MA162: Finite mathematics

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Schedule:

- HW 3.1-3.3, 4.1 (Late)
- HW 2.5-2.6 due Friday, Mar 01, 2013
- Exam 2, Monday, Mar 04, 2013, from 5pm to 7pm
- HW 5.1 due Friday, Mar 08, 2013
- Spring Break, Mar 09-17, 2013
- HW 5.2-5.3 due Friday, Mar 22, 2013

Today we will cover 2.5: applications of matrix multiplication, and Ch 4: shadow prices

2.5: Matrices as conversion tables

- A table lets you convert from one type of thing to another
- This table lets you convert from a client to his stock holdings:

$$\begin{array}{c} \\ \textit{Bill} \\ \textit{Jim} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{IBM} & \textit{Google} & \textit{Toyota} & \textit{Texaco} \end{array} \\ \begin{pmatrix} 18 & 16 & 12 & 14 \\ 12 & 18 & 11 & 12 \end{pmatrix} \end{array}$$

  Bill has 18 shares of IBM

- This table lets you convert from a stock to its value:

$$\begin{array}{c} \\ \textit{IBM} \\ \textit{Google} \\ \textit{Toyota} \\ \textit{Texaco} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Today} & \textit{Yesterday} & \textit{Daybefore} & \ldots \end{array} \\ \begin{pmatrix} 3 & 3.01 & 2.99 & \ldots \\ 4 & 3.99 & 3.99 & \ldots \\ 5 & 5.01 & 5.01 & \ldots \\ 1 & 1.02 & 1.03 & \ldots \end{pmatrix} \end{array}$$

  Google sold for \$3.99/share yesterday

- The source is on the left, and the destination is on the top

2.5: Matrix multiplication to combine conversions

- We can combine this into a single conversion table
- (Client $\to$ Stocks) $\times$ (Stocks $\to$ Value) $=$ Client $\to$ Value

$$\begin{array}{c} \\ \textit{Bill} \\ \textit{Jim} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{IBM} & \textit{Google} & \textit{Toyota} & \textit{Texaco} \end{array} \\ \begin{pmatrix} 18 & 16 & 12 & 14 \\ 12 & 18 & 11 & 12 \end{pmatrix} \end{array} \times \begin{array}{c} \\ \textit{IBM} \\ \textit{Google} \\ \textit{Toyota} \\ \textit{Texaco} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Today} & \textit{Yesterday} & \textit{Daybefore} & \ldots \end{array} \\ \begin{pmatrix} 3 & 3.01 & 2.99 & \ldots \\ 4 & 3.99 & 3.99 & \ldots \\ 5 & 5.01 & 5.01 & \ldots \\ 1 & 1.02 & 1.03 & \ldots \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ \textit{Bill} \\ \textit{Jim} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Today} & \textit{Yesterday} & \textit{Daybefore} & \ldots \end{array} \\ \begin{pmatrix} (18)(3)+(16)(4)+(12)(5)+(14)(1) & \ldots & \ldots & \ldots \\ (12)(3)+(18)(4)+(11)(5)+(12)(1) & \ldots & \ldots & \ldots \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ \textit{Bill} \\ \textit{Jim} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Today} & \textit{Yesterday} & \textit{Daybefore} & \ldots \end{array} \\ \begin{pmatrix} 192 & 192.42 & 192.20 & \ldots \\ 175 & 175.29 & 175.17 & \ldots \end{pmatrix} \end{array}$$

2.5: Comparing pricing contracts

- We need to buy some supplies

	Resource Usage			Resource price		
	Prod X	Prod Y	Prod Z	Store K	Store L	Store M
Res A	1	1	1	\$1.00	\$0.75	\$2.00
Res B	5	4	8	\$1.25	\$1.50	\$1.00
Res C	3	3	3	\$1.50	\$1.25	\$1.75
Res D	1	1	2	\$2.00	\$1.25	\$1.00
Res E	2	1	1	\$1.00	\$1.50	\$2.00
Production Level	10	40	100			

- So product Z uses 8 units of resource B
- Each store has offered us an exclusive price contract (Store L offers resource A as \$0.75 per unit, but only if we promise not to buy from Store K or Store M)
- We plan on producing 40 units of product Y
- Which store's pricing contract will be cheaper?

2.5: Comparing pricing contracts

- Want to convert Products to Store (Price)
- (Product $\rightarrow$ Resource) $\times$ (Resource $\rightarrow$ Store)

	Res A	Res B	Res C	Res D	Res E
Prod X	1	5	3	1	2
Prod Y	1	4	3	1	1
Prod Z	1	8	3	2	1

$\times$

	Store K	Store L	Store M
Res A	\$1.00	\$0.75	\$2.00
Res B	\$1.25	\$1.50	\$1.00
Res C	\$1.50	\$1.25	\$1.75
Res D	\$2.00	\$1.25	\$1.00
Res E	\$1.00	\$1.50	\$2.00

$=$

	Store K	Store L	Store M
Prod X	**\$15.75**	\$16.25	\$17.25
Prod Y	\$13.50	**\$13.25**	\$14.25
Prod Z	\$20.50	\$20.50	**\$19.25**

- Except each store is cheapest for one of the products!
- need to take into account how much of each product we make

2.5: Comparing pricing contracts

- Want to convert Production Level to Store (Price)
- (Level $\rightarrow$ Product) $\times$ (Product $\rightarrow$ Resource $\rightarrow$ Store)

	Prod X	Prod Y	Prod Z
Level	10	40	100

$\times$

	Store K	Store L	Store M
Prod X	\$15.75	\$16.25	\$17.25
Prod Y	\$13.50	\$13.25	\$14.25
Prod Z	\$20.50	\$20.50	\$19.25

$=$

	Store K	Store L	Store M
Level	\$2747.50	\$2742.50	\$2667.50

- For the projected production levels,

  Store M offers the cheaper package

2.5: Square matrix, migration

- This table (from the US Census) converts residents from 2011 to 2012

	Northeast	*Midwest*	*South*	*West*
NE	98.92%	0.09%	0.65%	0.33%
MW	0.08%	99.01%	0.56%	0.35%
So	0.16%	0.27%	99.20%	0.37%
We	0.05%	0.28%	0.46%	99.19%

- It says that 0.65% of people in the Northeast Census Region moved to the South Census Region

- While population changes occur due to a variety of factors, apparently "internal" migration is 25% to 50% of it, while birth/death is only about 50%

- If we pretend the matrix doesn't change from year to year, we could predict future years too!

2.5: Square matrix, migration

- If we multiply this table by itself 10 times,

  it estimates converting 2011 residents to 2021 residents

	Northeast	*Midwest*	*South*	*West*
NE	89.76%	0.93%	6.05%	3.14%
MW	0.77%	90.63%	5.25%	3.32%
So	1.48%	2.54%	92.46%	3.50%
We	0.59%	2.72%	4.36%	92.31%

- Distribution:

	NE	MW	SO	WE
2012	18.01%	21.77%	36.91%	23.31%
2021	17.02%	21.48%	37.38%	24.10%
∞	9.10%	20.63%	39.55%	30.72%

2.5: Another example

- (Products $\rightarrow$ Resource requirements) $\times$ (Resource $\rightarrow$ value) = (Products $\rightarrow$ Value)
- Very useful calculation, but perhaps tricky

	Prod X	Prod Y	Prod Z	Budget
Res A	1	1	1	100
Res B	5	4	8	500
Res C	3	3	3	1000
Res D	1	1	2	150
Res E	2	1	1	120
Profit	1	2	3	

Raw resource prices:

Res A	Res B	Res C	Res D	Res E
0.25	0.10	0.10	0.10	0.25

- What are some problems with "just multiply"?

2.5: Another example

- (Products $\rightarrow$ Resource requirements) $\times$ (Resource $\rightarrow$ value) = (Products $\rightarrow$ Value)
- Very useful calculation, but perhaps tricky

	Prod X	Prod Y	Prod Z	Budget
Res A	1	1	1	100
Res B	5	4	8	500
Res C	3	3	3	1000
Res D	1	1	2	150
Res E	2	1	1	120
Profit	1	2	3	

Raw resource prices:

Res A	Res B	Res C	Res D	Res E
0.25	0.10	0.10	0.10	0.25

- What are some problems with "just multiply"?

  Among others: the tables are "sideways", the sizes and labels don't match

2.5: An answer

- This is closer, now the sizes and labels match:

	Res A	Res B	Res C	Res D	Res E
Prod X	1	5	3	1	2
Prod Y	1	4	3	1	1
Prod Z	1	8	3	2	1
Budget	100	500	1000	150	120

$\times$

	Value
Res A	\$0.25
Res B	\$0.10
Res C	\$0.10
Res D	\$0.10
Res E	\$0.25

	Value
Prod X	\$1.65
Prod Y	\$1.30
Prod Z	\$1.80
Budget	\$220.00

- What does "value of product X is \$1.65" actually mean?

- What does "value of the budget is \$220.00" actually mean?

2.5: An answer

- This is closer, now the sizes and labels match:

	Res A	Res B	Res C	Res D	Res E
Prod X	1	5	3	1	2
Prod Y	1	4	3	1	1
Prod Z	1	8	3	2	1
Budget	100	500	1000	150	120

$\times$

	Value
Res A	\$0.25
Res B	\$0.10
Res C	\$0.10
Res D	\$0.10
Res E	\$0.25

	Value
Prod X	\$1.65
Prod Y	\$1.30
Prod Z	\$1.80
Budget	\$220.00

- What does "value of product X is \$1.65" actually mean?

  It is the total cost of its used resource

- What does "value of the budget is \$220.00" actually mean?

  This is the tax liability of the raw resources

4.1: A different answer for the budget

- Is \$220.00 a good price for the resources?
- Remember from last week, if we made 75 product Ys and 25 product Zs, we got \$225.00

$$\left[\begin{array}{ccc|c|cccc|c|c} X & Y & Z & A & B & C & D & E & P & RHS \\ \hline 3/4 & \textcircled{1} & 0 & 2 & -1/4 & 0 & 0 & 0 & 0 & 75 \\ 1/4 & 0 & \textcircled{1} & -1 & 1/4 & 0 & 0 & 0 & 0 & 25 \\ 0 & 0 & 0 & -3 & 0 & \textcircled{1} & 0 & 0 & 0 & 700 \\ 1/4 & 0 & 0 & 0 & -1/4 & 0 & \textcircled{1} & 0 & 0 & 25 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & \textcircled{1} & 0 & 20 \\ \hline 5/4 & 0 & 0 & 1 & 1/4 & 0 & 0 & 0 & \textcircled{1} & 225 \end{array}\right]$$

- We shouldn't sell the needed resources for less than \$225.00!

4.1: Marginal value of our resources

- How much should we pay for just a little more of resource A?
- How much should we charge to sell just a little bit of resource B?
- We look at our profit function:

$$\left[\begin{array}{ccc|cccccc|c} X & Y & Z & A & B & C & D & E & P & RHS \\ \hline 5/4 & 0 & 0 & 1 & 1/4 & 0 & 0 & 0 & \textcircled{1} & 225 \end{array}\right]$$

$P = \$225.00 - \$1.25X - \$1.00A - \$0.25B$

- Every A we don't use making Y and Z costs us \$1.00, so we should not sell for anything less than \$1.00 or we will lose money
- Every B we don't use costs us \$0.25 . . .

  but we can buy them for \$0.10 . . .

4.1: Buying resources for increased profit

- We can buy more B at a profit!
- If we buy 100 more units of B, the revenue goes up \$25 to \$250 but we spent \$10 on the B

X	Y	Z	A	B	C	D	E	P	RHS
1	1	1	①	0	0	0	0	0	100
5	4	8	0	①	0	0	0	0	600
3	3	3	0	0	①	0	0	0	1000
1	1	2	0	0	0	①	0	0	150
2	1	1	0	0	0	0	①	0	120
−1	−2	−3	0	0	0	0	0	①	0

Same as last week →

X	Y	Z	A	B	C	D	E	P	RHS
3/4	①	0	2	−1/4	0	0	0	0	50
1/4	0	①	−1	1/4	0	0	0	0	50
0	0	0	−3	0	①	0	0	0	700
1/4	0	0	0	−1/4	0	①	0	0	0
1	0	0	−1	0	0	0	①	0	20
5/4	0	0	1	1/4	0	0	0	①	250

- Start with 600 B; $P = 250$, make 50 Ys and Zs,

  use all A and B and D, 700 C leftover, 20 E leftover
- If we buy more than 100 units of B, we waste money:

  we start to run out of resource D

4.1: Marginal, shadow prices

- Look at the bottom line,

  those are the prices we can buy/sell resources at

  or increase in product price needed before it is profitable to make
- Careful: marginal is for "just a little bit more"
- How much more?
- Until we pivot, so we need to check the pivot ratio!

X	Y	Z	A	B	C	D	E	P	RHS
3/4	①	0	2	−1/4	0	0	0	0	75
1/4	0	①	−1	1/4	0	0	0	0	25
0	0	0	−3	0	①	0	0	0	700
1/4	0	0	0	−1/4	0	①	0	0	25
1	0	0	−1	0	0	0	①	0	20
5/4	0	0	1	1/4	0	0	0	①	225

- B column: smallest non-positive ratio is 25/(-1/4) = 100,

  so that is the increase until D pivots