Test Review 1 **MA 433** **S08**

The test will be over chapers 1 and 2, except for sections 26 and 27. From chapter 1 you should know how to work with and manipulate complex numbers using absolute values, complex conjugates, and the polar form of a complex number. Know how to find powers and roots of a complex number. Be able to identify the boundary and interior of a set as well as tell when a set is bounded, connected, open, and closed. From chapter 2, study mapping properties of $w = z^n, n = 2, 3, \ldots,$ and $w = e^z$. Know what is meant by limit, continuity, differentiability, and analyticity of a function of a complex variable. Be able to use the Cauchy Riemann equations (in real, polar, or complex forms) to determine when a function is not differentiable at a point and also when a function has continuous first partials, to determine whether it is differentiable at a point. Finally in chapter 2 study harmonic functions and their various properties, including how to find a conjugate of a given harmonic function in a disk or **C**. Here are two tests complete with answers and quite similar to the actual test for you to practice on.

Practice Test 1

10 pts 1. Show that if $P(z) = a_0 + a_1 z + \ldots + a_n z^n,$ and the a_i's are real numbers, $0 \le i \le n$, then
(a) $\overline{P(\bar{z})} = P(z)$.
(b) For P as in (a) show that if z_0 is a root or zero of P then so also is $\bar{z}_0$.

40 pts 2. Find the following:
(a) All solutions to $(z^2+1)^2 = -1$. You can leave your answer in polar form.
(b) $\lim_{z \to i} \dfrac{f(z) - f(i)}{z - i}$ when $f(z) = z^4 + z^2$.
(c) The interior and boundary of $\{z = x + iy : x \le y \le 1\} \cap \{z : |z| < 1\}$. Sketch your answers.
(d) Give real and imaginary parts of $\frac{(1+i)^3}{(1-i)^2}$.
(e) $\lim_{z \to \infty} \dfrac{(3z^2+1)^4}{z^8 + 11z^5 + 19}$.
(f) The image of $\{z = x + iy : x > 0, y > 0\}$ under $w = z^2$.

25 pts 3. Given $f(z) = e^{-2x+3}[\cos(2y) - i\sin(2y)]$ for $z = x + iy$.
(a) Does f satisfy the Cauchy Riemann Equations for each z in **C**.
(b) Show that f is entire (i.e analytic in **C**).
(c) Find $f'(z)$ and express this function in terms of f.
(d) Let $g(z) = f(z^2)$. Determine all z ' s for which g is analytic and find $g'(z)$.

25 pts 4. If $z = r^{i\theta}$ let $f(z) = r^{1/2} e^{i(\theta/2+\pi/4)}$ when $r > 0$, $|\theta| < \pi/2$. Find $u =$ Re $f, v =$ Im f, as functions of r, θ. (a) Show that u, v satisfy the polar CR equations when $r > 0, |\theta| < \pi/2$.
(b) Given that the Laplacian of a function h in polar coordinates is, $\Delta h = r^{-1}(r h_r)_r + r^{-2} h_{\theta\theta}$. Show that v is harmonic in $\{z = x + iy : x > 0\}$.

10 pts **Extra Credit:** Given that if g is analytic in a domain D and $g' \equiv 0$ in D, then $g \equiv$ constant in D. Show that if $f = u + iv$ is analytic in a domain D and $2u + 3v \equiv 0$ in D then $f \equiv$ constant in D.

Answers Practice Test 1

1. (a) Follows from properties of conjugates, since the conjugate of a sum or product is the sum or product of the conjugates and the fact that P has real coefficients.
(b) Use (a), in fact $0 = \overline{P(z_0)} = P(\bar{z}_0)$.

2. (a) z in $\{\pm 2^{1/4}\, e^{5\pi i/8},\ \pm 2^{1/4}\, e^{3\pi i/8}\}$.
(b) - 2 i.
(c) Interior is equal to $\{z = x + iy : x < y\} \cap \{z : |z| < 1\}$.
Boundary is equal to $\{z = x+iy : x = y, |z| < 1\} \cup \{z : |z| = 1, y > x\}$. Can sketch in class if asked.
(d) $-1 - i$.
(e) 81.
(f) Image is $\{w = u + iv : v > 0\}$.

3. (a) Yes. $f_x = -2f = -if_y$ as is easily shown.
(b) Since f has continuous first partials it follows from a theorem in the book that f is analytic for all z so by definition, entire.
(c) $f'(z) = f_x = -2f(z)$.
(d) z^2 and f are analytic everywhere, so by classwork g is analytic everywhere (entire). Moreover by the chain rule, $g'(z) = 2zf'(z^2) = -4zf(z^2)$.

4. If $u(z) + iv(z) = f(z) = r^{1/2}e^{i(\theta/2+\pi/4)}$, then $u(r,\theta) = r^{1/2}\cos(\theta/2 + \pi/4)$ and $v(r,\theta) = r^{1/2}\sin(\theta/2+\pi/4)$. (a) Polar Cauchy Riemann equations are $u_r = (1/2)r^{-1/2}\cos(\theta/2+\pi/4) = r^{-1}v_\theta$ and $r^{-1}u_\theta = -(1/2)r^{-1/2}\sin(\theta/2+\pi/4) = -v_r$.
(b) From (a), $r^{-1}(rv_r)_r = (1/4)r^{-3/2}\sin(\theta/2+\pi/4)$ and $r^{-2}\, v_{\theta\theta} = -(1/4)r^{-3/2}\sin(\theta/2+\pi/4)$. Thus $\Delta v = 0$ in $\{(r,\theta) : r > 0, |\theta| < \pi/2\} = \{z = x + iy : x > 0\}$.

EC. From the CR equations $u_x = v_y, u_y = -v_x$. Also by assumption, $2u + 3v = 0$, so $u_x = -3v_x/2$, $u_y = -3v_y/2$. Use in the CR equations to get $v_y = -3v_x/2$, $v_x = 3v_y/2$ Solve these equations for v_x, v_y to get $v_x = v_y \equiv 0$. Similarly $u_x \equiv u_y \equiv 0$ so $g'(z) = \frac{\partial g}{\partial x}(z) \equiv 0$ in D. By the given theorem, $g \equiv$ complex constant.

Practice Test 2

35 pts 1. (a) Find the real and imaginary part of $(1 + i\sqrt{3})^{-10}$.
(b) Write the circle, K with center (2,3) and radius 4 in complex notation.
(c) If K is as in (b), write the tangent line to this circle and through (2,7) in complex notation.
(d) Determine where $f(z) = 3(\bar{z} - 4)^3 + 1$ has a z derivative.
(e) Use $e^{3i\theta} = (e^{i\theta})^3$ to find $\cos 3\theta$ in terms of $\cos\theta, \sin\theta$.

20 pts 2. Given $w = u + iv = e^z = e^x(\cos y + i\sin y)$ when $z = x + iy$ and the sets,
$S_1 = \{z = x+iy : 0 < y < \pi\}, S_2 = \{z = x+iy : x > 0, 0 < y < \pi\}$. You may assume that f is one to one on both sets. Find the image of S_1, S_2 under e^z.

20 pts 3. True or False and justify your reasoning !
(a) If $f(z) = z^2/\bar{z}^2, z \neq 0$, then $\lim_{z\to 0} f(z) = 1$.
(b) If $z = x+iy$ and $f(z) = \dfrac{x^2y^2}{(x^2+y^2)^{3/2}}, (x,y) \neq (0,0)$, while $f(0) = 0$, then f satisfies the Cauchy

Riemann equations but $f'(0)$ does not exist.

25 pts 4. Given the function $\text{Log } z = \ln r + i\theta$, when $z = re^{i\theta}$, $-\pi < \theta < \pi$, $r > 0$.
(a) Sketch the domain D of definition of $\text{Log } z$.
(b) Use the polar form of the Cauchy Riemann equations to show that $\text{Log } z$ is analytic in D.
(c) Find $\frac{d}{dz}$ $\text{Log } z$ for $z \in D$.
(d) Given that a composition of analytic functions is analytic, (provided the composition makes 'sense'). Show that $\text{Log }(1-z)$ is analytic in $B(0,1) = \{z : |z| < 1\}$.

10 pts **Extra Credit:** Let $u(x, y) = 2x(1 - y)$.
(a) Show that u is harmonic in the complex plane.
(b) Find all harmonic conjugates v of u.
(c) Find an entire function h such that $u = \text{Re } h$.

Answers Practice Test 2

1. (a) $2^{-11}(-1 + i\sqrt{3})$.
(b) $K = \{z : |z - 2 - 3i| = 4\}$.
(c) $\{z : \text{ Im } z = 7\}$.
(d) z derivative only at $z = 4$.
(e) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

2. If $f(z) = e^z$, then $f(S_1) = \{w = u + iv : v > 0\}$ and $f(S_2) = \{w = u + iv : |w| > 1 \text{ and } v > 0\}$. Can sketch $f(S_1), f(S_2)$ in class if asked.

3. (a) False since if $z = re^{i\theta}$, then $f(re^{i\theta}) = e^{4i\theta}$, so that $f \to -1$ as $z \to 0$ along the ray $\{z = re^{i\pi/4} : 0 < r < \infty\}$ while $f(z) \to 1$ as $z \to 0$ along the real and/or imaginary axes.
(b) True since $f(x, 0) = f(0, y) \equiv 0$ implies $f_x(0,0) = 0 = -if_y(0,0)$ and if $z = re^{i\theta}$ then $f(z)/z = \cos^2\theta\sin^2\theta e^{-i\theta}$ which does not have a limit as $z \to 0$.

4. (a) Domain is $D = \mathbf{C} \setminus (-\infty, \mathbf{0}]$.
(b) Let $u(r,\theta) = \ln r$, $v(r,\theta) = \theta$. Then $u_r = r^{-1} = r^{-1}v_\theta$ and $r^{-1}u_\theta = -v_r = 0$. Thus $f(z) = \text{ Log } z$ satisfies the polar Cauchy Riemann equations in D. Since u, v have continuous first partials in D it follows that f is analytic in D.
(c) $f'(z) = e^{-i\theta}f_r = z^{-1}$ in D.
(d) Note for z in $B(0,1)$ that $\text{Re }(1 - z) > 0$ and so $1 - z \in D$ whenever $z \in B(0,1)$. Since a composition of analytic functions is analytic it follows that $\text{Log }(1 - z)$ is analytic in D.

EC. (a) Show $u_{xx} + u_{yy} = 0$.
(b) $v = 2y + x^2 - y^2 + c$.
(c) $h(z) = 2z + iz^2$.