1 = 100$, B: $500/4 = 125$, C: $1000/3$ ¿ 300,

  D: $150/1 = 150$, E: $120/1 = 120$,

- We run out of A first while trying to make Y
- So the profitless, A-wasting product A is going to be set to 0
- So we should make A to be FREE and make Y a ①

4.1: Easy example

X	Y	Z	A	B	C	D	E	P	RHS
1	1	1	①	0	0	0	0	0	100
5	4	8	0	①	0	0	0	0	500
3	3	3	0	0	①	0	0	0	1000
1	1	2	0	0	0	①	0	0	150
2	1	1	0	0	0	0	①	0	120
−1	−2	−3	0	0	0	0	0	①	0

$$\xrightarrow[R_4 - R_1,\ R_5 - R_1,\ R_6 + 2R_1]{R_2 - 4R_1,\ R_3 - 3R_1}$$

X	Y	Z	A	B	C	D	E	P	RHS
1	①	1	1	0	0	0	0	0	100
1	0	4	−4	①	0	0	0	0	100
0	0	0	−3	0	①	0	0	0	700
0	0	1	−1	0	0	①	0	0	50
1	0	0	−1	0	0	0	①	0	20
1	0	−1	2	0	0	0	0	①	200

- Now we have $P = 200 - X + Z - 2A$, $X = \text{FREE}$, $Z = \text{FREE}$, $A = \text{FREE}$
- Probably want to make some Z, as it is still profitable
- (only \$1 per Z instead of \$3 per Z; this is because to make a Z it to NOT make a Y)

4.1: Easy example

X	Y	Z	A	B	C	D	E	P	RHS
1	①	1	1	0	0	0	0	0	100
1	0	4	−4	①	0	0	0	0	100
0	0	0	−3	0	①	0	0	0	700
0	0	1	−1	0	0	①	0	0	50
1	0	0	−1	0	0	0	①	0	20
1	0	−1	2	0	0	0	0	①	200

- Z is our pivot column; which row?

$Y = 100 - Z$, so $Z \leq 100$; $B = 100 - 4Z$ so $Z \leq 25$;

$C = 700$, so Z is whatever; $D = 50 - Z$, so $Z \leq 50$;

$E = 20$, so Z is whatever

- Most restrictive law is B: we'll make B zero first, so $B =$ FREE

4.1: Easy example

X	Y	Z	A	B	C	D	E	P	RHS
1	①	1	1	0	0	0	0	0	100
1	0	4	−4	①	0	0	0	0	100
0	0	0	−3	0	①	0	0	0	700
0	0	1	−1	0	0	①	0	0	50
1	0	0	−1	0	0	0	①	0	20
1	0	−1	2	0	0	0	0	①	200

$R_1 - \frac{1}{4}R_2$
$\frac{1}{4}R_2$
$R3$ is good
$\longrightarrow$
$R_4 - \frac{1}{4}R2$
R_5 is good
$R_6 + \frac{1}{4}R_2$

X	Y	Z	A	B	C	D	E	P	RHS
3/4	①	0	2	−1/4	0	0	0	0	75
1/4	0	①	−1	1/4	0	0	0	0	25
0	0	0	−3	0	①	0	0	0	700
1/4	0	0	0	−1/4	0	①	0	0	25
1	0	0	−1	0	0	0	①	0	20
5/4	0	0	1	1/4	0	0	0	①	225

- $P = 225 - \frac{5}{4}X - A - \frac{1}{4}C$, $X = \text{FREE}$, $A = \text{FREE}$, $C = \text{FREE}$
- So we should make 0 Xs, 75 Ys, 25 Zs,
- We should waste no A, no C, waste 700 B, waste 25 D, waste 20 E
- And make \$225 in the process

4.1: Special cases for pivot column

- What if there are two negative numbers?

  You can choose either. No one knows the best choice. Popular strategies:

  (1) choose the left one, (2) choose the big one, (3) choose the one with bigger ratio, (4) choose randomly

- What if there are no negative numbers?

  You are DONE! None of the FREE variables are profitable, so don't make them

- What if there is a 0 in a FREE column?

  Then you can leave it FREE or not; it does not (currently) affect profit

- How do I tell which one is the bottom line?

  The row with a $\textcircled{1}$ in the P column In our class, it should always be the bottom row.

- What if the P column doesn't have a $\textcircled{1}$?

  Then something is terribly wrong. Either choose an RREF where P has a $\textcircled{1}$, or start over.

4.1: Special cases for pivot row

- What if there is a tie?

  You can choose either. Popular strategies: (1) choose the one whose ① is leftmost, (2) choose randomly

- What if the pivot column has a 0 or a negative?

  Ignore it. This is not the pivot row.

  $\left[\begin{array}{ccc|c} Y & Z & W & RHS \\ \hline -4 & ① & 0 & 10 \\ 0 & 0 & ① & 10 \end{array}\right]$ means $Z = 10 + 4Y$ and $W = 10$; big Y does not make Z or W negative

- What if the RHS has a 0?

  This is the pivot row! (As long as the pivot column is positive)

  $\left[\begin{array}{cc|c} Y & Z & RHS \\ \hline 2 & ① & 0 \end{array}\right]$ means $Z = -2Y$; big Y makes Z negative immediately!

- What if the RHS has a negative?

  Then something has gone horribly wrong. The FREE $= 0$ solution is not feasible.

  $\left[\begin{array}{cc|c} Y & Z & RHS \\ \hline 2 & ① & -10 \end{array}\right]$ means $Z = -10 - 2Y$; even $Y = 0$ makes Z negative!

- What if every entry in the pivot column is 0 or negative?

  Then the optimal solution involves ∞. "Unbounded"

4.1: Easy example using corners

- Method of corners?
- The feasible region has 8 corners; each corner is described by which variables are free.
- The feasible region is shaped like a solid cube, but sitting in 5- or 8-dimensional space
- Our pivoting rules took us from $\{X, Y, Z\}$ to $\{X, A, Z\}$ to $\{X, A, B\}$
- Each corner was better than the last
- The pivot column rule ensures we go to a nearby better corner
- The pivot row rule ensures that we land on a corner, not just at the intersection of some lines

4.1: How to find the corners

- To find the corners, I tried all $\binom{8}{4} = 70$ sets of free variables, and only kept the 8 legal ones
- That is way harder than what we did with pivot rules already!
- The order of pivotting matters
- Here is another path: $\{X, Y, Z\} \to \{E, Y, Z\} \to \{E, A, Z\} \to \{E, A, B\} \to \{X, A, B\}$
- Notice how we decided E was a bad product at first, and then changed our mind at the last step?
- Notice how this took 4 more RREFS instead of 2 more?
- Pivotting is complicated and surprising well-studied but not well-understood