MA162: Finite mathematics

Jack Schmidt

University of Kentucky

March 18th, 2013

SCHEDULE:

- HW 1.1-1.4, 2.1-2.6, 3.1-3.3, 4.1, 5.1 (Late)
- HW 5.2-5.3 due Friday, Mar 22, 2013
- HW 6A due Friday, Mar 29, 2013
- HW 6B-6C due Friday, Apr 5, 2013
- Exam 3, Monday, Apr 8, 2013

Today we will cover annuities as sinking funds

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
  - Compound interest
  - Sinking funds
  - Amortized loans

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations

5.1: Time value of money

- **Interest rate** describes how the value of money changes over time
- "1% per month" (aka "12% APR compounded monthly") means:

  Now $\xrightarrow{\times 101\%}$ one month from now

- We can travel farther in time:

  Now $\xrightarrow{\times 101\%}$ one month from now $\xrightarrow{\times 101\%}$ two months from now

  Now $\xrightarrow{\times 101\% \times 101\%}$ Two months from now

- To move money forward time one period, multiply by $1 + i$
- To move it back, divide by $1 + i$

5.2: Annuities

- "Annuity" can refer to a wide variety of financial instruments, often associated with retirement
- For us: it is a steady flow of cash into an interest bearing account

  "$100 invested at the end of every month"

  "money in the account earns 1% per month" (12% APR compounded monthly)
- Such a cash flow is worth $1200+$68.25 at the end of the year
- The $1200 part is just the 12 payments of $100
- How do we figure out the "+$68.25" part?

5.2: Work on the front of the worksheet

- Answer question #1 (a-e) on the worksheet
- There is an interest bearing account
- Nominal interest rate of 12% APR, compounded monthly

  So actual interest rate is 1% per month, compounded monthly
- Several $100 bills were deposited
- How much they are worth depends on how long they've been there

5.2: Sum method for annuity

- \$100 is worth $\$100(1.01)^n$ if it has been there n months

  (a) $\$100(1.01)^0 = \$100$

  (b) $\$100(1.01)^1 = \$101$

  (c) $\$100(1.01)^2 = \$102.01$

  (d) is just $\$100 + \$101 + \$102.01 = \$303.01$

- So twelve \$100s (one for eleven, one for ten, . . . , one for one, one just now) is

$$\$100(1.01)^{11} + \$100(1.01)^{10} + \cdots + \$100(1.01)^1 + \$100(1.01)^0$$

- Only new part from eleven \$100s is the $\$100(1.01)^{11} = \$111.57$

  (e) Total is $\$1156.68 + \$111.57 = \$1268.25$

5.2: Spreadsheet method for annuity

- Four columns: Old balance, Interest, Payment, New Balance

Date	Old	Int	Pay	New
Jan	$0.00	$0.00	$100.00	$100.00
Feb	$100.00	$1.00	$100.00	$201.00
Mar	$201.00	$2.01	$100.00	$303.01
Apr	$303.01	$3.03	$100.00	$406.04
May	$406.04	$4.06	$100.00	$510.10
Jun	$510.10	$5.10	$100.00	$615.20
Jul	$615.20	$6.15	$100.00	$721.35
Aug	$721.35	$7.21	$100.00	$828.56
Sep	$828.56	$8.29	$100.00	$936.85
Oct	$936.85	$9.37	$100.00	$1046.22
Nov	$1046.22	$10.46	$100.00	$1156.68
Dec	$1156.68	$11.57	$100.00	$1268.25

	A	B	C	D
1	0	`=A1*0.01`	100	`=A1+B1+C1`
2	`=D1`	`=A2*0.01`	100	`=A2+B2+C2`
3	`=D2`	`=A3*0.01`	100	`=A3+B3+C3`

5.2: Formula method

$$A = R((1+i)^n - 1)/i$$

- where the **Recurring payment** is how much is deposited at the end of each period, like \$100
- the **interest rate** per period, like $12\%/12 = 0.01$

  (nominal interest rate of 12% APR compounded monthly)
- the **number of periods**, like 12 months (one year)
- the **accumulated amount**, like

$$A = \$100((1+0.01)^{12} - 1)/(0.01) = \$1268.25$$

$$A = 100 \star ((1+0.01) \wedge 12 - 1)/(0.01) = 1268.250301$$

5.2: Examples of formula

$$A = R((1+i)^n - 1)/i$$

- After two years of investing \$100 at the end of every month at a nominal interest rate of 12% APR compounded monthly:

$$\begin{aligned} R &= \$100 \\ i &= 12\%/12 \\ &= 0.01 \\ n &= 24 \text{ months} \\ A &= \$100((1+12\%/12)^{24} - 1)/(12\%/12) \approx \$2697.35 \end{aligned}$$

- After three years of investing \$100 at the end of every month at a nominal interest rate of 12% APR compounded monthly:

$$\begin{aligned} R &= \$100 \\ i &= 0.01 \\ n &= 36 \text{ months} \\ A &= \$100((1+12\%/12)^{36} - 1)/(12\%/12) \approx \$4307.69 \end{aligned}$$

5.2: Now finish the worksheet

2. Continue the previous two examples, but to 5 years

3. Need to invest more if you want \$10,000 in only 3 years

4. How long to get \$50,000 at the original rate?

5.2: Now finish the worksheet

2. Continue the previous two examples, but to 5 years

$\$100((1+0.12/12)^{(5\cdot 12)}-1)/(0.12/12)=\$8166.97$

3. Need to invest more if you want \$10,000 in only 3 years

$\$10000/(((1+0.12/12)^{3\cdot 12}-1)/(0.12/12))=\$232.14$

4. How long to get \$50,000 at the original rate?

$\log(\$50000/\$100\cdot(0.12/12)+1)/\log(1+0.12/12)=180.07$ months, 15.00 years

5.2: Total payment

- **Total payment** is a popular measure of a financial instrument
- The total payment of $100 per month for one year is $1200
- It is literally the total of the payments in
- It mixes the just now $100 with the one month ago $100 with the eleven months ago $1200
- It is nearly useless for valuing cash flows, but consumers like it psychologically

5.2: Total payment versus time value of money

- The following cash flows have the same total payment of $1200
- Pay $1200 now
- Pay $1200 in twelve months
- Pay $100 at the end of the next twelve months
- Which is the cheapest option, taking into account the time value of money?

5.2: Total payment versus time value of money

- The following cash flows have the same total payment of \$1200
- Pay \$1200 now
- Pay \$1200 in twelve months
- Pay \$100 at the end of the next twelve months
- Which is the cheapest option, taking into account the time value of money?
- \$1200 in twelve months is cheapest

5.2: Summary

- Today we learned about **annuities**, **present value**, **future value**, and **total payout**
  - Future value of annuity, paying out n times at per-period interest rate i
  $$A = R\frac{(1+i)^n - 1}{i}$$
  - Present value of annuity is just future value divided by $(1+i)^n$
  - Total payout is just nR, n payments of R each
- You are now ready to complete HW 5.2 and should have already completed HW 5.1
- Make sure to take advantage of office hours:
  today 4pm-5pm in Mathskeller (CB63, basement of White Hall Classroom Building)