Commutator formulas

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This expository note mentions some interesting formulas using commutators. It touches on Hall's collection process and the associated Hall polynomials. It gives an alternative expression that is linear in the number of commutators and shows how to find such a formula using staircase diagrams. It also shows the shortest possible such expression.

Future versions could touch on isoperimetric inequalities in geometric group theory, powers of commutators and Culler's identity as well as its effect on Schur's inequality between $[G : Z(G)]$ and $|G'|$.

1 Powers of products versus products of powers

In an abelian group one has $(xy)^n = x^n y^n$ so in a general group one has $(xy)^n = x^n y^n d_n(x, y)$ for some product of commutators $d_n(x, y)$. This section explores formulas for $d_n(x, y)$.

1.1 A nice formula in a special case

1.1 A nice formula in a special case is given by certain binomial coefficients:

$$(xy)^n = x^{\binom{n}{1}} y^{\binom{n}{1}} [y, x]^{\binom{n}{2}} [[y, x], x]^{\binom{n}{3}} [[[y, x], x], x]^{\binom{n}{4}} \cdots [y,\ _{n-1}x]^{\binom{n}{n}}$$

The special case is G' is abelian and commutes with y.

The commutators involved are built inductively: From y and x, one gets $[y, x]$. From $[y, x]$ and x, one gets $[[y, x], x]$. From $[y,\ _{n-2}x]$ and x, one gets $[y,\ _{n-1}, x]$. In general, one would also need to consider $[y, x]$ and $[[y, x], x]$, but the special case assumes commutators commute, so $[[y, x], [[y, x], x]] = 1$. In general, one would also need to consider $[y, x]$ and y, but the special case assumes commutators commute with y, so $[[y, x], y] = 1$.

To avoid an excess of brackets in the future, we use the left normed convention: $[a, b, c, \ldots, z] = [[\ldots[[a, b], c], \ldots], z]$. For example, $[[[y, x], x], x] = [y, x, x, x]$ and $[[y, x], [[y, x], x]] = [[y, x], [y, x, x]]$.

The exponents involved use binomial coefficients. The binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is a polynomial in n of degree $\min(k, n-k)$ that takes on integer values when n is an integer, but the polynomial itself has rational coefficients. There is a theorem that every polynomial with rational coefficients that only takes integer values at integers is a $\mathbb{Z}$-linear combination of binomial coefficients.

The next section generalizes this formula to the case where commutators vanish if they are nested deeply enough.

1.2 Hall polynomials for nilpotent groups are a nice way to express $d_n(x, y)$ in terms of nested commutators.

To define how deeply nested a commutator is, we define the "weight" of various simple expression. The short version is that a nested commutator has weight equal to how many things get commutated. We say x and y are "commutators" of weight 1. If w is a commutator of weight i and v is a commutator of weight j, then $[w, v]$ is a commutator of weight $i + j$.

A nilpotent group of class c is one in which all commutators of weight greater than c vanish. Thus in a nilpotent group, we are interested in expressions for $d_n(x,y)$ sorted by weight. Hall's commutator collection process is particularly suited to this. In the free nilpotent group of class c certain commutators of each weight are selected as "basic commutators". These basic commutators are always chosen to be commutators of basic commutators of lower weight, and form a basis of the free abelian group generated by all commutators of their weight modulo all commutators of higher weight. In terms of these basic commutators, $d_n(x,y)$ is a product of powers of basic commutators, and the exponents on each one are integer-valued rational-coefficient polynomials of n, the so-called Hall polynomials.

The expansion up to weight 6 is in figure 2. The expansion up to weight 3 is:

$$(xy)^n = x^n y^n [y,x]^{\binom{n}{2}} [y,x,x]^{\binom{n}{3}} [y,x,y]^{2\binom{n}{3}+\binom{n}{2}} \mod \gamma_4(\langle x,y\rangle)$$

where $\gamma_4(\langle x,y\rangle)$ is the subgroup generated by all commutators of weight 4 or greater.

For $n=2$, most of the binomial coefficients are 0, and so this gives the pleasant finite formula:

$$(xy)^2 = x^2y^2[y,x][[y,x],y]$$

However, for $n=3$ and up the formula is no longer finite. In fact, allowing a less nice formula, but using the same commutator collection process, we get for $n=3$:

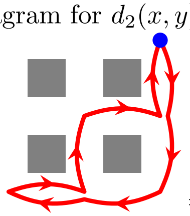
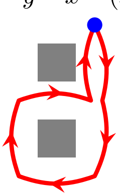
$$\begin{aligned}(xy)^3 &= x^3y^3[y,x][y,x,y][y,x,y][y,x,y,y][y,x][y,x,y][y,x,y]\\ &\cdot[y,x,y,y][y,x,x][y,x,x,y][y,x,x,y][y,x,x,y,y][y,x][y,x,y]\end{aligned}$$

Yuck, $d_3(x,y)$ is expressed as a product of 14 commutators, in unattractive order.

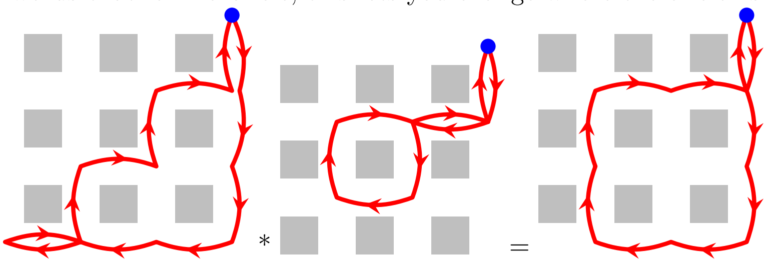
1.3 Shorter expressions would be highly desirable. How many commutators does $d_3(x,y)$ take? Is 14 commutators really efficient? In fact a simple argument using diagrams shows us how to write $d_n(x,y)$ as a product of $n-1$ commutators, each one of a very nice form.

The diagram itself lies on what is called the Cayley graph of the integer lattice in the plane. An explanation that produces identical diagrams is that we start at the origin, and every time we read an x^k we travel k in the x-direction and every time we read a y^k we travel k in the y-direction. Hence $xyxy$ takes us along a zig-zag path to the point with coordinates $(2,2)$. To make the picture more modern, we imagine sky-scrapers built at all $(m+\frac{1}{2}, n+\frac{1}{2})$ for $m,n\in\mathbb{Z}$ and that we are attached to the origin by a (very) extendible leash that trails behind us (attached to a monkey vest). Equality of paths means that if you follow the left hand side of the equality, and then retrace your steps on the right hand side, then the leash doesn't get stuck around a building (though other pedestrians and small plants beware). In less modern language, we only take the 1-skeleton; there are no 2-cells.

We want an expression for $d_2(x,y)$ using only a single commutator. We look at the diagram for $d_2(x,y) = y^{-2}x^{-2}(xy)^2$:

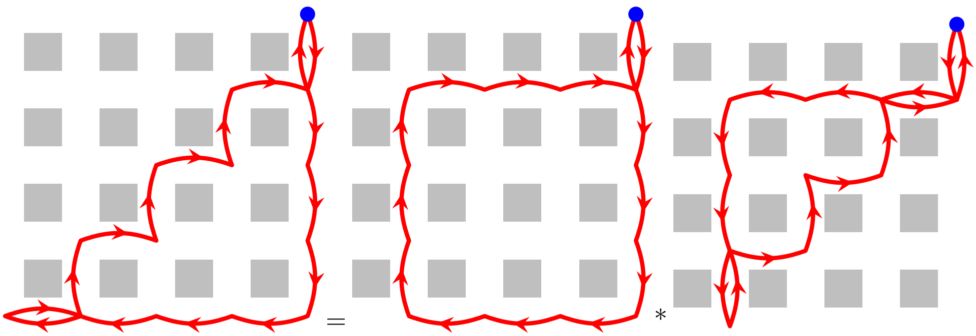
 =  and this latter is $y^{-1}[y,x]y = [y,x]^y$, showing us that $(xy)^2 = x^2y^2[y,x]^y$.

A few key points about the diagrams: conjugation is how you change the starting position of diagram: w^v means "first go backwards along v to get to the new starting position, now travel w as if this was the origin, now travel v back to the true origin." For words that form a circle, this lets you change where the circle "starts."



In other words, $d_3(x, y) * [y, x]^{xy} = [y^2, x^2]^y$, so using that $[x, y]^{-1} = [y, x]$, we get

$$(xy)^3 = x^3 y^3 [y^2, x^2]^y [x, y]^{xy}$$



In general we get

$$d_n(x, y) = ([y^n, x^n] d_{n-1}(y, x))^y$$

so that the commutator length of $d_n(x, y)$ is at most $1 + d_{n-1}(x, y)$. Unrolling the recursion shows the length of d_n is at most $n - 1$.

1.4 The shortest possible expression in terms of number of commutators used can be found with only a bit more trickery. Instead of using $d_{n-1}(y, x)$ to turn $d_n(x, y)$ into a box, which is a commutator of powers, we need to use d_{n-2} to get a faster recursion. The result is a weird shape, but if you look closely (and can distinguish red versus blue in the following diagram), you can see the result is also a commutator. Long story short,

$$d_n(x, y) = \left([xy^{n-1}, y^{-1}x^{n-2}] d_{n-2}(y, x)\right)^y$$

so that the commutator length of $d_n(x, y)$ is at most $1 + d_{n-2}(x, y)$. Unrolling the recursion shows the length of d_n is at most $\lfloor n/2 \rfloor$.

1.5 Is there any shorter expression? This is asking about the so-called **commutator length**. I hope the diagrams indicate that a geometric approach is useful. The answer to the commutator length question makes more substantial use of topology. Algorithmically, I would like to point out it uses integer programming (and the stable version of the problem, which asks about the limit of $\frac{1}{n}$th of the commutator length of the nth power, uses linear programming).

See (Culler, 1981), (Bavard, 1991), (Calegari, 2009).

For those of us who have not yet learned the topological methods, there is software to handle the calculations, scallop.

```
$ ./scallop -c -C AAAAAABBBBBBabababababab
cl_{a*b}(1AAAAAABBBBBBabababababab ) = 5/2 = 2.5
```

The `-c` option tells it to use methods for free products of cyclic groups, like a free group. The `-C` option tells it to compute commutator length (harder) rather than stable commutator length. The answer it gives is currently always 0.5 less than what I would consider the correct answer (it is computing a topological invariant, rather than the stable or not commutator length). At any rate, this means the minimum number of commutators when expressing $d_6(a, b)$ is 3.

See Danny Calegari's monograph scl for details.

References

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2 Figures

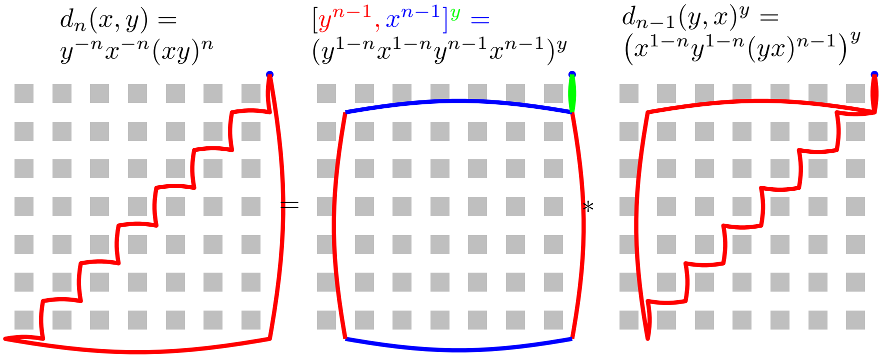
The figures have been moved to the following pages.

Jack Schmidt

$$
\begin{aligned}
(xy)^n = \quad & x^n y^n \\
& \cdot [y,x]^{n_2} \\
& \cdot [y,x,x]^{n_3} \\
& \cdot [y,x,y]^{2n_3+n_2} \\
& \cdot [y,x,x,x]^{n_4} \\
& \cdot [y,x,x,y]^{3n_4+2n_3} \\
& \cdot [y,x,y,y]^{3n_4+2n_3} \\
& \cdot [y,x,x,x,x]^{n_5} \\
& \cdot [y,x,x,x,y]^{4n_5+3n_4} \\
& \cdot [y,x,x,y,y]^{6n_5+6n_4+n_3} \\
& \cdot [y,x,y,y,y]^{4n_5+3n_4} \\
& \cdot [[y,x,x],[y,x]]^{6n_5+7n_4+n_3} \\
& \cdot [[y,x,y],[y,x]]^{12n_5+18n_4+6n_3} \\
& \cdot [y,x,x,x,x,x]^{n_6} \\
& \cdot [y,x,x,x,x,y]^{5n_6+4n_5} \\
& \cdot [y,x,x,x,y,y]^{10n_6+12n_5+3n_4} \\
& \cdot [y,x,x,y,y,y]^{10n_6+12n_5+3n_4} \\
& \cdot [y,x,y,y,y,y]^{5n_6+4n_5} \\
& \cdot [[y,x,x,x],[y,x]]^{10n_6+13n_5+3n_4} \\
& \cdot [[y,x,x,y],[y,x]]^{30n_6+52n_5+24n_4+2n_3} \\
& \cdot [[y,x,y,y],[y,x]]^{30n_6+54n_5+27n_4+3n_3} \\
& \cdot [[y,x,y],[y,x,x]]^{20n_6+36n_5+21n_4+4n_3} \\
& \cdot [y,x,x,x,x,x,x]^{n_7} \\
& \cdot [y,x,x,x,x,x,y]^{6n_7+5n_6} \\
& \cdot [y,x,x,x,x,y,y]^{15n_7+20n_6+6n_5} \\
& \cdot [y,x,x,x,y,y,y]^{20n_7+30n_6+12n_5+n_4} \\
& \cdot [y,x,x,y,y,y,y]^{15n_7+20n_6+6n_5} \\
& \cdot [y,x,y,y,y,y,y]^{6n_7+5n_6} \\
& \cdot [[y,x,x,x,x],[y,x]]^{15n_7+21n_6+6n_5} \\
& \cdot [[y,x,x,x,y],[y,x]]^{60n_7+115n_6+64n_5+9n_4} \\
& \cdot [[y,x,x,y,y],[y,x]]^{90n_7+190n_6+126n_5+27n_4+n_3} \\
& \cdot [[y,x,y,y,y],[y,x]]^{60n_7+120n_6+72n_5+12n_4} \\
& \cdot [[[y,x,x],[y,x]],[y,x]]^{45n_7+100n_6+67n_5+12n_4} \\
& \cdot [[[y,x,y],[y,x]],[y,x]]^{90n_7+225n_6+184n_5+51n_4+2n_3} \\
& \cdot [[y,x,x,x],[y,x,x]]^{20n_7+34n_6+16n_5+n_4} \\
& \cdot [[y,x,x,y],[y,x,x]]^{60n_7+125n_6+84n_5+18n_4} \\
& \cdot [[y,x,y,y],[y,x,x]]^{60n_7+130n_6+96n_5+27n_4+2n_3} \\
& \cdot [[y,x,x,x],[y,x,y]]^{40n_7+75n_6+40n_5+5n_4} \\
& \cdot [[y,x,x,y],[y,x,y]]^{120n_7+270n_6+196n_5+48n_4+2n_3} \\
& \cdot [[y,x,y,y],[y,x,y]]^{120n_7+280n_6+220n_5+63n_4+4n_3} \\
& \cdots
\end{aligned}
$$

Here $[y,x] = y^{-1}x^{-1}yx$ is the group theoretic commutator, $[y,x,x] = [[y,x],x]$ is the left normed commutator, n_k is the binomial coefficient $\frac{n!}{(n-k)!k!}$ and the "$\cdots$" refers to a product of nested commutators of strictly larger weight, that is, an element of $\gamma_8(\langle x,y\rangle)$ where $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$.

Figure 1: Expansion of $(xy)^n$ in terms of Hall's basic commutators



This gives the simple recurrence:

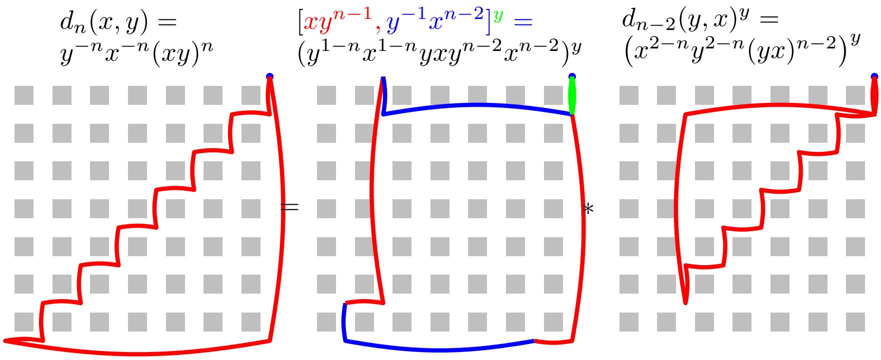
$$\left(d_n(x,y) = [y^{n-1}, x^{n-1}] \cdot d_{n-1}(y,x)\right)^y$$

Figure 2: Simple recursive expression in terms of commutators

$$\begin{aligned}
(xy)^2 &= x^2y^2[y,x]^y \\
(xy)^3 &= x^3y^3[y^2,x^2]^y[x,y]^{xy} \\
(xy)^4 &= x^4y^4[y^3,x^3]^y[x^2,y^2]^{xy}[y,x]^{yxy} \\
(xy)^5 &= x^5y^5[y^4,x^4]^y[x^3,y^3]^{xy}[y^2,x^2]^{yxy}[x,y]^{xyxy} \\
(xy)^6 &= x^6y^6[y^5,x^5]^y[x^4,y^4]^{xy}[y^3,x^3]^{yxy}[x^2,y^2]^{xyxy}[y,x]^{yxyxy} \\
(xy)^7 &= x^7y^7[y^6,x^6]^y[x^5,y^5]^{xy}[y^4,x^4]^{yxy}[x^3,y^3]^{xyxy}[y^2,x^2]^{yxyxy}[x,y]^{xyxyxy} \\
\dots \\
(xy)^n &= x^ny^n\prod_{i=1}^{n-1} t(n-i,i)
\end{aligned}$$

Here $t(k,i) = \left([x^k,y^k]^{(-1)^i}\right)^{\cdots xy}$ and $\cdots xy$ is the alternating product of x and y of length i, ending in y.

Figure 3: Expansion of $(xy)^n$ as nice conjugates of $[x^k,y^k]^{\pm 1}$



This gives the faster recurrence:

$$\left(d_n(x,y) = [xy^{n-1}, y^{-1}x^{n-2}] \cdot d_{n-2}(y,x)\right)^y$$

Figure 4: Faster recursive expression in terms of commutators

$$\begin{aligned}
d_1 &= 1 \\
d_2 &= [xy^1,\ y^{-1}x^0]^y \\
d_3 &= [xy^2,\ y^{-1}x^1]^y \\
d_4 &= [xy^3,\ y^{-1}x^2]^y[yx^1, x^{-1}y^0]^{xy} \\
d_5 &= [xy^4,\ y^{-1}x^3]^y[yx^2, x^{-1}y^1]^{xy} \\
d_6 &= [xy^5,\ y^{-1}x^4]^y[yx^3, x^{-1}y^2]^{xy}[xy^1, y^{-1}x^0]^{yxy} \\
d_7 &= [xy^6,\ y^{-1}x^5]^y[yx^4, x^{-1}y^3]^{xy}[xy^2, y^{-1}x^1]^{yxy} \\
d_8 &= [xy^7,\ y^{-1}x^6]^y[yx^5, x^{-1}y^4]^{xy}[xy^3, y^{-1}x^2]^{yxy}[yx^1, x^{-1}y^0]^{xyxy} \\
d_9 &= [xy^8,\ y^{-1}x^7]^y[yx^6, x^{-1}y^5]^{xy}[xy^4, y^{-1}x^3]^{yxy}[yx^2, x^{-1}y^1]^{xyxy} \\
d_{10} &= [xy^9,\ y^{-1}x^8]^y[yx^7, x^{-1}y^6]^{xy}[xy^5, y^{-1}x^4]^{yxy}[yx^3, x^{-1}y^2]^{xyxy}[xy^1, y^{-1}x^0]^{yxyxy} \\
d_{11} &= [xy^{10}, y^{-1}x^9]^y[yx^8, x^{-1}y^7]^{xy}[xy^6, y^{-1}x^5]^{yxy}[yx^4, x^{-1}y^3]^{xyxy}[xy^2, y^{-1}x^1]^{yxyxy}
\end{aligned}$$

We leave it to the reader to formulate d_n. These are expressions of the d_n as a product of the fewest commutators, $\lfloor \frac{n}{2} \rfloor$.

Figure 5: Shortest expressions

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