Sylow subgroups of GL(3,q)

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We describe the Sylow p-subgroups of $\mathrm{GL}(n, q)$ for $n \leq 4$. These were described in (Carter & Fong, 1964) and (Weir, 1955).

1 Overview

The groups $\mathrm{GL}(n, q)$ have three types of Sylow p-subgroups: p divides q, $p > n$, and $p \leq n$.

1.1 p **divides** q**:** The Sylow p-subgroups of the first type are easy to describe as the upper triangular matrices with 1s on the diagonal (and anything above):

$$P = \left\{ \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} : * \in \mathbb{F}_q \right\}$$

In the remainder, we assume p does not divide q.

1.2 $p > n$**:** The Sylow p-subgroups of the second type are abelian and are contained within a direct product T of extension fields of $\mathbb{F}_q$. If e is the order of q mod p, then p divides $q^e - 1$ and so the Sylow p-subgroup of $\mathbb{F}_{q^e}^\times$ is a non-identity cyclic group. If $e > n$, then p does not divide the order of $\mathrm{GL}(n, q)$, so both T and the Sylow p-subgroup will be the identity subgroup. If $e = 1$, then T is the group of diagonal matrices.

$$T = \left\{ \begin{bmatrix} \zeta_1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & 0 & 0 & \zeta_5 \end{bmatrix} : \zeta_i \in \mathbb{F}_q^\times \right\}$$

Since T is a direct product, its Sylow p-subgroup is the direct product of the Sylow p-subgroups $(\mathbb{F}_q^\times)_p$ of its factors $\mathbb{F}_q^\times$.

$$P = \left\{ \begin{bmatrix} \zeta_1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & 0 & 0 & \zeta_5 \end{bmatrix} : \zeta_i \in (\mathbb{F}_q^\times)_p \right\}$$

Now if $e > 1$ we work with block diagonal matrices. $\mathrm{GL}(e, q)$ is the set of linear transformations of $\mathbb{F}_q^e$. If we choose an isomorphism of $\mathbb{F}_q$-vector spaces $V \cong \mathbb{F}_q^e$ (that is, if we choose an ordered basis of V), then we get an isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}(e,q)$. The vector space V we are interested in is $V = \mathbb{F}_{q^e}$. Every extension field of $\mathbb{F}_q$ is a $\mathbb{F}_q$-vector space. Every element of $\mathbb{F}_{q^e}^\times$ acts via multiplication on V in a $\mathbb{F}_q$-linear way (proof is the distributive and commutative law). Hence $\mathbb{F}_{q^e}^\times \leq \mathrm{GL}(V) \cong \mathrm{GL}(e,q)$. Now we consider a larger vector space $W \cong \mathbb{F}_q^n$. W is a direct sum of copies of $\mathbb{F}_{q^e}$ and then copies of $\mathbb{F}_q$. The subgroup T of $\mathrm{GL}(W)$ is defined by letting copies of $\mathbb{F}_{q^e}^\times$ act on the copies of $\mathbb{F}_{q^e}$, and letting the identity act on the copies of $\mathbb{F}_q$.

For example, if $e = 3$ and $n = 13$ then

$$T_e = \left\{ \begin{bmatrix} \zeta_1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} : \zeta_i \in \mathbb{F}_{q^3}^\times \right\}$$

where each ζ_i is actually a $3{\times}3$ matrix block coming from $\mathrm{GL}(V) = \mathrm{GL}(\mathbb{F}_{q^e}) \cong \mathrm{GL}(3,q)$. This happens for example when finding the Sylow 19-subgroup of $\mathrm{GL}(3,7)$.

Then P again just restricts the ζ_i to come from the Sylow p-subgroup of $\mathbb{F}_{q^e}^\times$.

In all cases, we get an abelian (homocyclic) subgroup isomorphic to a Sylow p-subgroup of a direct product of extension fields containing at least a primitive pth root of unity. Indeed, the field $\mathbb{F}_{q^e}$ is the splitting field of $x^p - 1$ over $\mathbb{F}_q$.

1.3 $p \leq n$**:** The primes less than or equal to n are similar to the other primes that don't divide q, but there is an extra complication from the permutation matrices. Again we look at the order e of q mod p. Since $p \leq n$, $e < n$. If $e = 1$, then the subgroup containing the Sylow is easy to describe: it is the group M of monomial matrices. These matrices have one nonzero entry in each row and column, so they are the product of a permutation matrix and a diagonal matrix.

$$M = \mathrm{Sym}(n) \ltimes T, \qquad \begin{bmatrix} 0 & \zeta_1 & 0 & 0 & 0 \\ \zeta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_3 \\ 0 & 0 & 0 & \zeta_4 & 0 \\ 0 & 0 & \zeta_5 & 0 & 0 \end{bmatrix} \in M$$

Now if $e > 1$, then the $\zeta_i \in \mathbb{F}_{q^e}^\times$ are $e \times e$ matrix blocks, so we get

$$M_e = \mathrm{Sym}(\lfloor n/e \rfloor) \ltimes T_e.$$

As long as $p > 2$, this is fine.

If $p = 2$, then e needs to be defined to be the order of q mod 4, rather than mod 2. Hence it is possible that e is divisible by p, so M_e is not quite big enough. We need to take a larger subgroup of $\mathrm{GL}(V)$ than just $\mathbb{F}_{q^e}^\times$. We take one more $\mathbb{F}_q$-linear transformation of $V = \mathbb{F}_{q^e}$, the Frobenius automorphism, $f : \mathbb{F}_{q^e} \to \mathbb{F}_{q^e} : v \mapsto v^q$. Since f is an invertible $\mathbb{F}_q$-linear transformation of V, $f \in \mathrm{GL}(V)$. I like to call $\langle f \rangle$ by its fancy name, $\mathrm{Gal}(\mathbb{F}_{q^e}/\mathbb{F}_q)$. Since $\langle f \rangle$ normalizes $\mathbb{F}_{q^e}^\times$ and intersects it trivial, the subgroup generated by f and $\mathbb{F}_{q^e}^\times$ is a semidirect product, $\Gamma\mathrm{L}(1,q^e) = \mathrm{Gal}(\mathbb{F}_{q^e}/\mathbb{F}_q) \ltimes \mathbb{F}_{q^e}^\times$, and we define

$$\Gamma M_e = \mathrm{Sym}(\lfloor n/e \rfloor) \ltimes (\mathrm{Gal}(\mathbb{F}_{q^e}/\mathbb{F}_q) \ltimes \mathbb{F}_{q^e}^\times)^{\lfloor n/e \rfloor}$$

Then a Sylow p-subgroup of ΓM_e is formed by taking Sylow p-subgroups of each of its ingredients. If P_1 is a Sylow p-subgroup of $\mathrm{Sym}(\lfloor n/e \rfloor)$, and P_2 is a Sylow p-subgroup of $\mathrm{Gal}(\mathbb{F}_{q^e}/\mathbb{F}_q)$, and P_3 is a Sylow p-subgroup of $\mathbb{F}_{q^e}^\times$, then

$$P = P_1 \ltimes (P_2 \ltimes P_3)^{\lfloor n/e \rfloor}$$

When n is small, it is very difficult for all of this to happen at once, so often things will be simpler than in the general case.

1.4 Proofs For detailed proofs of similar descriptions, see (Carter & Fong, 1964) and (Weir, 1955). However, if n, q, and p are explicit numbers, then one can usually prove these results in a fairly elementary manner. More or less by definition, each of the claimed Sylow p-subgroups is at least a p-subgroup. Each one is defined as the Sylow p-subgroup of some subgroup of $\mathrm{GL}(n, q)$. These subgroups are chosen very carefully: (1) they have a structure where Sylow p-subgroups can be specified fairly explicitly in terms of Sylow p-subgroups of cyclic groups and a $\mathrm{Sym}(n)$, and (2) their index is relatively prime to p, so that a Sylow p-subgroup of M_e is a Sylow p-subgroup of $\mathrm{GL}(n, q)$.

The general description (for p not dividing q) is as block monomial matrices with entries from $\Gamma\mathrm{L}(1, q^e)$. We compute the order of these groups:

$|\mathbb{F}_{q^e}^\times| = q^e - 1$

$|T_e| = (q^e - 1)^{\lfloor n/e \rfloor}$

$|\mathrm{GL}(n, q)| = q^{\binom{n}{2}}(q^n - 1)(q^{n-1} - 1)\cdots(q^2 - 1)(q - 1)$

If $p > n$, then the verification is some neat and elementary number theory. We can assume $e \leq n$, since otherwise p does not divide any of the factors in the above expression for $|\mathrm{GL}(n, q)|$. Which factors does p divide? Clearly it divides $q^{ek} - 1$ since $q^{ek} = (q^e)^k \equiv 1^k = 1 \mod p$. Does p divide $(q^{ek} - 1)/(q^e - 1) = (q^e)^{k-1} + (q^e)^{k-2} + \ldots + q^e + 1 \equiv 1^{k-1} + 1^{k-2} + \ldots + 1^1 + 1^0 = k \mod p$? Well, only if $p \leq k \leq n$, so not in this case; we are safe. In particular, p does not divide $[\mathrm{GL}(n, q) : T_e]$, since each $q^i - 1$ that p divides is of the form $q^{ek} - 1$, and for each of those there is a $q^e - 1$ in $|T_e|$, and the quotient is not divisible by p.

The problem occurs when $p \leq n$. Then it is possible that p divides $(q^{ek} - 1)/(q^e - 1)$, so T_e does not contain a Sylow p-subgroup. There are two main issues: p divides k or $p = 2$ divides e.

For example, if $e = 1$ and $n = p > 2$, then $|T_e| = (q - 1)^n$ and we only have $(q^p - 1)/(q - 1) \equiv p \mod p^2$ to worry about. Hence a Sylow p-subgroup of $\mathrm{Sym}(p)$ with order p takes care of the excess.

When $p = 2$, we no longer have $(q^p - 1)/(q - 1) \equiv p \mod p^2$, and this necessitates the Galois group.

2 GL(1,q)

For $n = 1$, we have the situation with $\mathrm{GL}(1, q) \cong \mathbb{F}_q^\times$ is cyclic, and so in principle its Sylow p-subgroups are all easy to describe. For later parallels, I'll mention that $\mathbb{F}_q$ is a 1-dimensional $\mathbb{F}_q$-vector space, and $\mathbb{F}_q^\times$ acts on it by multiplication. In this case, the matrix of an element ζ is easy to describe: it is $\begin{bmatrix}\zeta\end{bmatrix}$.

$$P = \left\{ \begin{bmatrix}\zeta_1\end{bmatrix} : \zeta_1 \in (\mathbb{F}_q^\times)_p \right\}$$

In many ways though, $n = 1$ is so small the general features are obscured. In terms of the overview, all primes fall into the case $p > n$ with $e = 1$, so they are all Sylow p-subgroups of $T = \mathrm{GL}(1, q)$, which is not a very impressive answer for "what are the Sylow p-subgroups of $\mathrm{GL}(1, q)$?"

3 GL(2,q)

For $n = 2$, we have that

$$|\mathrm{GL}(2, q)| = (q^2 - 1)(q^2 - q) = q(q - 1)^2(q + 1)$$

and the latter factorization is the driving force in our case-by-case analysis.

3.1 Case 0 (p divides q): If p divides q, then the upper triangular matrices with ones on the diagonal form an elementary abelian Sylow q-subgroup of order q.

$$P = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{F}_q \right\}$$

3.2 Case 1a ($p > n, e = 1$): If $p > n$ divides $q - 1$, then we view $\mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong \mathrm{GL}(1, q) \times \mathrm{GL}(1, q) \leq \mathrm{GL}(2, q)$. The index $[\mathrm{GL}(2, q) : \mathrm{GL}(1, q) \times \mathrm{GL}(1, q)] = q(q+1)$ is not divisible by p, so a Sylow p-subgroup of $\mathrm{GL}(1, q) \times \mathrm{GL}(1, q)$ is a Sylow p-subgroup of $\mathrm{GL}(2, q)$. A direct product of Sylow p-subgroups is a Sylow p-subgroup of the direct product, so this case is easy.

$$P = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} : \zeta_i \in (\mathbb{F}_q^\times)_p \right\}$$

3.3 Case 2a ($p > n, e = 2$): If $p > n$ divides $q + 1$, then we consider $\mathbb{F}_{q^2}^\times \leq \mathrm{GL}(2, q)$ with index $[\mathrm{GL}(2, q) : \mathbb{F}_{q^2}^\times] = q(q - 1)$ not divisible by p. Hence a Sylow p-subgroup of $\mathbb{F}_{q^2}^\times$ is a cyclic Sylow p-subgroup of $\mathrm{GL}(2, q)$.

$$P = \left\{ \begin{bmatrix}\zeta_1\end{bmatrix} : \zeta_1 \in (\mathbb{F}_{q^2}^\times)_p \right\}$$

3.4 Case 1b ($p = n, e = 1$): If $p = n$ divides $(q - 1)/2$, then we consider $\mathrm{Sym}(2) \ltimes (\mathrm{GL}(1, q) \times \mathrm{GL}(1, q))$ consisting of monomial matrices, a subgroup of index $[\mathrm{GL}(2, q) :$

$\mathrm{Sym}(2) \ltimes (\mathrm{GL}(1,q) \times \mathrm{GL}(1,q))] = q(q+1)/2$ which is not divisible by p. A Sylow p-subgroup is thus a wreath product of a Sylow p-subgroup of $\mathrm{Sym}(n)$ with a Sylow p-subgroup of $\mathrm{GL}(1,q)$. In this case, this just means monomial matrices whose entries are 2^kth roots of unity.

$$P = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} : \zeta_i \in (\mathbb{F}_q^\times)_p \right\} \cup \left\{ \begin{bmatrix} 0 & \zeta_1 \\ \zeta_2 & 0 \end{bmatrix} : \zeta_i \in (\mathbb{F}_q^\times)_p \right\}$$

Or as a semidirect product:

$$P = \mathrm{Sym}(2) \ltimes (\mathbb{F}_q^\times)_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \ltimes \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} : \zeta_i \in (\mathbb{F}_q^\times)_p \right\}$$

3.5 Case 2b ($p = n, e = 2$): If $p = n$ divides $(q+1)/2$, then we consider $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \ltimes \mathbb{F}_{q^2}$, a subgroup of index $[\mathrm{GL}(2,q) : \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \ltimes \mathbb{F}_{q^2}] = q(q-1)/2$ which is not divisible by p. A Sylow p-subgroup is thus a Sylow p-subgroup of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ acting on a Sylow p-subgroup of $\mathbb{F}_{q^2}^\times$.

In more detail, $\mathbb{F}_{q^2}$ is a two-dimensional vector space, so choose a basis. The Frobenius automorphism $f : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2} : x \mapsto x$ is an invertible $\mathbb{F}_q$-linear transformation of this two-dimensional vector space, so it has an associated 2×2 matrix, $\left[f\right]$. Similarly, every element ζ_1 of $\mathbb{F}_{q^2}^\times$ acts via multiplication as $\mathbb{F}_q$-linear transformation of the two-dimensional vector space $\mathbb{F}_{q^2}$, and so has an associated 2×2 matrix $\left[\zeta_1\right]$. In these terms we have the following explicit description of the Sylow p-subgroup:

$$P = \left\{ \left[\zeta_1\right] : \zeta_1 \in (\mathbb{F}_{q^2}^\times)_p \right\} \cup \left\{ \left[f\right] \left[\zeta_1\right] : \zeta_1 \in (\mathbb{F}_{q^2}^\times)_p \right\}$$

or as a semidirect product:

$$P = \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \ltimes (\mathbb{F}_{q^2}^\times)_p = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \left[f\right] \right\} \ltimes \left\{ \left[\zeta_1\right] : \zeta_1 \in (\mathbb{F}_{q^2}^\times)_p \right\}$$

4 GL(3,q)

For $n = 3$, we have that

$$|\,\mathrm{GL}(3,q)| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)^3(q+1)(q^2+q+1)$$

and the latter factorization is the driving force in our case-by-case analysis.

4.1 Case 0 (p divides q): If p divides q, then the upper triangular matrices with ones on the diagonal form a Sylow q-subgroup of order q^3, with derived subgroup, center, and Frattini subgroup all of order q.

4.2 Case 1a ($p > n$ and $e = 1$): If $p > n$ divides $q - 1$, then we view $\mathbb{F}_q^\times \times \mathbb{F}_q^\times \times \mathbb{F}_q^\times \cong \mathrm{GL}(1,q) \times \mathrm{GL}(1,q) \times \mathrm{GL}(1,q) \leq \mathrm{GL}(3,q)$. The index $[\mathrm{GL}(3,q) : \mathrm{GL}(1,q)^3] = q^3(q+1)(q^2+q+1)$ is not divisible by p, so a Sylow p-subgroup of $\mathrm{GL}(1,q)^3$ is a Sylow

p-subgroup of $\mathrm{GL}(3,q)$. A direct product of Sylow p-subgroups is a Sylow p-subgroup of the direct product, so this case is easy.

4.3 Case 2a ($p > n$ and $e = 2$): If $p > n$ divides $q+1$, then we consider $1 \times \mathbb{F}_{q^2}^\times \leq \mathrm{GL}(3,q)$ with index $[\mathrm{GL}(3,q) : \mathbb{F}_{q^2}^\times] = q^3(q-1)^2(q^2+q+1)$ not divisible by p. Hence a Sylow p-subgroup of $\mathbb{F}_{q^2}^\times$ is a cyclic Sylow p-subgroup of $\mathrm{GL}(3,q)$.

4.4 Case 3a ($p > n$ and $e = 3$): If $p > n$ divides q^2+q+1, then we consider $\mathbb{F}_{q^3}^\times \leq \mathrm{GL}(3,q)$ with index $q^3(q-1)^2(q+1)$, not divisible by p. Hence a Sylow p-subgroup of $\mathbb{F}_{q^3}^\times$ is a cyclic Sylow p-subgroup of $\mathrm{GL}(3,q)$.

Now we handle $p = 2$ (case b) and $p = 3$ (case c).

4.5 Case b ($p = 2$): For $p = 2$, we know that q is odd lest we are in case 0. Hence in all case b, we have a Sylow p-subgroup of $\mathrm{GL}(1,q) \times \mathrm{GL}(2,q)$ is a Sylow $2p$-subgroup of $\mathrm{GL}(3,q)$ since the index $[\mathrm{GL}(3,q) : \mathrm{GL}(1,q) \times \mathrm{GL}(2,q)] = q^2(q^2+q+1)$ is not divisible by p. Note that there is no case 3b, since $q^2+q+1 \equiv 1 \mod 2$ is never divisible by 2. Hence we just have a direct product of Sylow 2-subgroups of $\mathrm{GL}(1,q)$ and $\mathrm{GL}(2,q)$ which follow from the previous section.

4.6 Case c ($p = 3$): For $p = 3$, we are either in case 1c or case 2c. Case 3c is handled by case 1c. Actually case 2c is handled by case 2a; only $p = 2$ was an exception. In case 1c, we consider the subgroup $\mathrm{Sym}(3) \ltimes \mathrm{GL}(1,q)^3$ of monomial matrices which has index $q^3(q+1)(q^2+q+1)/6$ which is not divisible by p, since q^2+q+1 is not divisible by 9. Again we get monomial matrices whose permutation pattern comes from a Sylow 3-subgroup of $\mathrm{Sym}(3)$, and whose entries come from a Sylow 3-subgroup of $\mathbb{F}_q^\times$.

5 GL(4,q)

For $n = 4$, we have that

$$|\mathrm{GL}(4,q)| = (q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3) = q^6(q-1)^4(q+1)^2(q^2+q+1)(q^2+1)$$

and the latter factorization is the driving force in our case-by-case analysis.

Case 0, 1a, 2a, 3a are suspiciously similar. Even case 4a should be no surprise.

Case 0: upper unitriangular matrices

Case 1a: Look inside $\mathrm{GL}(1,q)^4$.

Case 2a: Look inside $\mathbb{F}_{q^2}^\times \times \mathbb{F}_{q^2}^\times$

Case 3a: Look inside $\mathbb{F}_{q^3}^\times$

Case 4a: Look inside $\mathbb{F}_{q^4}^\times$

For $p = 2$ we have case b. For $p = 3$ we have case c.

For $p = 2$, we always live inside $\mathrm{Sym}(2) \ltimes \mathrm{GL}(2,q)^2$, and so case 1b and 2b from $n = 2$ suffice. There is no need for 3b or 4b.

For $p = 3$, in 1c, we look inside $\mathrm{GL}(1,q) \times \mathrm{GL}(3,q)$ and use case 1c from $n = 3$, and in 2c we look inside $\mathbb{F}_{q^2}^\times \times \mathbb{F}_{q^2}^\times$ and use 2c from $n = 2$. There is no need for 3c or 4c.

Higher dimensions n behave fairly similarly. Case 0 is always p divides q and is always the upper unitriangular matrices. Case a is always $p > n$, and always results in an abelian Sylow p-subgroup contained within some maximal torus (not necessarily split, that is what the case 1a, 2a, 3a, 4a, etc. check. Case b,c,d handle the primes $p \leq n$ and involve a mix of Galois groups and permutation matrices (with the Galois groups popping up in case 2,3,4, etc. when the torus is not split, and the permutation matrices popping up whenever one deals with a subgroup of the form $\mathrm{GL}(1,q)^p \leq \mathrm{Sym}(p) \ltimes \mathrm{GL}(1,q)^p$).

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