MA162: Finite mathematics

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Schedule:

- HW 1.1-1.4, 2.1-2.6, 3.1-3.3, 4.1, 5.1 (Late)
- HW 5.2-5.3 due Friday, Mar 22, 2013
- HW 6A due Friday, Mar 29, 2013
- HW 6B-6C due Friday, Apr 5, 2013
- Exam 3, Monday, Apr 8, 2013

Today we will cover amortized loans

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
  - Compound interest
  - Sinking funds
  - Amortized loans

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations

5.2: Summary

- Monday we learned about **annuities**, **present value**, **future value**, and **total payout**
  - Future value of annuity, paying out n times at per-period interest rate i

$$A = R\frac{(1+i)^n - 1}{i}$$

  - Present value of annuity is just future value divided by $(1+i)^n$
  - Total payout is just nR, n payments of R each
- We should be done with 5.1 and 5.2.
- Today we handle 5.3.

5.3: Buying annuities

- How much would you pay today for an annuity paying you back $100 per month for 12 months?
- No more than $1200 for sure

  (if you had $1200 you could just pay yourself)
- What is the least amount of money you would accept?
- This depends on the interest rate you have available.

5.3: Buying annuities (worksheet #1)

- Suppose you have a 12% APR (1% per month) account

  How much would that account need to have in it now

  to pay out $100 every month for a year?

- Surely $1200 is enough.

- Work this out on the worksheet:

  How much would we have leftover?

5.3: Buying annuities

- Suppose you have a 12% APR (1% per month) account

  How much would that account need to have in it now

  to pay out \$100 every month for a year?

- We can use the time value of money to figure this out

- How much is the cash flow worth?

- We have a formula
  $\$100((1 + 0.12/12)^{12} - 1)/(0.12/12) = \$1268.25$

- But that is what it is worth a year from now

5.3: Buying annuities

- We solve a 5.1 problem:

$$\begin{aligned} P &= ? \\ i &= 0.12/12 = 0.01 \text{ per month} \\ n &= 12 \text{ months} \\ A &= \$1268.25 \\ A &= P(1+i)^n \\ \$1268.25 &= P(1.01)^{12} \\ P &= \$1268.25/(1.01)^{12} = \$1125.50 \end{aligned}$$

- \$1125.50 right now can be invested to end up with \$1268.25
- \$100 every month can be invested to end up with \$1268.25
- So \$100 every month is worth the same as \$1125.50 now

5.3: Pricing annuities again

- What if we don't want to invest it?
  What if we want to spend $100 every month?
- Well, put $1125.50 in the bank and remove $100 every month
- How much is left at the end of the year?

Date	Old Balance	Interest on Old	Withdrawal	New Balance
Jan	$1125.50	$11.26	$100.00	$1036.76
Feb	$1036.76	$10.37	$100.00	$ 947.12
Mar	$ 947.12	$ 9.47	$100.00	$ 856.59
Apr	$ 856.59	$ 8.57	$100.00	$ 765.16
May	$ 765.16	$ 7.65	$100.00	$ 672.81
Jun	$ 672.81	$ 6.73	$100.00	$ 579.54
Jul	$ 579.54	$ 5.80	$100.00	$ 485.33
Aug	$ 485.33	$ 4.85	$100.00	$ 390.19
Sep	$ 390.19	$ 3.90	$100.00	$ 294.09
Oct	$ 294.09	$ 2.94	$100.00	$ 197.03
Nov	$ 197.03	$ 1.97	$100.00	$ 99.00
Dec	$ 99.00	$ 0.99	$100.00	$ -0.01

5.3: Pricing an annuity

- To price an annuity using our old formulas:
- Find the future value $A = R((1+i)^n - 1)/(i)$
- Find the present value by solving $A = P(1+i)^n$

$$P = A/((1+i)^n)$$

- If you like new formulas, the book divides the $(1+i)^n$ using algebra:

$$P = R\left(1 - (1+i)^{(-n)}\right)/(i)$$

5.3: Examples

- How much is a cash flow of \$100 per month for four years worth if you have a 12% APR account?

$$\begin{aligned} \mathrm{P} &= ? \\ \mathrm{R} &= \$100 \\ \mathrm{i} &= 0.12/12 = 0.01 \\ \mathrm{n} &= 4(12) = 48 \end{aligned}$$

$P = \$100(1 - (1 + 0.01)^{-48})/(0.01) = \$3797.40$

- How much is a cash flow of \$100 per month for four hundred years worth if you have a 12% APR account?

$P = \$100(1 - (1 + 0.01)^{-4800})/(0.01) = \$10000.00$

- Cash flows "max out" at some point

5.3: Worksheet (#2-#5)

2. How much to pay now to cover \$100 per month for five years at 12% APR?

3. How much does \$10,000 cover per month for three years at 12% APR?

4. How long does \$6970.05 take to pay back at \$100 per month and 12% APR?

5. How long does \$50000.00 take to pay back at \$100 per month and 12% APR?

5.3: Worksheet (#2-#5)

2. How much to pay now to cover \$100 per month for five years at 12% APR?

   $P = \$100(1 - (1 + 0.01)^{-60})/0.01 = \$4495.50$

3. How much does \$10,000 cover per month for three years at 12% APR?

   $R = \$10000/((1 - (1 + 0.01)^{-36})/(0.01)) = \$332.14$

4. How long does \$6970.05 take to pay back at \$100 per month and 12% APR?

   $n = -\log(1 - 6970.05/100 \cdot 0.01)/\log(1 + 0.01) = 120.00$ months, 10 years

5. How long does \$50000.00 take to pay back at \$100 per month and 12% APR?

   The interest is \$500 per month, so \$100 per month won't cut it

5.3: Perspective on annuities

- Bart borrows \$100 per month from Alex for a year
- Alex is well known to charge 12% APR
- Bart will owe Alex \$1268.25 at the end of the year

- Alex invests \$100 per month in Bart for a year
- Bart is well known to have a 12% annual rate of return
- Alex expects to receive \$1268.25 by the end of the year

- Bart is a bank (like a savings account)

5.3: Perspective on amortized loans

- Bart invests \$1125.51 in Alex at the beginning of the year
- Alex is well known to have a 12% annual rate of return
- Bart expects Alex to yield \$100 per month for a year

- Alex borrows \$1125.51 from Bart at the beginning of the year
- Bart is well known to charge 12% APR
- Alex has to pay back \$100 per month for a year

- Bart is a credit card company or a bank's mortgage office