Regularity of Lipschitz Free Boundaries in Two-phase Problems for the p-Laplace Operator

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Abstract

In this paper we study the regularity of the free boundary in a general two-phase free boundary problem for the p-Laplace operator and we prove, in particular, that Lipschitz free boundaries are $C^{1,\gamma}$-smooth for some $\gamma \in (0,1)$. As part of our argument, and which is of independent interest, we establish a Hopf boundary type principle for non-negative p-harmonic functions vanishing on a portion of the boundary of a Lipschitz domain.

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1 Introduction

In [C1,C2,C3] a theory for general two-phase free boundary problems for the Laplace operator was developed. In [C1] Lipschitz free boundaries were shown to be $C^{1,\gamma}$-smooth for some $\gamma \in (0,1)$ and in [C2] it was shown that free boundaries which are well approximated by Lipschitz graphs are in fact Lipschitz. Finally, in [C3] the existence part of the theory was developed.

In this paper, which is the first in a sequel, we begin our study of the corresponding problems for the p-Laplace operator by generalizing the results in [C1] to the p-Laplace operator when $p \neq 2$, $1 < p < \infty$. This generalization is highly nontrivial due to the non-linear and degenerate character of the p-Laplace operator for $p \neq 2$. Indeed, what enables us to proceed on these problems in the case $p \neq 2$, $1 < p < \infty$, are the recent results in [LN,LN1,LN2] (see also [LN3]). To briefly outline these results we recall that in [LN] we established the boundary Harnack inequality for positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$ and we carried out an in depth analysis of p-capacitary functions in starlike Lipschitz ring domains. The study in [LN] was continued in [LN1] where we established, as one of our results, the Hölder continuity for ratios of positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$. Finally, in [LN2] several results concerning the boundary behaviour of the gradient of a p-harmonic function, vanishing on a portion of the boundary of a Lipschitz or C^1-domain, were proved. The analysis in this paper is a 'tour de force' of the techniques developed in [LN,LN1,LN2].

To properly state our results we need to introduce some notation. Points in Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and we let $\mathbf{S}^{n-1}$ denote the $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$. Let $\bar{E}, \partial E$, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$ and let $d(y, E)$ be equal to the distance from $y \in \mathbf{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and $|x| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of x. Let $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ whenever $x \in \mathbf{R}^n$, $r > 0$, and let dx be Lebesgue n-measure on $\mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O. Let

$$\|f\|_{W^{1,q}(O)} = \|f\|_{L^q(O)} + \| \, |\nabla f| \, \|_{L^q(O)}$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_{L^q(O)}$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Finally let $C(E)$ be the set of continuous functions on E.

Given $D \subset \mathbf{R}^n$ a bounded domain (i.e, a connected open set) and $1 < p < \infty$, we say that u is p-harmonic in D provided $u \in W^{1,p}(D)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(D)$. Observe that if u is smooth enough and $\nabla u \neq 0$ in D, then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \text{ in } D \tag{1.2}$$

so u is a classical solution in D to the p-Laplace partial differential equation. Here, as in the sequel, $\nabla \cdot$ is the divergence operator. u is said to be a p-subsolution (p-supersolution) in D

provided $u \in W^{1,p}(D)$ and provided (1.1) holds with $=$ replaced by $\leq$ ($\geq$) whenever $\theta \geq 0$ a.e. in D. Let $u \in C(\bar{D})$ and suppose that u changes sign in D. Put $D^+(u) = \{x \in D : u(x) > 0\}$, $F(u) = \partial D^+(u) \cap D$, and let $D^-(u)$ be the interior of the set $\{x \in D : u(x) \leq 0\}$. Set $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$. Assuming that $w \in F(u)$ and that $F(u)$ is smooth in a neighborhood of w we let $\nu = \nu(w)$ denote the unit normal, to $F(u)$ at w, pointing into $D^+(u)$. Moreover, we let $u_\nu^+(w)$ and $u_\nu^-(w)$ denote the normal derivatives of u^+ and u^- at w in the direction of ν. Note that $u_\nu^+, -u_\nu^- \geq 0$. In this paper we consider weak solutions, defined and continuous in $\bar{D}$, to the following general two-phase free boundary problem,

$$
\begin{aligned}
(i) &\quad \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \text{ in } D^+(u) \cup D^-(u),\\
(ii) &\quad u_\nu^+(w) = G(-u_\nu^-(w)) \text{ whenever } w \in F(u),\\
(iii) &\quad u = f \in C(\partial D) \text{ on } \partial D.
\end{aligned} \tag{1.3}
$$

In (1.3) (ii) the function $G : [0, \infty) \to [0, \infty)$ defines the free boundary condition and the interface $F(u)$ is referred to as the free boundary. If we make no a priori classical regularity assumptions on the interface $F(u)$ then the free boundary condition in (1.3) (ii) must be interpreted in a weak sense and in particular a notion of weak solutions to the problem in (1.3) has to be introduced. Let $\langle \cdot, \cdot \rangle^+ = \max\{\langle \cdot, \cdot \rangle, 0\}$, $\langle \cdot, \cdot \rangle^- = -\min\{\langle \cdot, \cdot \rangle, 0\}$. We will work with the following notion of weak solutions to the problem in (1.3).

Definition 1.4. *Let $D \subset \mathbf{R}^n$ be a bounded domain, $u \in C(\bar{D})$ and $1 < p < \infty$, be given. u is a (weak) solution to the problem in (1.3) if u is p-harmonic in $D^+(u) \cup D^-(u)$, $u = f$ on ∂D and if the free boundary condition in (1.3) (ii) is satisfied in the following sense. Assume that $w \in F(u)$ and there exists a ball $B(\hat{w}, \hat{\rho})$, $\hat{w} \in D^+(u) \cup D^-(u)$ with $w \in \partial B(\hat{w}, \hat{\rho})$. If $\nu = (\hat{w} - w)/|\hat{w} - w|$, then the following holds, as $x \to w$ non-tangentially, for some α, $\beta \in [0, \infty]$ with $\alpha = G(\beta)$,*

$$
\begin{aligned}
(i) &\quad \text{if } B(\hat{w}, \hat{\rho}) \subset D^+(u), \text{ then } u(x) = \alpha \langle x - w, \nu \rangle^+ - \beta \langle x - w, \nu \rangle^- + o(|x - w|),\\
(ii) &\quad \text{if } B(\hat{w}, \hat{\rho}) \subset D^-(u), \text{ then } u(x) = \alpha \langle w - x, \nu \rangle^+ - \beta \langle w - x, \nu \rangle^- + o(|x - w|).
\end{aligned}
$$

Recall that $\phi : E \to \mathbf{R}$ is said to be Lipschitz on E provided there exists $b, 0 < b < \infty$, such that $|\phi(z) - \phi(w)| \leq b\,|z - w|$ whenever $z, w \in E$. The infimum of all b such that this holds is called the Lipschitz norm of ϕ on E, denoted $\|\phi\hat{\|}_E$. It is well known that if $E = \mathbf{R}^{n-1}$, then ϕ is differentiable almost everywhere on $\mathbf{R}^{n-1}$ and $\|\phi\hat{\|}_{\mathbf{R}^{n-1}} = \|\,|\nabla\phi|\,\|_{L^\infty(\mathbf{R}^{n-1})}$.

We can now state the first main result proved in this paper.

Theorem 1. *Let $D \subset \mathbf{R}^n$ be a bounded domain, assume that $u \in C(\bar{D})$ and that u is a solution in D, for some $1 < p < \infty$, to the problem in (1.3) in the sense of Definition 1.4. Moreover, suppose that $G > 0$ is strictly increasing on $[0, \infty)$ and, for some $N > 0$, that $s^{-N} G(s)$ is decreasing on $(0, \infty)$. Assume that $0 \in F(u)$, $\bar{B}(0, 2) \subset D$, $\max_{B(0,2)} |u| = 1$ and that*

$$
\begin{aligned}
D^+(u) \cap B(0,2) &= \Omega \cap B(0,2),\ F(u) \cap B(0,2) = \partial\Omega \cap B(0,2),\\
\Omega &= \{y = (y', y_n) \in \mathbf{R}^n : y_n > \psi(y')\},
\end{aligned}
$$

in an appropriate coordinate system where ψ is Lipschitz on $\mathbf{R}^{n-1}$ with $M = \|\psi\hat{\|}_{\mathbf{R}^{n-1}}$. Then there exists $\sigma = \sigma(p, n, M, N) \in (0, 1)$ such that $\nabla\psi$ is Hölder continuous of order σ on

$\{x' : (x', \psi(x'))\} \in B(0, 1/8)$. *The* C^σ *Hölder norm of* $\nabla\psi$ *depends only on* p, n, M, N.

We say that $F(u) \cap B(0, 1/8)$ is $C^{1,\sigma}$ whenever the conclusion of Theorem 1 holds. Theorem 1 is completely new if $1 < p < \infty, p \neq 2$, while, as previously stated, Theorem 1 was proved in [C1] for $p = 2$ and the Laplace operator. We also note that the work in [C1] was generalized in [W] to solutions of fully nonlinear PDEs of the form $F(\nabla^2 u) = 0$, where F is homogeneous. Further analogues of the work in [C1] were obtained for a class of nonisotropic operators in [F] and for fully nonlinear PDEs of the form $F(\nabla^2 u, \nabla u) = 0$, where F is homogeneous in both arguments, in [F1]. Finally generalizations of the results in [C1] were made to non-divergence form linear PDE with variable coeficients in [CFS] and generalized in [Fe] to fully nonlinear PDEs of the form $F(\nabla^2 u, x) = 0$. Each of the above generalizations is concerned with non-divergence form PDE. Generalizations of the work in [C1] to linear divergence form PDEs with variable coefficients were obtained in [FS], [FS1].

To briefly outline the proof of Theorem 1 we note that our argument combines the geometric approach developed in [C1] with the analytic techniques for p-harmonic functions in Lipschitz domains developed in [LN,LN1,LN2]. In particular, let $\nu \in \mathbf{R}^n$, $|\nu| = 1$, and consider $\tilde\nu \in \mathbf{R}^n$. Let $\theta(\nu, \tilde\nu)$ denote the angle between ν and $\tilde\nu$ and for $\theta_0 \in [0, \pi]$, $\epsilon_0 \in \mathbf{R}_+ = (0, \infty)$, we introduce

$$\Gamma(\nu, \theta_0, \epsilon_0) := \{\tilde\nu \in \mathbf{R}^n : \ \theta(\nu, \tilde\nu) < \theta_0, \ |\tilde\nu| \leq \epsilon_0\}. \tag{1.5}$$

Then $\Gamma(\nu, \theta_0, \epsilon_0)$ is a cone of directions and if $\epsilon_0 = 1$ we write $\Gamma(\nu, \theta_0) = \Gamma(\nu, \theta_0, \epsilon_0)$. Let $O \subset \mathbf{R}^n$ be an open set and let $u \in C(\bar O)$. Let $\nu \in \mathbf{R}^n$, $|\nu| = 1$, $\theta_0 \in (0, \pi/2]$ and $\epsilon_0 \in \mathbf{R}_+$ be given. Put

$$O(\nu, \theta_0, \epsilon_0) = \{x \in O : B(x - \tau, |\tau| \sin(\theta_0/2)) \subset O \text{ for every } \tau \in \Gamma(\nu, \theta_0/2, \epsilon_0)\}.$$

Then u is said to be monotone in O with respect to the directions in the cone $\Gamma(\nu, \theta_0, \epsilon_0)$ if

$$\sup_{B(x, |\tau| \sin(\theta_0/2))} u(y - \tau) \leq u(x) \text{ whenever } \tau \in \Gamma(\nu, \theta_0/2, \epsilon_0) \text{ and } x \in O(\nu, \theta_0, \epsilon_0). \tag{1.6}$$

If (1.6) is true then $\Gamma(\nu, \theta_0, \epsilon_0)$ is referred to as a cone of monotonicity for u in O.

Let u, Ω, $0 \in F(u)$ be as in the statement of Theorem 1 and let M denote the Lipschitz constant of Ω. To prove Theorem 1 we establish the following steps.

Step 0. (Existence of a cone of monotonicity) Using the Lipschitz character of Ω it follows that u is monotone in $B(0, r_1)$, $r_1 = 1/(4c_2)$ where c_2 is the constant defined in Theorem 2.4 of section 2, with respect to the directions in the cone $\Gamma(e_n, \theta_0, \epsilon_0)$ for some $\theta_0 = \theta_0(p, n, M) \in (0, \pi/2]$ and for some small $\epsilon_0 = \epsilon_0(p, n, M) > 0$.

Step 1. (Enlargement of the cone of monotonicity in the interior) If $\tau \in \Gamma(e_n, \theta_0/2, \epsilon_0)$ for some (θ_0, ϵ_0), put $\epsilon = |\tau| \sin(\theta_0/2)$ and set

$$v_\epsilon(x) = v_{\epsilon,\tau}(x) = \sup_{B(x,\epsilon)} u(y - \tau)$$

whenever $u(y - \tau)$ is defined in $B(x, \epsilon)$. Let $\nu = \nabla u(\frac{r_1 e_n}{32})/|\nabla u(\frac{r_1 e_n}{32})|$. In Lemma 4.2 we prove the existence of positive $\mu = \mu(p, n, M)$ and $\rho = \rho(M)$ such that if $\epsilon = |\tau| \sin(\theta_0/2), \lambda = \cos(\theta_0/2 + \theta(\nu, \tau))$ and $0 < \epsilon \leq \epsilon_0 \rho$, then

$$v_{(1+\mu\lambda)\epsilon}(x) \leq (1 - \mu\lambda) u(x) \text{ whenever } x \in B(\tfrac{e_n}{32}, \rho r_1).$$

Step 2. (Enlargement of the cone of monotonicity at the free boundary) Using the notation stated in Steps 0,1, we in Lemma 4.3 prove that there exists $\bar{\mu} > 0$, depending only on p, n, M, and N, such that

$$v_{(1+\bar{\mu}\lambda)\epsilon}(x) \leq u(x) \text{ whenever } x \in B(0, r_1/100).$$

It is shown in [C1, Lemma 17] that Step 2 implies the existence of $\omega \in \mathbf{S}^{n-1}, \bar{\theta} \in (0, \pi/2]$, $c_-, c_+ > 1$, depending only on p, n, M and N such that

$$\pi/2 - \bar{\theta} = c_-^{-1}(\pi/2 - \theta_0),\ \Gamma(e_n, \theta_0, \epsilon_0) \subset \Gamma(\omega, \bar{\theta}, \epsilon_0/c_+),\ \text{ and } u \text{ monotone in } \Gamma(\omega, \bar{\theta}, \epsilon_0/c_+).$$

Using this fact, as well as invariance of the p-Laplace equation under scalings and translations, we can replace $u(x)$ by $u(x_0 + \eta x)/\eta$ and given Step 0, repeat Steps 1, 2, in order to conclude, as in [C1, p.157], the $C^{1,\sigma}$-smoothness of $F(u) \cap B(0, 1/8)$ for some $\sigma = \sigma(p, n, M, N) \in (0, 1)$. Hence to prove Theorem 1 we only have to prove the statements in Step 0-2. The proof will use the full strength of the toolbox developed in [LN,LN1,LN2]. In particular, we establish, a Hopf boundary type principle for p-harmonic functions vanishing on a portion of a Lipschitz domain.

In order to describe some crucial ideas and an operator, $\hat{L}$, considered throughout this paper, we assume that Ω' is Lipschitz with constant M', $w' \in \partial\Omega', r' > 0$, and that u', v', are non-negative p-harmonic functions in $\Omega' \cap B(w', 2r')$. Also assume that u', v', are continuous on the closure of $\Omega' \cap B(w', 2r')$ and u', v' vanish continuously on $\partial\Omega' \cap B(w', 2r')$. We say that $|\nabla u'|$ and $|\nabla v'|$ satisfy a uniform non-degeneracy condition in $\Omega' \cap B(w', 2r')$ if there exists a constant $b > 1$ such that, for all $y \in \Omega' \cap B(w', 2r')$,

$$b^{-1}\frac{\tilde{u}(y)}{d(y, \partial\Omega')} \leq |\nabla \tilde{u}(y)| \leq b\frac{\tilde{u}(y)}{d(y, \partial\Omega')},\ \ \tilde{u} \in \{u', v'\}. \tag{1.7}$$

In general, in our applications, $b = b(p, n, M')$. We note that (1.7), (1.8) (see Lemma 2.2) imply u', v' are infinitely differentiable in $\Omega' \cap B(w, 2r')$. Let

$$e(y) = u'(y) - v'(y) \text{ whenever } y \in \bar{\Omega}' \cap \bar{B}(w', 2r') \tag{1.8}$$

and introduce

$$u(y, \tau) = \tau u'(y) + (1 - \tau)v'(y) \text{ whenever } y \in \bar{\Omega}' \cap \bar{B}(w', 2r') \text{ and } \tau \in [0, 1]. \tag{1.9}$$

Using the fact that u', v' are classical solutions to the p-Laplace operator in (1.2) and the fact that

$$|\xi|^{p-2}\xi - |\eta|^{p-2}\eta = \int_0^1 \frac{d\{|t\xi + (1-t)\eta|^{p-2}[t\xi + (1-t)\eta]\}}{dt} dt$$

whenever $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$, it follows that

$$\hat{L}e := \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(\, \hat{b}_{ij}(y) e_{y_j}(y)\,\right) = 0 \text{ whenever } y \in \Omega' \cap B(w', 2r') \tag{1.10}$$

where

$$
\begin{aligned}
\hat{b}_{ij}(y) &= \int_0^1 b_{ij}(y,\tau)d\tau, \\
b_{ij}(y,\tau) &= |\nabla u(y,\tau)|^{p-4}((p-2)u_{y_i}(y,\tau)u_{y_j}(y,\tau)+\delta_{ij}|\nabla u(y,\tau)|^2),
\end{aligned} \tag{1.11}
$$

for $i,j\in\{1,...,n\}$ and δ_{ij} is the Kronecker δ. We observe that the operator $\hat{L}$ in (1.10)-(1.11) is a symmetric linear operator in divergence form in $\Omega'\cap B(w',2r')$ and that, in particular, the function e, representing the difference between the p-harmonic functions u' and v', satisfies the linear pde in (1.10). To estimate the ellipticity of $\hat{L}$ at $y\in\Omega'\cap B(w',2r')$ we note that

$$
\min\{p-1,1\}|\xi|^2\hat{\lambda}(y)\le\sum_{i,j=1}^n\hat{b}_{ij}(y)\xi_i\xi_j\le\max\{p-1,1\}|\xi|^2\hat{\lambda}(y) \tag{1.12}
$$

whenever $\xi\in\mathbf{R}^n$ and

$$
\hat{\lambda}(y)=\int_0^1|\nabla u(y,\tau)|^{p-2}d\tau\approx\Big(|\nabla u'(y)|+|\nabla v'(y)|\Big)^{p-2}. \tag{1.13}
$$

Here, as in the sequel, $A\approx B$ means that A/B is bounded above and below by constants which, unless otherwise stated, may only depend on p,n and M. In Lemmas 2.12 and 2.14, we establish interior and boundary Harnack inequalities for non-negative solutions to $\hat{L}$ (assuming u',v' satisfy (1.7)). Similar arguments are used to establish Theorem 2.22 which plays a fundamental role in the proof of Theorem 1. As mentioned earlier, the proofs in section 2 use the toolbox developed in [LN, LN1, LN2].

As an application of Theorem 1 to free boundary-inverse type problems below the continuous threshold, we show, see Theorem 5.1 in section 5, that Theorem 3 in [LN2] remains true without any smallness assumption on the Lipschitz constant of the domain. A full statement of this theorem together with an outline of its proof is given in section 5. Given Theorem 1 in this paper and the results in [LN2], our task is to show that a certain blow-up limit p-harmonic function, u_∞, is a weak solution to a one phase free boundary problem in the sense described in Definition 1.4.

This paper is organized in the following way. In section 2 we state a number of results from [LN,LN1,LN2] concerning p-harmonic functions in Lipschitz domains. Moreover, in this section we focus on the operator $\hat{L}$ constructed as in (1.7)-(1.13) and we develop, as described above, a number of new results using the toolbox developed in [LN,LN1,LN2]. In section 3 we then construct, in analogy with [C1], appropriate continuous p-subsolutions to be used for comparison. In section 4 we establish Step 0-2 using the results stated and established in sections 2 and 3. In section 5 we prove Theorem 5.1 mentioned above. Finally in section 6, we briefly discuss a generalization of Theorem 1.

2 p-Harmonic functions in Lipschitz domains

This section is devoted to the boundary behaviour of p-harmonic functions vanishing on a portion of a Lipschitz domain. In particular, in the following we let $\Omega\subset\mathbf{R}^n$ be a bounded Lipschitz domain, i.e., we assume that there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned} \Omega \cap B(x_i, 4r_i) &= \{y = (y', y_n) \in \mathbf{R}^n : y_n > \phi_i(y')\} \cap B(x_i, 4r_i), \\ \partial\Omega \cap B(x_i, 4r_i) &= \{y = (y', y_n) \in \mathbf{R}^n : y_n = \phi_i(y')\} \cap B(x_i, 4r_i), \end{aligned}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constant of Ω is defined to be $M = \max_i \||\nabla\phi_i|\|_\infty$. If Ω is Lipschitz and $r_0 = \min r_i$, then for each $w \in \partial\Omega$, $0 < r < r_0$, we can find points $a_r(w) \in \Omega \cap \partial B(w, r)$ with $d(a_r(w), \partial\Omega) \geq c^{-1}r$ for a constant $c = c(M)$. In the following we let $a_r(w)$ denote one such point. Furthermore, if $w \in \partial\Omega$, $0 < r < r_0$, then we let $\Delta(w, r) = \partial\Omega \cap B(w, r)$ be the naturally defined surface ball. We let $e_i, 1 \leq i \leq n$, denote the point in $\mathbf{R}^n$ with one in the i th coordinate position and zeroes elsewhere. Moreover, throughout the paper c will denote, unless otherwise stated, a positive constant ≥ 1, not necessarily the same at each occurrence, which only depends on p, n and M. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1, not necessarily the same at each occurrence, which depends on p, n, M and $a_1, \dots, a_n$. With this notation we state,

Lemma 2.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Given $p, 1 < p < \infty$, $w \in \partial\Omega, 0 < r < r_0$, suppose $\hat{u} > 0$ is p-harmonic in $\Omega \cap B(w, 4r)$, and continuous in $B(w, 4r)$ with $\hat{u} \equiv 0$ on $B(w, 4r) \setminus \Omega$. There exists, $c, c' \geq 1, \alpha \in (0, 1)$, depending only on p, n, M, such that if $x \in \Omega \cap B(w, r/c)$, then*

$$(a) \qquad \max_{B(x, \frac{1}{2} d(x, \partial\Omega))} \hat{u} \leq c' \hat{u}(x)\,.$$

$$(b) \quad |\hat{u}(z) - \hat{u}(y)| \leq c' \left(\tfrac{|z-y|}{d(x,\partial\Omega)} \right)^\alpha \hat{u}(x) \ \textit{whenever } z, y \in B(x, 2d(x, \partial\Omega)).$$

Lemma 2.2. *Let $\Omega, w, p, \hat{u}, r$, be as in Lemma 2.1. Then $\hat{u}$ has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular there exists $\sigma \in (0, 1], c \geq 1$, depending only on p, n, such that if $x, y \in B(\tilde{w}, \tilde{r}/2)$, $B(\tilde{w}, 4\tilde{r}) \subset \Omega \cap B(w, 4r)$, then*

$$c^{-1} |\nabla\hat{u}(x) - \nabla\hat{u}(y)| \leq (|x - y|/\tilde{r})^\sigma \max_{B(\tilde{w}, \tilde{r})} |\nabla\hat{u}| \leq c(|x - y|/\tilde{r})^\sigma u(\tilde{w})/d(\tilde{w}, \partial\Omega).$$

If $\nabla u(\tilde{w}) \neq 0$, then u is infinitely differentiable in a neighborhood of $\tilde{w}$. Moreover, if for some $\beta \in (1, \infty)$,

$$\frac{\hat{u}(y)}{d(y, \partial\Omega)} \leq \beta |\nabla\hat{u}(y)| \ \textit{whenever } y \in B(\tilde{w}, 2\tilde{r}),$$

then there exists $\bar{c} \geq 1$, depending only on p, n, β, such that

$$\max_{B(\tilde{w}, \frac{\tilde{r}}{2})} \sum_{i,j=1}^n |\hat{u}_{y_i y_j}| \leq \bar{c} \left(\tilde{r}^{-n} \int_{B(\tilde{w}, \tilde{r})} \sum_{i,j=1}^n |\hat{u}_{y_i y_j}|^2 \, dy \right)^{1/2} \leq \bar{c}^2 \hat{u}(\tilde{w})/d(\tilde{w}, \partial\Omega)^2.$$

For the proof of Lemma 2.1, see [LN, Lemmas 2.1, 2.2]. A proof of the first display in Lemma 2.2 can be found for example in [L]. The rest of the proof of Lemma 2.2 follows from the first display and Schauder type estimates. The following theorem is given in [LN1, Theorem 2].

Theorem 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that $\hat{u}, \hat{v} > 0$, are p-harmonic in $\Omega \cap B(w, 2r)$, continuous in $B(w, 2r)$ with $u \equiv v \equiv 0$ on $B(w, 2r) \setminus \Omega$. Under these assumptions there exist $c_1, 1 \leq c_1 < \infty$, and $\tilde{\sigma}$, $\tilde{\sigma} \in (0, 1)$, both depending only on p, n, and M, such that if $\tilde{r} = r/c_1$ and $y_1, y_2 \in \Omega \cap B(w, r/c_1)$, then*

$$\left| \frac{\hat{u}(y_1)}{\hat{v}(y_1)} - \frac{\hat{u}(y_2)}{\hat{v}(y_2)} \right| \leq c_1 \frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))} \left(\frac{|y_1 - y_2|}{r} \right)^{\tilde{\sigma}}.$$

We note that the proof in [LN1] of Theorem 2.3 uses an iteration-induction type argument which assumes boundedness in the above inequality, i.e.,

$$\left| \frac{\hat{u}(y_1)}{\hat{v}(y_1)} - \frac{\hat{u}(y_2)}{\hat{v}(y_2)} \right| \leq c \frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))}.$$

This inequality was proved in [LN]. Moreover, the proof of Theorem 2.3 in [LN1] also yields,

Theorem 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Let $w \in \partial\Omega, 0 < r < r_0$, and suppose that $\Omega \cap B(w, 2r) = \{x : x_n > \phi(x')\} \cap B(w, 2r)$, where ϕ is Lipschitz with norm $\leq M$. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that $\hat{u} > 0$ is p-harmonic in $\Omega \cap B(w, 2r)$ and continuous in $B(w, 2r)$ with $\hat{u} \equiv 0$ on $B(w, 2r) \setminus \Omega$. There exist θ_0, $\theta_0 \in (0, \pi/2]$, and $c_2 > 1$, which both depend only on p, n, M, such that if $\tilde{r} = r/c_2$, then*

$$c_2^{-1} \frac{\hat{u}(y)}{d(y, \partial\Omega)} \leq \langle \nabla \hat{u}(y), \xi \rangle \leq |\nabla \hat{u}(y)| \leq c_2 \frac{\hat{u}(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap B(w, \tilde{r})$ and $\xi \in \Gamma(e_n, \theta_0) \cap \mathbf{S}^{n-1}$.

We note that assuming Theorem 2.3, Theorem 2.4 can also be derived as in [C, Lemma 5]. The following lemma gives us a criteria for determining when the non degeneracy inequality in Theorem 2.4. holds at a point.

Lemma 2.5. *Let O be an open set, $w \in \partial O, r > 0$, and suppose that $\hat{u}, \hat{v}$ are positive p-harmonic functions in O. Let $a \geq 1$, $y \in O$, $\xi \in \mathbf{R}^n$, $|\xi| = 1$ and assume that*

$$\frac{1}{a} \frac{\hat{v}(y)}{d(y, \partial O)} \leq \langle \nabla \hat{v}(y), \xi \rangle \leq |\nabla \hat{v}(y)| \leq a \frac{\hat{v}(y)}{d(y, \partial O)}.$$

Let $\tilde{\epsilon}^{-1} = (ca)^{(1+\sigma)/\sigma}$, where σ is as in Lemma 2.2 and $c = c(p, n)$. Then the following statement is true for $c = c(p, n)$ suitably large. If

$$(1 - \tilde{\epsilon})\tilde{L} \leq \frac{\hat{v}}{\hat{u}} \leq (1 + \tilde{\epsilon})\tilde{L}$$

in $B(y, \frac{1}{4}d(y, \partial O))$ *for some* $\tilde{L}, 0 < \tilde{L} < \infty$, *then*

$$\frac{1}{ca} \frac{\hat{u}(y)}{d(y, \partial O)} \leq \langle \nabla \hat{u}(y), \xi \rangle \leq |\nabla \hat{u}(y)| \leq ca \frac{\hat{u}(y)}{d(y, \partial O)}.$$

Proof. The stated lemma is similar to Lemmas 4.3 and 5.4 in [LN1] and Lemma 3.1 in [LN4]. We omit the details. □

We will also need one of the main theorems proved in [LN2]. To state this theorem, we need some more notation. Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and $w \in \partial\Omega, 0 < r < r_0$. If $0 < b < 1$ and $x \in \Delta(w, 2r)$ then we let

$$\Gamma(x) = \Gamma_b(x) = \{y \in \Omega : \ d(y, \partial\Omega) > b|x - y|\} \cap B(w, 4r). \tag{2.6}$$

Let σ denote surface area or Hausdorff $(n-1)$-measure on $\partial\Omega$. Given a measurable function k on $\cup_{x \in \Delta(w,2r)} \Gamma(x)$ we define the non tangential maximal function $N(k) : \Delta(w, 2r) \to \mathbf{R}$ for k as

$$N(k)(x) = \sup_{y \in \Gamma(x)} |k|(y) \text{ whenever } x \in \Delta(w, 2r). \tag{2.7}$$

Next let $L^q(\Delta(w, 2r))$, $1 \leq q \leq \infty$, be the space of q th power integrable functions, with respect to σ, on $\Delta(w, 2r)$. Furthermore, given a measurable function f on $\Delta(w, 2r)$ we say that f is of bounded mean oscillation on $\Delta(w, r)$, $f \in BMO(\Delta(w, r))$, if there exists A, $0 < A < \infty$, such that

$$\int_{\Delta(x,s)} |f - f_\Delta|^2 d\sigma \leq A^2 \sigma(\Delta(x, s)) \tag{2.8}$$

whenever $x \in \Delta(w, r)$ and $0 < s \leq r$. Here f_Δ denotes the average of f on $\Delta = \Delta(x, s)$ with respect to σ. The least A for which (2.8) holds is denoted by $\|f\|_{BMO(\Delta(w,r))}$. The following theorem is Theorem 1 in [LN2].

Theorem 2.9. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded Lipschitz domain with constant* M. *Given* $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, *suppose that* $\hat{u}$ *is a positive* p*-harmonic function in* $\Omega \cap B(w, 4r)$, $\hat{u}$ *is continuous in* $\bar{\Omega} \cap B(w, 4r)$, *and* $\hat{u} = 0$ *on* $\Delta(w, 4r)$. *Then*

$$\lim_{y \in \Gamma(x), y \to x} \nabla \hat{u}(y) = \nabla \hat{u}(x)$$

for σ *almost every* $x \in \Delta(w, 4r)$. *Furthermore, there exist* $q > p$ *and a constant* c, $1 \leq c < \infty$, *which both only depend on* p, n *and* M, *such that*

$$(i) \qquad N(|\nabla \hat{u}|) \in L^q(\Delta(w, 2r)),$$

$$(ii) \qquad \int_{\Delta(w,2r)} |\nabla \hat{u}|^q d\sigma \leq c r^{(n-1)(\frac{p-1-q}{p-1})} \Bigg(\int_{\Delta(w,2r)} |\nabla \hat{u}|^{p-1} d\sigma \Bigg)^{q/(p-1)},$$

$$(iii) \qquad \log |\nabla \hat{u}| \in BMO(\Delta(w, r)), \quad \| \log |\nabla \hat{u}| \|_{BMO(\Delta(w,r))} \leq c.$$

2.1 Degenerate elliptic equations

Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain with constant M, $w \in \partial\Omega$, and $0 < r < r_0$. Suppose that $u', v' > 0$ are p-harmonic in $\Omega \cap B(w, 2r)$ and continuous on $B(w, 2r)$ with $u' \equiv v' \equiv 0$ on $B(w, 2r) \setminus \Omega$. Let $r' = r/(4\max\{c_1, c_2\})$ where c_1 is defined in Theorem 2.3 and c_2 is defined in Theorem 2.4. From these theorems we deduce, for some $c = c(p, n, M)$, $1 \leq c < \infty$, and all $y \in \Omega \cap B(w, 2r')$, that

$$\begin{aligned} &(i) \qquad c^{-1}\frac{\tilde{u}(y)}{d(y, \partial\Omega')} \leq |\nabla \tilde{u}(y)| \leq c\frac{\tilde{u}(y)}{d(y, \partial\Omega')} \text{ for } \tilde{u} \in \{u', v'\}, \\ &(ii) \qquad c^{-1}\frac{u'(a_{r'}(w))}{v'(a_{r'}(w))} \leq \frac{u'(y)}{v'(y)} \leq c\frac{u'(a_{r'}(w))}{v'(a_{r'}(w))}. \end{aligned} \tag{2.10}$$

We define

$$e(y) = u'(y) - v'(y) \text{ whenever } y \in \bar{\Omega} \cap \bar{B}(w, 2r') \tag{2.11}$$

and let $\hat{L}$ be the operator defined in (1.10)-(1.11). Recall that the ellipticity of $\hat{L}$ is estimated in (1.12)-(1.13). We first state an interior Harnack inequality for positive solutions to $\hat{L}$.

Lemma 2.12. *Let Ω, M, $w \in \partial\Omega$, r, u', v', r' and $\hat{L}$ be as above. Assume that $B(\tilde{w}, 2\tilde{r}) \subset \Omega \cap B(w, 2r')$ for some $\tilde{w} \in \Omega$ and for some $\tilde{r} > 0$. There exists a constant $c = c(p, n, M) > 1$ such that if h is non-negative and $\hat{L}h = 0$ in $B(\tilde{w}, 2\tilde{r})$, then*

$$\max_{B(\tilde{w},\tilde{r})} h \leq c \min_{B(\tilde{w},\tilde{r})} h.$$

Proof. Using (2.10) (i), the Harnack inequality for p-harmonic functions, and (1.12), (1.13), we see that $\hat{L}$ is uniformly elliptic in $B(\tilde{w}, 3\tilde{r}/2)$. The stated Harnack inequality then follows from classical arguments, see [LSW]. □

Lemma 2.13. *Let Ω, M, $w \in \partial\Omega$, r, u', v', r' and $\hat{L}$ be as above. Let $\zeta \in \Omega \cap \partial B(w, 3r'/2)$ with $d(\zeta, \partial\Omega) \geq \eta r$. Let h^* be a solution to $\hat{L}$ in $\Omega \cap B(w, 3r'/2)$ with continuous boundary values. Suppose that $h^* \geq 0$ on $\partial[\Omega \cap B(w, 3r'/2)]$ and $h^* \geq 1$ on $\Omega \cap \partial B(w, 3r'/2) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$. Then there exists $c = c(p, n, M, \eta)$ such that*

$$ch^*(a_{r'}(w)) \geq 1.$$

Proof. Lemma 2.13 follows from Lemma 2.12 and standard arguments as in ([CFMS]) for uniformly elliptic PDE. □

Next we prove the boundary Harnack inequality for the operator $\hat{L}$.

Lemma 2.14. *Let Ω, M, $w \in \partial\Omega$, r, u', v', r' and $\hat{L}$ be as above. There exists $\hat{c}$, $1 \leq \hat{c} < \infty$, depending only on p, n, M, such that if if e_1, e_2, are positive solutions to the operator $\hat{L}$ in $\Omega \cap B(w, 2r')$ and e_1, e_2, are continuous in $\bar{B}(w, 2r')$ with $e_1 \equiv 0 \equiv e_2$ on $B(w, 2r') \setminus \Omega$, then*

$$\hat{c}^{-1}\frac{e_1(a_{r''}(w))}{e_2(a_{r''}(w))} \leq \frac{e_1(y)}{e_2(y)} \leq \hat{c}\frac{e_1(a_{r''}(w))}{e_2(a_{r''}(w))} \text{ where } r'' = r'/\hat{c}$$

whenever $y \in \Omega \cap B(w, r'')$. Moreover, if $0 < \rho < r$, then the continuous Dirichlet problem for $\hat{L}$ in $\Omega \cap B(w, \rho)$ always has a solution.

Proof. The proof of Lemma 2.14 is similar to the proof of Theorem 1 in [LN]. To outline the proof observe from Theorem 2.9 (iii) that

$$\log |\nabla \tilde{u}| \in BMO(\Delta(w, r)) \text{ whenever } \tilde{u} \in \{u', v'\}$$

with norms $\leq c(p, n, M)$. Using this fact and arguing as in [LN4, Lemma 4.1] (this lemma is a refined version of Lemma 2.45 in [LN]), it follows that there exists, given $z \in \partial\Omega$, $B(z, 4s) \subset B(w, 2r')$, a starlike Lipschitz domain $\tilde{\Omega} \subset \Omega \cap B(z, s)$ with center $\tilde{z}$, $d(\tilde{z}, \partial\Omega) \geq c^{-1}s$, such that

$$\begin{array}{ll} (a) & c\,\sigma(\partial\tilde{\Omega} \cap \Delta(z, s)) \geq s^{n-1}, \\ (b) & c^{-1}s^{-1}\,\tilde{u}(\tilde{z}) \,\leq\, |\nabla \tilde{u}(x)| \,\leq\, cs^{-1}\,\tilde{u}(\tilde{z}) \text{ whenever } x \in \tilde{\Omega},\ \tilde{u} \in \{u', v'\}. \end{array} \tag{2.15}$$

In $(a) - (b)$ the constant c depends only on p, n and M. Next we define, for $x \in \tilde{\Omega}$, the measure

$$d\gamma(x) = d(x, \partial\tilde{\Omega}) \left(\max_{B(x, \frac{1}{2}d(x, \partial\tilde{\Omega}))} \sum_{i,j=1}^{n} |\nabla \hat{b}_{ij}(\cdot)|^2 \right) dx \tag{2.16}$$

where $\hat{b}_{ij}(\cdot)$ is defined in (1.11) relative to u', v'. From the conclusion of Theorem 2.4 we see that

$$|\nabla \tilde{u}| \approx \langle \nabla \tilde{u}, \xi \rangle \text{ for } \tilde{u} \in \{u', v'\} \text{ and some } \xi \in \mathbf{S}^{n-1}\ .$$

Using this fact and Lemma 2.2 we see that the integral defining $\hat{b}_{ij}(x)$ can be differentiated with respect to $x_i, 1 \leq i \leq n$, under the integral sign. Doing this and using (2.10) $(i), (ii)$, we deduce that

$$\sum_{i,j=1}^{n} |\nabla \hat{b}_{ij}(x)|^2 \leq c\left((|\nabla u'(x)| + |\nabla v'(x)|)^{2p-6} \sum_{i,j=1}^{n} (u'_{x_i x_j}(x))^2 + (v'_{x_i x_j}(x))^2 \right) \tag{2.17}$$

for some $c = c(p, n, M)$, $1 \leq c < \infty$. We assume, as we may, that $u(\tilde{z}) \geq v(\tilde{z})$. Then from (2.15) (b), (2.17) we deduce that

$$\sum_{i,j=1}^{n} |\nabla \hat{b}_{ij}(x)|^2 \leq c|\nabla u'(x)|^{2p-6} \sum_{i,j=1}^{n} (u'_{x_i x_j}(x))^2 + c(u'(\tilde{z})/v'(\tilde{z}))^{2p-6}\, |\nabla v'(x)|^{2p-6} \sum_{i,j=1}^{n} (v')^2_{x_i x_j}(x). \tag{2.18}$$

We define the following measures whenever $x \in \tilde{\Omega}$,

$$\begin{aligned} d\gamma_1(x) &= d(x, \partial\tilde{\Omega}) \left(\max_{B(x, \frac{1}{2}d(x, \partial\tilde{\Omega}))} |\nabla u'(\cdot)|^{2p-6} \sum_{i,j=1}^{n} (u'_{x_i x_j}(\cdot))^2 \right) dx, \\ d\gamma_2(x) &= d(x, \partial\tilde{\Omega}) \left(\max_{B(x, \frac{1}{2}d(x, \partial\tilde{\Omega}))} |\nabla v'(\cdot)|^{2p-6} \sum_{i,j=1}^{n} (v'_{x_i x_j}(\cdot))^2 \right) dx. \end{aligned}$$

Using this display, (2.15) (b), and the estimate for second derivatives in Lemma 2.2, we get by arguing as in [LN, Lemma 2.54],

$$\gamma_i(\tilde{\Omega} \cap B(y, t)) \,\leq\, c\, t^{n-1}\, (\zeta_i(\tilde{z})/s)^{2p-4}$$

whenever $i = 1, 2, y \in \tilde{\Omega}$, and $0 < t < s/4$. Here $\zeta_1 = u'$ and $\zeta_2 = v'$. From this display and (2.18) we see first that $\gamma \leq c(\gamma_1 + (u'(\tilde{z})/v'(\tilde{z}))^{2p-6}\gamma_2)$ and thereupon that γ is a Carleson measure on $\partial\tilde{\Omega}$ in the sense that

$$\gamma(\tilde{\Omega} \cap B(z,t)) \leq c\, t^{n-1} \,(u'(\tilde{z})/s)^{2p-4}, \tag{2.19}$$

whenever $z \in \partial\tilde{\Omega}$ and $0 < t < s/4$. From (2.15)(b) and (1.11) we also see that

$$c^{-1}(u'(\tilde{z})/s)^{p-2}\,|\xi|^2 \leq \sum_{i,j=1}^{n} \hat{b}_{ij}(y)\xi_i\xi_j \leq c\,(u'(\tilde{z})/s)^{p-2}\,|\xi|^2 \tag{2.20}$$

whenever $\xi \in \mathbf{R}^n$ and $y \in \tilde{\Omega}$. For $1 \leq i, j \leq n$, set $b^*_{ij} = (u'(\tilde{z})/s)^{2-p}\,\hat{b}_{ij}(y)$ when $y \in \tilde{\Omega}$. Let $L^* = (u'(\tilde{z})/s)^{2-p}\,\hat{L}$ and define γ^* as in (2.16) relative to (b^*_{ij}). From (2.20) we observe that L^* is uniformly elliptic in $\tilde{\Omega}$ with constants depending only on p, n, M and $L^* e_i = 0, i = 1, 2$. From (2.19) we also have

$$\gamma^*(\tilde{\Omega} \cap B(z,t)) \leq c\, t^{n-1}$$

whenever $z \in \partial\tilde{\Omega}$ and $0 < t < s/4$. From this discussion and a theorem in [KP] it follows that if $\omega^*(\cdot, \tilde{z})$ is elliptic measure defined with respect to $L^*, \tilde{z}$, in $\tilde{\Omega}$, then $\omega^*(\cdot, \tilde{z})$ is an A^∞ weight with respect to surface area on $\partial\tilde{\Omega}$ (see [LN, Theorem 3.11]). Since $\omega^*(\partial\tilde{\Omega} \cap B(z,s), \tilde{z}) \geq c^{-1}$, it follows from the A^∞ condition and (2.15) (a) that

$$\omega^*(\partial\tilde{\Omega} \cap \Delta(z,s), \tilde{z}) \geq c_*^{-1}, \tag{2.21}$$

where $c_* = c_*(p, n, M)$. Finally one can use arbitrariness of z, s, in (2.21), as well as Lemma 2.12, and some arguments on elliptic measure from [LN, Lemma 3.13], [HL, ch 3, sec 4], to conclude that Lemma 2.14 is true. $\Box$

2.2 The Hopf boundary principle

The following theorem is a refinement of Theorem 2.3.

Theorem 2.22. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that $\hat{u}$ and $\hat{v}$ are non-negative p-harmonic functions in $\Omega \cap B(w, 2r)$ with $\hat{v} \leq \hat{u}$. Assume also that $\hat{u}, \hat{v}$, are continuous in $B(w, 2r)$ with $\hat{u} \equiv 0 \equiv \hat{v}$ on $B(w, 2r) \setminus \Omega$. There exists $c_3, 1 \leq c_3 < \infty$, depending only on p, n, M, such that if $\tilde{r} = r/c_3$, then*

$$c_3^{-1}\frac{\hat{u}(a_{\tilde{r}}(w)) - \hat{v}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))} \leq \frac{\hat{u}(y) - \hat{v}(y)}{\hat{v}(y)} \leq c_3\frac{\hat{u}(a_{\tilde{r}}(w)) - \hat{v}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))}$$

whenever $y \in \Omega \cap B(w, \tilde{r})$.

Proof. We first prove the left hand inequality in Theorem 2.22. To do so we argue as in [W]. Let $r' = r/(4\max\{c_1, c_2, \hat{c}\})$ where $c_1, c_2, \hat{c}$ are defined in Theorems 2.3, 2.4 and Lemma 2.14, respectively. Put $\hat r = r'/(4\hat c)$. We prove the existence of λ, $1 \leq \lambda < \infty$, depending on p, n, M, such that if

$$e(y) = \lambda\left(\frac{\hat u(y) - \hat v(y)}{\hat u(a_{\hat r}(w)) - \hat v(a_{\hat r}(w))}\right) - \frac{\hat v(y)}{\hat v(a_{\hat r}(w))} \tag{2.23}$$

for $y \in \Omega \cap B(w, 2r')$, then

$$e(y) \geq 0 \text{ whenever } y \in \Omega \cap B(w, 2\hat r). \tag{2.24}$$

To do this, we initially allow $\lambda \geq 1$ to vary in (2.23). λ is then fixed near the end of the argument. Set

$$u'(y) = \frac{\lambda\, \hat u(y)}{\hat u(a_{\hat r}(w)) - \hat v(a_{\hat r}(w))},$$

$$v'(y) = \frac{\lambda\, \hat v(y)}{\hat u(a_{\hat r}(w)) - \hat v(a_{\hat r}(w))} + \frac{\hat v(y)}{\hat v(a_{\hat r}(w))}.$$

Observe from (2.23) that $e = u' - v'$. Let $\hat L$ be defined as in (1.9)-(1.11) using u', v', and let e_1, e_2 be the solutions to $\hat L e_i = 0, i = 1, 2$, in $\Omega \cap B(w, 3r'/2)$, with continuous boundary values:

$$\begin{aligned} e_1(y) &= \frac{\hat u(y) - \hat v(y)}{\hat u(a_{\hat r}(w)) - \hat v(a_{\hat r}(w))} \\ e_2(y) &= \frac{\hat v(y)}{\hat v(a_{\hat r}(w))} \end{aligned} \tag{2.25}$$

whenever $y \in \partial(\Omega \cap B(w, 3r'/2))$. Existence of e_1, e_2, follows from Lemma 2.14. From Lemma 2.14 we also get that

$$\hat c^{-1} \frac{e_1(a_{\hat r}(w))}{e_2(a_{\hat r}(w))} \leq \frac{e_1(y)}{e_2(y)} \leq \hat c\, \frac{e_1(a_{\hat r}(w))}{e_2(a_{\hat r}(w))} \tag{2.26}$$

whenever $y \in \Omega \cap B(w, 2\hat r)$. We now put

$$\lambda = \hat c\, \frac{e_2(a_{\hat r}(w))}{e_1(a_{\hat r}(w))}$$

and observe from (2.26) that

$$\lambda e_1(y) - e_2(y) \geq 0 \text{ whenever } y \in \Omega \cap B(w, 2\hat r). \tag{2.27}$$

Let $\hat e = \lambda e_1 - e_2$ and note from linearity of $\hat L$ that $\hat e, e$, both satisfy the same linear pde in $\Omega \cap B(w, 3r'/2)$ and also that these functions have the same continuous boundary values on $\partial(\Omega \cap B(w, 3r'/2))$. Hence, using the maximum principle for the operator $\hat L$, it follows that $e = \hat e$ and then by (2.27) that $e(y) \geq 0$ in $\Omega \cap B(w, 2\hat r)$. To complete the proof of the left hand inequality in Theorem 2.22 with $\tilde r$ replaced by $\hat r$, we show that

$$\lambda \leq c(p, n, M). \tag{2.28}$$

In fact let $\tilde{L}$ denote the operator corresponding to $\hat{u} - \hat{v}$ defined as in (1.9) - (1.11) with u', v' replaced by $\hat{u}, \hat{v}$. Then from the Harnack inequality in Lemmas 2.12 for $\tilde{L}$, applied to $\hat{u} - \hat{v}$, and the definition of $\hat{r}$, we deduce the existence of $\zeta \in \Omega \cap \partial B(w, 3r'/2)$ with $d(\zeta, \partial\Omega) \geq r/c$ and $e_1 \geq c^{-1}$ on $\Omega \cap \partial B(w, 3r'/2) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$. Lemmas 2.12, 2.13 can now be applied with h, h^* replaced by e_1, in order to get $e_1(a_{\hat{r}}(w)) \geq \bar{c}^{-1}$. Also from Lemma 2.1 applied to $\hat{v}$ we get $e_2(a_{\hat{r}}(w)) \leq \bar{c}$ for some $\bar{c} = \bar{c}(p, n, M)$. Thus (2.28) is true and the proof of the left hand inequality in Theorem 2.22 is valid.

To prove the right hand inequality in this theorem, one proceeds similarly only in this case one needs to show for e_1, e_2 as above that

$$e_1(a_{\hat{r}}(w)) \leq \bar{c} \text{ and } e_2(a_{\hat{r}}(w)) \geq \bar{c}.$$

The first inequality follows from the proof that (2.21) implies Lemma 2.14 for $\tilde{L}$ in [LN, section 3] while the second inequality follows from Lemma 2.1 and Lemma 2.13. This finishes the proof of Theorem 2.22. $\Box$

Finally we note the following consequence of Theorem 2.22.

Corollary 2.29. *Let $\Omega, \hat{u}, \hat{v}, w, p, r, r', \tilde{r}$, be as in Theorem 2.22. Assume $B(\hat{w}, \hat{\rho}) \subset \Omega \cap B(w, \tilde{r}/2)$ and $\zeta \in \partial B(\hat{w}, \hat{\rho}) \cap \partial\Omega$. There exists $c_4 = c_4(p, n, M) \geq 1$ such that if $y \in B(\hat{w}, \hat{\rho})$, then*

$$c_5(\hat{u}(y) - \hat{v}(y)) \geq (\hat{u}(\hat{w}) - \hat{v}(\hat{w})) \frac{|y - \zeta|}{\hat{\rho}}.$$

Proof. Let v^* be the p-harmonic function in $B(\hat{w}, \hat{\rho}) \setminus B(\hat{w}, \hat{\rho}/4)$ which has continuous boundary values zero on $\partial B(\hat{w}, \hat{\rho})$ and $\hat{v}(\hat{w})$ on $\partial B(\hat{w}, \hat{\rho}/4)$. From Harnack's inequality and the maximum principle for p harmonic functions we see that $v^* \leq c'\hat{v}$ in $B(\hat{w}, \hat{\rho}) \setminus B(\hat{w}, \hat{\rho}/4)$ where $c' = c'(p, n)$. Using this fact it is easily seen (e.g., direct calculation) that if $y \in B(\hat{w}, \hat{\rho}) \setminus B(\hat{w}, \hat{\rho}/4)$, then

$$\hat{v}(\hat{w}) \, \tfrac{|\zeta - y|}{\hat{\rho}} \, \leq \, cv^*(y) \, \leq c^2 \hat{v}(y) \tag{2.30}$$

on $B(\hat{w}, \hat{\rho}) \setminus B(\hat{w}, \hat{\rho}/4)$. From Harnack's inequality we see that (2.30) holds for $\hat{v}$ in $B(\hat{w}, \hat{\rho})$. Using (2.30) and the conclusion of Theorem 2.22 we deduce the last display in Corollary 2.29. $\Box$

3 Construction of subsolutions

Recall that given a bounded domain D and $1 < p < \infty$, we say that u is a p-subsolution in D provided $u \in W^{1,p}(D)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx \leq 0 \tag{3.1}$$

whenever $\theta \in W_0^{1,p}(D)$ and $\theta \geq 0$ a.e. on D. Moreover, we say that u is a p-supersolution provided $-u$ is a p-subsolution. Finally, let $C^2(D)$ denote functions with continuous second partials on D. If $\phi \in C^2(D)$ we let $\nabla^2\phi(x)$ denote the Hessian matrix of ϕ at $x \in D$. We shall need the following criteria for p-subharmonicity.

Lemma 3.2. *Let u be continuous in the bounded open set D and suppose that if $u \leq h$ on ∂G, then $u \leq h$ in G, whenever G is an open set with $\bar{G} \subset D$ and h is p-harmonic on G with continuous boundary values. Then u is a p-subsolution in the open set G whenever $\bar{G} \subset D$.*

Proof. see [HKM, Theorem 7.25]. □

Let $S(n)$ denote the set of all symmetric $n \times n$ matrices and let P be the Pucci type extremal operator (see [CC]) defined for $M \in S(n)$ by

$$P(M) = \inf_{A \in A_p} \sum_{i,j=1}^{n} a_{ij} M_{ij} \,. \tag{3.3}$$

Here A_p denotes the set of all symmetric $n \times n$ matrices $A = \{a_{ij}\}$ which satisfy

$$\min\{p-1, 1\}\, |\xi|^2 \;\leq\; \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \;\leq\; \max\{p-1, 1\}\, |\xi|^2 \text{ whenever } \xi \in \mathbf{R}^n. \tag{3.4}$$

In this section we first prove the following lemma.

Lemma 3.5. *Let $D \subset \mathbf{R}^n$, let $\phi > 0$ be in $C^2(D), \|\nabla \phi\|_{L^\infty(D)} \leq 1/2$, p fixed, $1 < p < \infty$, and suppose that*

$$\phi(x) P(\nabla^2 \phi(x)) \geq \; 50pn\, |\nabla \phi(x)|^2 \; \textit{whenever } x \in D.$$

Let u be continuous in an open set O containing the closure of $\bigcup_{x \in D} B(x, \phi(x))$ and define

$$v(x) = \max_{\bar{B}(x, \phi(x))} \; u$$

whenever $x \in D$. If u is p-harmonic in $O \setminus \{u = 0\}$, then v is continuous and a p-subsolution in $\{v \neq 0\} \cap G$ whenever G is an open set with $\bar{G} \subset D$.

Proof. We note that v is continuous in D as follows easily from the fact that u, ϕ, are continuous in D. We first prove Lemma 3.5 with D replaced by $D^+(v)$. The proof is by contradiction. Indeed, if Lemma 3.5 is false, relative to $D^+(v)$, then there exists G open with $\bar{G} \subset D^+(v)$ and a p-harmonic function h on G that is continuous on $\bar{G}$ with

$$v \leq h \text{ on } \partial G \text{ and } \max_{G} (v - h) > 0.$$

From continuity it follows for small $\epsilon > 0$ that there exists $\hat{x} \in G$ with

$$\max_{\partial G} (v^{1+\epsilon} - h)(y) < \max_{G} (v^{1+\epsilon} - h)(y) = (v^{1+\epsilon} - h)(\hat{x}). \tag{3.6}$$

Using translation and rotation invariance of the p-Laplacian, as well as the maximum principle for p-harmonic functions, we may assume that

$$\hat{x} = 0 \text{ and } \max_{\bar{B}(0, \phi(0))} u = u(\phi(0) e_n). \tag{3.7}$$

Also, we may assume $\phi(0) = 1$, since otherwise we put

$$\begin{aligned}\tilde{u}(x) &= u(\phi(0)x),\ \tilde{h}(x) = h(\phi(0)x)\\ \tilde{\phi}(x) &= \phi(\phi(0)x)/\phi(0),\ \tilde{v}(x) = \max_{\bar{B}(x,\tilde{\phi}(x))} \tilde{u}.\end{aligned}$$

Then (3.6) holds with v, h replaced by $\tilde{v}, \tilde{h}$ in a neighborhood of 0. Also, $\tilde{u}, \tilde{h}$ are p-harmonic and $\tilde{\phi}$ satisfies the same condition as ϕ in Lemma 3.5. Clearly $\tilde{\phi}(0) = 1$. Repeating the following argument, we get a contradiction to (3.6) with v, h replaced by $\tilde{v}, \tilde{h}$. Thus we assume that (3.7) holds and $\phi(0) = 1$. We claim that $\nabla u(e_n) \neq 0$. Indeed, otherwise, since $u(e_n) = \max_{\bar{B}(0,1)} u$ we can use the maximum principle for p-harmonic functions and estimate $u(e_n) - u$ from below by a p-harmonic function ψ in $B(te_n, 1-t) \setminus \bar{B}(te_n, (1-t)/2)$ where $0 < t < 1$ is so small that $\bar{B}(te_n, 1-t) \subset O \setminus \{u = 0\}$. Moreover, ψ has continuous boundary value 0 on $\partial B(te_n, 1-t)$ and $\psi \equiv \min\limits_{\bar{B}(te_n,(1-t)/2)} (u(e_n) - u)$, continuously on $\partial B(te_n, (1-t)/2)$. ψ can be written explicitly. Doing this and using a Hopf boundary maximum principle type argument it follows that either our claim is true or $u \equiv u(e_n)$ in $B(te_n, 1-t)$. In the latter case one readily concludes that $u \neq 0$ in $B(0, 1)$ and thereupon that $u(0) = u(e_n)$. Then from (3.6) and $u \leq v$ we deduce

$$\max_{\partial G}(u^{1+\epsilon} - h)(y) < \max_{G}(u^{1+\epsilon} - h)(y) = (u^{1+\epsilon} - h)(0).$$

Now $u^{1+\epsilon}$ is a p-subsolution as is easily checked. Using this fact and the boundary maximum principle for p-subsolutions we see that the above inequality cannot hold. Thus $\nabla u(e_n) \neq 0$.

To continue the proof of Lemma 3.5 we note that $\nabla u(e_n) = |\nabla u(e_n)| e_n$. and following [C1] we choose the system of coordinates so that

$$\nabla \phi(0) = \alpha e_1 + \beta e_n \tag{3.8}$$

for some constants α, $\beta \in \mathbf{R}$ and we introduce the direction

$$\sigma = \frac{\sigma^*}{|\sigma^*|},\ \sigma^* = \sigma^*(x) = e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum_{i=2}^{n-1} x_i e_i. \tag{3.9}$$

The constant γ is chosen below (3.15). Then

$$|\sigma^*|^2 = 1 + (\beta x_1 - \alpha x_n)^2 + \gamma^2 \sum_{i=2}^{n-1} x_i^2. \tag{3.10}$$

Let $y(x) = x + \phi(x)\sigma(x)$ and note that $y(0) = e_n$, as well as, $v(x) \geq u(y(x))$ for x in a neighborhood of zero. In view of (3.6), (3.7), it follows for some $t > 0$ that if

$$f(x) = u^{1+\epsilon}(y(x)) - h(x), x \in \bar{B}(0, t),\ \text{then } 0 < f(0) = \max_{\bar{B}(0,t)} f. \tag{3.11}$$

It turns out, as we will see below, that f has continuous second partials in a neighborhood of 0. Moreover, we will be able to use the second derivative test for a relative maximum in order to obtain a contradiction to (3.11). With this game plan in mind, we again follow [C1] and use Taylor's formula at $x = 0$ to derive that

$$y(x) = e_n + y_1(x) + y_2(x) + o(|x|^2) \tag{3.12}$$

where

$$\begin{aligned} y_1(x) &= x + (\alpha x_1 + \beta x_n)e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum_{i=2}^{n-1} x_i e_i, \\ y_2(x) &= \left(\frac{1}{2} \sum_{i,j=1}^{n} \phi_{x_i x_j}(0) x_i x_j - \frac{1}{2}(\beta x_1 - \alpha x_n)^2 - \frac{1}{2}\gamma^2 \sum_{i=2}^{n-1} x_i^2 \right) e_n \\ &\quad + \langle \nabla\phi(0), x\rangle ((\beta x_1 - \alpha x_n)e_1 + \gamma \sum_{i=2}^{n-1} x_i e_i). \end{aligned} \tag{3.13}$$

Using the chain rule and (3.12), (3.13), we see that if $g(x) = u(y(x)), x \in B(0,t)$, then

$$\nabla g(0) = |\nabla u(e_n)| [\alpha e_1 + (1+\beta)e_n] \neq 0, \tag{3.14}$$

since $|\nabla\phi(0)| \leq 1/2$ and $\nabla u(e_n) \neq 0$.

From (3.11) and (3.14) we find that

$$0 = \nabla f(0) = (1+\epsilon) u(e_n)^{\epsilon} \nabla g(0) - \nabla h(0).$$

Thus

$$\xi = \nabla g(0)/|\nabla g(0)| = \nabla h(0) \; |\nabla h(0)|.$$

From this display, (3.14), and Lemma 2.2 we obtain first that h is infinitely differentiable in a neighborhood of 0 and thereupon from rotational invariance of the p-Laplace equation that

$$(p-2) h_{\xi\xi}(0) + \Delta h(0) = 0 \tag{3.15}$$

where Δ denotes the Laplacian and $h_{\xi\xi}$ is the second directional derivative of h in the direction of ξ. We now choose γ so that $(1+\gamma)^{-2}[(1+\beta)^2 + \alpha^2] = 1$ and note by an easy calculation that $y_1(x) = \Gamma x$ where Γ is an orthogonal matrix. Hence $y_1(x)$ can be interpreted as the composition of a rotation and a dilation. Again using translation, rotation, and dilation invariance of the p-Laplacian it follows that if $g_1(x) = u(e_n + y_1(x))$, then g_1 is p-harmonic in a neighborhood of 0. Also $\nabla g_1(0) = \nabla g(0) \neq 0$, so as in (3.15) we have,

$$(p-2)(g_1)_{\xi\xi}(0) + \Delta g_1(0) = 0. \tag{3.16}$$

Let $g_2(x) = g(x) - g_1(x)$. Using Taylor's formula at e_n for u (permissible by Lemma 2.2), we see that

$$\begin{aligned} g_2(x) &= \langle \nabla u(e_n), y_2(x) \rangle + O(|x|^3) \\ &= \frac{1}{2} |\nabla u(e_n)| \left(\sum_{i,j=1}^{n} \phi_{x_i x_j}(0) x_i x_j - (\beta x_1 - \alpha x_n)^2 - \gamma^2 \sum_{i=2}^{n-1} x_i^2 \right) + O(|x|^3). \end{aligned} \tag{3.17}$$

Thus,

$$
\begin{aligned}
|\nabla u(e_n)|^{-1}[(p-2)(g_2)_{\xi\xi}(0) \quad & +\Delta g_2(0)] = (p-2)\phi_{\xi\xi}(0) + \Delta\phi(0) \\
& -\alpha^2(1+\gamma)^{-2}(p-2) - (\alpha^2+\beta^2+(n-2)\gamma^2) \qquad (3.18) \\
& \geq P(\nabla^2\phi(0)) - 50pn|\nabla\phi(0)|^2 \geq 0,
\end{aligned}
$$

where the last inequality was obtained from using the definition of $P, \alpha, \beta, \gamma, |\nabla\phi(0)| \leq 1/2$, and a ball park estimate. From (3.16), (3.18) we see that

$$
(p-2)g_{\xi\xi}(0) + \Delta g(0) \geq 0. \tag{3.19}
$$

Finally, using (3.19) and (3.11) we compute,

$$
(p-2)f_{\xi\xi}(0) + \Delta f(0) \geq (p-1)\epsilon(1+\epsilon)u(e_n)^{\epsilon-1}|\nabla g(0)|^2 > 0, \tag{3.20}
$$

which is a contradiction to the second derivative test for maxima. From this contradiction we conclude that $v \leq h$ in G. Hence v satisfies the hypotheses of Lemma 3.2 in $D^+(v)$. Applying Lemma 3.2 we get Lemma 3.5 in $D^+(v)$. The proof of Lemma 3.5 in $D^-(v)$ is similar. In fact the only place we used positivity of v was in (3.6). If we replace $v^{1+\epsilon}$ in this display by $-(-v)^{1-\epsilon}$, then the proof for $D^-(v)$ is essentially unchanged from the proof for $D^+(v)$. $\Box$

The next lemma gives the asymptotic development of the p-subsolution constructed in Lemma 3.5.

Lemma 3.21. *Let D, u, ϕ, O, and $v = v_\phi$ be as in the statement of Lemma 3.5 and assume that $(i), (ii)$ of Definition 1.4 hold for some α, β whenever $w \in O \cap \partial\{u > 0\}$ and there exists $B(\hat{w}, \hat{\rho}) \subset O \setminus \partial\{u > 0\}$ with $w \in \partial B(\hat{w}, \hat{\rho})$. If $\tilde{w} \in F(v)$, then there exist $w^* \in D^+(v)$ and $\rho^* > 0$ such that $B(w^*, \rho^*) \subset D^+(v)$ and $\tilde{w} \in \partial B(w^*, \rho^*)$. Also, there exist $\tilde{\alpha}$, $\tilde{\beta} \in [0, \infty)$, such that the following holds, as $x \to \tilde{w}$ non-tangentially with $\tilde{\nu} = (w^* - \tilde{w})/|w^* - \tilde{w}|$,*

$$
\begin{aligned}
&(a) \qquad v(x) \geq \tilde{\alpha}\langle x - \tilde{w}, \tilde{\nu}\rangle^+ - \tilde{\beta}\langle x - \tilde{w}, \tilde{\nu}\rangle^- + o(|x - \tilde{w}|), \\
&(b) \qquad \frac{\tilde{\alpha}}{1 - |\nabla\phi(\tilde{w})|} \geq G\left(\frac{\tilde{\beta}}{1 + |\nabla\phi(\tilde{w})|}\right).
\end{aligned}
$$

Furthermore, assume that $O \cap \partial\{u > 0\}$ is a Lipschitz graph, with Lipschitz constant M. If $\|\nabla\phi\|_{L^\infty(D)} \leq b$ and $b = b(M) > 0$ is sufficiently small, then $F(v)$ is a Lipschitz graph with Lipschitz constant M' where $M' \leq M + c\|\nabla\phi\|_{L^\infty(D)}$.

Proof. The proof of Lemma 3.21 for $p = 2$ can be found in Lemmas 10,11 of [C1]. The proof is based on a purely geometric argument using only smoothness of ϕ, the asymptotic expansion of u in balls tangent to $F(u)$, and Lipschitzness of $F(u)$, so is also valid here. $\Box$

Lemma 3.22. *Let $\rho \in (0, 10^{-2})$, $\gamma \in (0, 1/2)$, be given. There exists $h = h(\rho, p, n), 0 < h \leq 1$, and a family of C^2 functions $\{\phi_t\}$, $0 \leq t \leq 1$, defined in $B(0,2) \setminus B(e_n/8, \rho)$ such that*

(i)	$\phi_t(x) = 1$ *if* $x \in \bar{B}(0,2) \setminus B(0,1/2)$ *and* $\phi_t(x) \geq 1 + h\gamma t$ *whenever* $x \in B(0,1/16)$,
(ii)	$1 \leq \phi_t(x) \leq 1 + t\gamma,\ |\nabla \phi_t(x)| \leq \gamma t$ *whenever* $x \in \bar{B}(0,2) \setminus \bar{B}(e_n/8, \rho)$,
(iii)	$\phi_t(x) P(\nabla^2 \phi_t(x)) \geq 50pn |\nabla \phi_t(x)|^2$ *whenever* $x \in \bar{B}(0,2) \setminus B(e_n/8, \rho)$.

Proof. To prove this lemma we argue as in [F, Lemma 1.4]. Given $\rho \in (0, 10^{-2})$, $\gamma \in (0, 1/2)$, let $A = \{a_{ij}\}$ be an arbitrary symmetric $n \times n$ matrix satisfying (3.4) and let $L = \sum a_{ij} \frac{\partial}{\partial x_i \partial x_j}$ be the associated non divergence form operator. Let $f(x) = 1/|x|^{2N}$ whenever $x \in \mathbf{R}^n \setminus \{0\}$ and for some large positive N. By an explicit calculation it follows that if $c = c(n)$ is large enough and $N \geq c \max\{p, 1/(p-1)\}$, then for every operator L, as above,

$$Lf(x) \geq |\nabla f(x)| \text{ whenever } x \in B(0,4) \setminus \{0\}. \tag{3.23}$$

Next we let

$$\tilde{f}(x) = \max\{|e_n/8 - x|^{-2N} - 4^{2N}, 0\} \text{ whenever } x \in B(0,2) \setminus \bar{B}(e_n/8, \rho). \tag{3.24}$$

Then from (3.23), (3.24), we have $L\tilde{f} \geq |\nabla \tilde{f}|$ on $B(e_n/8, 1/4) \setminus B(e_n/8, \rho)$ and $\tilde{f} \equiv 0$ outside of $B(e_n/8, 1/4)$. Moreover, if we let $\hat{f} = \tilde{\gamma} \tilde{f}^4$ then $\hat{f}$ is C^2 on $B(0,2) \setminus \bar{B}(e_n/8, \rho)$. Choose $\tilde{\gamma} = \tilde{\gamma}(\rho, p, n) > 0$ small enough so that

$$0 \leq \max(\hat{f}, |\nabla \hat{f}|) \leq 1 \text{ on } B(0,2) \setminus B(e_n/8, \rho). \tag{3.25}$$

Now on $\{\tilde{f} > 0\}$, we have

$$L\hat{f} \ \geq\ 4\tilde{\gamma} \tilde{f}^3\, L\tilde{f} \ \geq\ 4\tilde{\gamma} \tilde{f}^3 |\nabla \tilde{f}| \text{ and } |\nabla \hat{f}| = 4\tilde{\gamma}\, \tilde{f}^3\, |\nabla \tilde{f}|.$$

Therefore,

$$L\hat{f} \geq |\nabla \hat{f}| \text{ in } B(0,2) \setminus B(e_n/8, \rho) \text{ and } \hat{f} \equiv 0 \text{ in } B(0,2) \setminus B(0,1/2). \tag{3.26}$$

Also,

$$\hat{f} \geq k_+^{-1} \text{ on } B(0,1/16) \text{ for some } k_+ = k_+(p,n,\rho) \geq 1. \tag{3.27}$$

To complete the construction we let $\phi_t(x) = 1 + \frac{t\gamma}{50pn} \hat{f}(x)$. From (3.25) - (3.27) we see that $\{\phi_t\}$, $t \in [0,1]$, satisfies $(i) - (iii)$ of Lemma 3.22. $\Box$

4 Regularity of the free boundary: establishing Step 0-2

The purpose of this section is to establish Step 0-2 stated in the introduction. In particular, we prove the following three lemmas.

Lemma 4.1. *Let* u*,* Ω*,* $0 \in F(u)$*,* p, M*, be as in the statement of Theorem 1. Let* $r_1 = (8 \max\{c_1, c_2, c_3\})^{-1}$ *where* c_1, c_2, c_3 *are defined in Theorems 2.3, 2.4, and 2.22, respectively.*

Then u is monotone in $B(0,r_1)$ with respect to the directions in the cone $\Gamma(e_n,\theta_0,\epsilon_0)$ for some $\theta_0=\theta_0(p,n,M)\in(0,\pi/2)$ and for some $\epsilon_0=\epsilon_0(p,n,M), 0<\epsilon_0\leq 1$.

Lemma 4.2. *Let u, Ω, $0\in F(u)$, p, M, r_1, θ_0, and ϵ_0 be as in the statement of Lemma 4.1. If $\tau\in\Gamma(e_n,\theta_0/2,\epsilon_0)$, let $\epsilon=|\tau|\sin(\theta_0/2)$ and put*

$$v_\epsilon(x)=v_{\epsilon,\tau}(x)=\sup_{B(x,\epsilon)} u(y-\tau) \text{ whenever } x\in B(0,2-2\epsilon).$$

Let $\nu=\nabla u(r_1e_n/32)/|\nabla u(r_1e_n/32)|$. There exists $c_4\geq 1000\max\{c_1,c_2,c_3\}$, depending only on p,n,M such that if $\rho=(100n[M+1])^{-1}, \mu=1/c_4, \lambda=\cos(\theta_0/2+\theta(\nu,\tau))$, and $0<|\tau|\leq\rho\epsilon_0r_1$, then

$$v_{(1+\mu\lambda)\epsilon}(x)\leq(1-\mu\lambda\epsilon)u(x) \text{ whenever } x\in B(r_1e_n/32,\rho r_1).$$

Lemma 4.3. *Let u, Ω, $0\in F(u)$, p, M, r_1, θ_0, ϵ_0, τ, ϵ, v_ϵ, ν, c_4, ρ, μ, λ be as in Lemma 4.2. There exists $c_5=c_5(p,n,M,N)\geq 1$ such that if $\bar\mu=1/c_5$, then*

$$v_{(1+\bar\mu\lambda)\epsilon}(x)\leq u(x) \text{ whenever } x\in B(0,r_1/100).$$

Proof of Lemma 4.1. Observe from the hypotheses of Theorem 1 and the maximum principle for p-harmonic functions that either $F(u)=D\cap\partial\{x\in D: u(x)<0\}$ or $u\geq 0$ in D. Using this observation we see that Lemma 4.1 is an easy consequence of Theorem 2.4. $\Box$

Proof of Lemma 4.2. Recall that $\theta(y,z)$ denotes the angle between the rays drawn from the origin to $y,z\in\mathbf{R}^n$. From Theorem 2.4 and the definition of r_1 we have

$$c_2^{-1}\frac{u(x)}{d(x,\partial\Omega)}\leq\langle\nabla u(x),\xi\rangle\leq|\nabla u(x)|\leq c_2\frac{u(x)}{d(x,\partial\Omega)} \tag{4.4}$$

whenever $x\in\Omega\cap B(0,4r_1)$ and $\xi\in\Gamma(e_n,\theta_0)\cap\mathbf{S}^{n-1}$, where $c_2\geq 1$ depends only on p, n and M. Let $\tau\in\Gamma(e_n,\theta_0/2,\epsilon_0)$ and let ϵ be as in the statement of Lemma 4.2. Let $y\in B(x,\epsilon)$, define $\bar\tau=\tau-(y-x)$, and note from basic geometry that

$$\theta(\tau,\bar\tau)<\theta_0/2,\ |\tau-\bar\tau|<|\tau|\sin(\theta_0/2),\ |\bar\tau|\geq|\tau|/2. \tag{4.5}$$

From Lemma 4.1 we see that u is monotone in $B(0,r_1)$ with respect to the directions in the cone $\Gamma(e_n,\theta_0,\epsilon_0)$. This fact, (4.5), and the definition of ρ in Lemma 4.2 imply that

$$\langle\nabla u(x),\bar\tau\rangle\geq 0 \text{ whenever } x\in B(r_1e_n/32,8\rho r_1). \tag{4.6}$$

Using the uniform non-degeneracy property of $|\nabla u|$ in (4.4) and Lemma 2.2, it follows from differentiation of (1.2), that if $\zeta=\langle\nabla u(x),\bar\tau/|\bar\tau|\rangle$, then ζ satisfies, at $x\in B(r_1e_n/32,8\rho r_1)$, the partial differential equation $L\zeta=0$, where

$$L=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{4.7}$$

and

$$b_{ij}(x) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](x), 1 \le i, j \le n. \tag{4.8}$$

In (4.8), δ_{ij} denotes the Kronecker δ. Furthermore, if $\xi \in \mathbf{R}^n \setminus \{0\}$, then

$$\min(p-1, 1)\,|\nabla u(x)|^{p-2}\,|\xi|^2 \;\le\; \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\xi_j \le \max\{p-1, 1\}|\nabla u(x)|^{p-2}\,|\xi|^2. \tag{4.9}$$

Therefore, using (4.4) we see first that L is uniformly elliptic in $B(r_1e_n/32, 8\rho r_1)$ and second that positive solutions satisfy an interior Harnack inequality. In particular, we can conclude there exists $c = c(p, n, M)$, $1 \le c < \infty$, such that

$$c^{-1}\langle\nabla u(r_1e_n/32), \bar\tau\rangle \le \langle\nabla u(x), \bar\tau\rangle \le c\langle\nabla u(r_1e_n/32), \bar\tau\rangle \text{ whenever } x \in B(r_1e_n/32, 4\rho r_1). \tag{4.10}$$

Using (4.10), as well as our assumptions on ϵ, ρ, and the mean value theorem from elementary calculus, we see that if $x \in B(r_1e_n/32, 2\rho r_1)$, then

$$u(x-\bar\tau) \le u(x) - \langle\nabla u(\hat x), \bar\tau\rangle \tag{4.11}$$

for some $\hat x \in B(r_1e_n/32, 4\rho r_1)$ located on the line connecting the points x and $x - \bar\tau$. Combining (4.11), (4.10), (4.4), and the Harnack inequality for p-harmonic functions, we see for some $c = c(p, n, M) \ge 1$ that

$$u(x-\bar\tau) \le u(x) - c^{-1}\langle\nabla u(r_1e_n/32), \bar\tau\rangle \le u(x) - c^{-2}\epsilon\cos(\theta/2 + \theta(\nu, \tau))\frac{u(x)}{r_1} \tag{4.12}$$

whenever $x \in B(r_1e_n/32, 2\rho r_1)$ and where ν is as stated in Lemma 4.2. Since $u(y-\tau) = u(x-\bar\tau)$ we can take the supremum over $y \in B(x, \epsilon)$ in (4.12) to get,

$$v_\epsilon(x) \le \left(1 - \lambda\epsilon/c'\right)u(x) \tag{4.13}$$

for some $c' = c'(p, n, M) \ge 1$ whenever $x \in B(r_1e_n/32, 2\rho r_1)$. Finally we use (4.13) to complete the proof of Lemma 4.2. In particular, for $\xi \in \mathbf{R}^n$, $|\xi| < 1$, and $x \in B(r_1e_n/32, 2\rho r_1)$ we define $e(x) = u(x) - u(x - \tau + \epsilon\xi)$ and let $\mu > 0$ be a small positive constant to be chosen. To complete the proof we intend to estimate $u(x) - u(x - \tau + (1+\mu\lambda)\epsilon\xi)$. Using (4.4), the mean value theorem, and Harnack's inequality for p-harmonic functions we see, for some $c = c(p, n, M) \ge 1$, that

$$\begin{aligned} u(x) - u(x - \tau + (1+\mu\lambda)\epsilon\xi) &= e(x) + u(x - \tau + \epsilon\xi) - u(x - \tau + (1+\mu\lambda)\epsilon\xi) \\ &\ge e(x) - c\mu\lambda\epsilon u(x) \end{aligned} \tag{4.14}$$

whenever $x \in B(r_1e_n/32, 2\rho r_1)$. From (4.13) we also have,

$$e(x) \ge (\lambda\epsilon/c')\,u(x) \text{ whenever } x \in B(r_1e_n/32, 2\rho r_1). \tag{4.15}$$

Combining (4.14)-(4.15) we get

$$\begin{aligned} u(x) - u(x - \tau + (1+\mu\lambda)\epsilon\xi) &\ge \lambda\epsilon/c'u(x) - c\mu\lambda\epsilon u(x) \\ &\ge c_+^{-1}\lambda\epsilon(1 - \mu c_+^2)u(x) \end{aligned} \tag{4.16}$$

whenever $x \in B(r_1 e_n/32, 2\rho r_1)$ provided $c_+ = c_+(p, n, M) \geq 1$ is large enough and $\mu < 1/c_+^2$. If $\mu = \frac{1}{2c_+^2}$, then from (4.16) we conclude that

$$(1 - \mu\lambda\epsilon)u(x) \geq u(x - \tau + (1 + \mu\lambda)\epsilon\xi) \tag{4.17}$$

whenever $x \in B(r_1 e_n/32, 2\rho r_1)$. Taking the supremum of the right hand side of (4.17) over $\xi \in B(0,1)$ we obtain Lemma 4.2. $\Box$

Proof of Lemma 4.3. Note that by scaling we can without loss of generality assume that $r_1 = 4$. Using Lemma 4.2 we see that

$$v_{(1+\mu\lambda)\epsilon}(x) \leq (1 - \mu\lambda\epsilon)u(x) \text{ whenever } x \in B(e_n/8, 4\rho) \tag{4.18}$$

where μ, λ, ϵ, and ρ are as in the statement of Lemma 4.2. From Lemma 4.1 we also know, after the rescaling, that

$$v_\epsilon(x) \leq u(x) \text{ whenever } x \in \bar{B}(0,4). \tag{4.19}$$

Let $\{\phi_t\}$, $t \in [0,1]$, be the C^2-regular family of functions introduced in Lemma 3.22, defined in $B(0,2) \setminus B(e_n/8, \rho)$ and adapted to the parameters ρ, γ, where $\gamma << 1$ is a parameter to be chosen. Let

$$\tilde{v}_t(x) := \sup_{B(x, \epsilon\phi_{\mu\lambda t}(x))} u(y - \tau) \text{ for } t \in [0,1] \text{ whenever } x \in B(0,2) \setminus B(e_n/8, \rho). \tag{4.20}$$

Then we see, using (4.18) - (4.20), Lemma 3.22, that

$$\begin{aligned} \tilde{v}_t(x) &\leq u(x) \text{ whenever } x \in B(0,2) \setminus B(0,1/2), \\ \tilde{v}_t(x) &\leq v_{(1+\mu\lambda)\epsilon}(x) \leq (1 - \mu\lambda\epsilon)u(x) \text{ whenever } x \in \partial B(e_n/8, \rho). \end{aligned} \tag{4.21}$$

Let

$$\begin{aligned} D^+(\tilde{v}_t) &= \{x : \tilde{v}_t(x) > 0\} \cap [B(0,2) \setminus \bar{B}(e_n/8, \rho)] \\ D^-(\tilde{v}_t) &= B(0,2) \setminus [\bar{B}(e_n/8, \rho) \cup \bar{D}^+(\tilde{v}_t)] \\ F(\tilde{v}_t) &= \partial\{x \in B(0,2) : \tilde{v}_t(x) > 0\} \cap [B(0,2) \setminus \bar{B}(e_n/8, \rho)]. \end{aligned}$$

We observe from the hypotheses on $u, \phi_t, t \in [0,1]$, that either $\tilde{v}_t(x) < 0$ for $x \in D^-(\tilde{v}_t)$ or $\tilde{v}_t \equiv 0$ in $D^-(\tilde{v}_t)$. From this observation and Lemma 3.5 we deduce that $\tilde{v}_t$, $t \in [0,1]$, is p-subharmonic in $D^+(\tilde{v}_t) \cup D^-(\tilde{v}_t)$. Using Lemma 3.21 we find for $\gamma = \gamma(p, n, M)$ small enough, that there exists $\psi_t : \mathbf{R}^{n-1} \to \mathbf{R}$ and $\Omega_t, t \in [0,1]$, with

$$\begin{gathered} \Omega_t = \{y = (y', y_n) \in \mathbf{R}^n : y_n > \psi_t(y')\}, \\ D^+(\tilde{v}_t) \cap B(0, 3/2) = \Omega_t \cap [B(0,3/2) \setminus \bar{B}(e_n/8, \rho)], \\ F(\tilde{v}_t) \cap B(0,3/2) = \partial\Omega_t \cap B(0,3/2), \end{gathered} \tag{4.22}$$

and $||\nabla\psi_t||_\infty \leq M + c\epsilon\gamma$. Also, from the definition of $\rho, \tilde{v}_t$, and the above observation we conclude that if $t \in [0,1]$, then

$$F(\tilde{v}_t) \cap B(e_n/8, 20\rho) = \emptyset \text{ and either } F(\tilde{v}_t) = B(0,2) \cap \partial\{\tilde{v}_t < 0\} \text{ or } \tilde{v}_t \equiv 0 \text{ in } D^-(\tilde{v}_t). \tag{4.23}$$

To proceed we let

$$\Gamma = \{t : t \in [0,1], \tilde{v}_t(x) \leq u(x) \text{ whenever } x \in B(0,1) \setminus B(e_n/8, \rho)\}. \tag{4.24}$$

Using the fact that $\phi_0 \equiv 1$ and (4.19) we see that $0 \in \Gamma$ and we intend to prove, for sufficiently small $\gamma = \gamma(p,n,M,N) > 0$, that $\Gamma = [0,1]$ by proving that Γ is both closed as well as relatively open in [0, 1]. Here N is as in Theorem 1. In fact, by the continuity of u and $\tilde{v}_t$ we immediately see that Γ is closed and hence it is enough to prove that Γ is open. Note that if $t \in \Gamma$, then $D^+(\tilde{v}_t) \subseteq D^+(u)$. To prove that Γ is open it is enough to prove that

$$F(u) \cap F(\tilde{v}_t) \cap (B(0,3/4) \setminus B(e_n/8, \rho)) = \emptyset \text{ whenever } t \in \Gamma,\ t \neq 1. \tag{4.25}$$

Indeed if (4.25) is true, then from (4.23) and another continuity argument, we deduce the existence of $\eta > 0$ such that

$$\bar{B}(0,5/8) \cap F(\tilde{v}_s) \subset D^+(u) \text{ whenever } s \in (t-\eta, t+\eta) \cap [0,1]. \tag{4.26}$$

From (4.26) and (4.21), we obtain

$$\tilde{v}_s \leq u \text{ on } \partial(B(0,1) \setminus [F(\tilde{v}_s) \cup B(e_n/8, \rho)]). \tag{4.27}$$

From (4.27), Lemma 3.5, and the maximum principle for p-harmonic functions, we have $s \in \Gamma$. Thus $(t-\eta, t+\eta) \cap [0,1] \subset \Gamma$ and Γ is relatively open. We conclude that $\Gamma = [0,1]$ when (4.25) holds.

To prove (4.25) we argue by contradiction and thus we assume that (4.25) does not hold for some t. Hence there exists

$$\tilde{w} \in F(u) \cap F(\tilde{v}_t) \cap B(0,3/4). \tag{4.28}$$

To obtain a contradiction to (4.28) we note from Lemma 3.21 that there exists $\hat{w} \in D^+(\tilde{v}_t)$, and $\hat{\rho} > 0$ such that $B(\hat{w}, \hat{\rho}) \subset D^+(\tilde{v}_t), \tilde{w} \in \partial B(\hat{w}, \hat{\rho})$. Moreover if $\tilde{\nu} = (\hat{w} - \tilde{w})/|\hat{w} - \tilde{w}|$, then there exists, $\bar{\alpha}, \bar{\beta}, \in [0, \infty)$, such that

$$\tilde{v}_t(x) \geq \bar{\alpha}\langle x - \tilde{w}, \tilde{\nu}\rangle^+ - \bar{\beta}\langle x - \tilde{w}, \tilde{\nu}\rangle^- + o(|x - \tilde{w}|), \tag{4.29}$$

non-tangentially near $\tilde{w}$. Furthermore,

$$\frac{\bar{\alpha}}{1 - \epsilon|\nabla\phi_{\mu\lambda t}(\tilde{w})|} \geq G\left(\frac{\bar{\beta}}{1 + \epsilon|\nabla\phi_{\mu\lambda t}(\tilde{w})|}\right). \tag{4.30}$$

Since $D^+(\tilde{v}_t) \subset D^+(u) \cap B(0,2)$, we see that $B(\hat{w}, \hat{\rho})$ is also a tangent ball for $D^+(u)$. Using the fact that u is a weak solution to the free boundary problem in (1.3), as defined in Definition 1.4, we obtain

$$u(x) = \alpha\langle x - \tilde{w}, \tilde{\nu}\rangle^+ - \beta\langle x - \tilde{w}, \tilde{\nu}\rangle^- + o(|x - \tilde{w}|, \tag{4.31}$$

as $x \to \tilde{w}$, non-tangentially for some $\alpha, \beta \in [0, \infty)$ with $\alpha = G(\beta)$.

We claim for some $c = c(p,n,M) \geq 1$ that

$$0 \leq \bar{\alpha} \leq \alpha(1 - \lambda\epsilon/c). \tag{4.32}$$

(4.30)-(4.32) easily lead to a contradiction for sufficiently small $\gamma = \gamma(p,n,M,N) > 0$. Here γ is as in Lemma 3.22. In fact, from (4.29), (4.31), and our assumption that $t \in \Gamma$ we see that $\bar{\alpha} \leq \alpha$ while $\beta \leq \bar{\beta}$. Using the assumptions on G in Theorem 1 and (4.32) we find that if $\bar{\beta} \neq 0$, then

$$\begin{aligned} G(\beta) \leq G(\bar{\beta}) \; &\leq \bar{\beta}^N \left(\tfrac{\bar{\beta}}{1+\epsilon|\nabla\phi_{\mu\lambda t}(\tilde{w})|} \right)^{-N} G\left(\tfrac{\bar{\beta}}{1+\epsilon|\nabla\phi_{\mu\lambda t}(\tilde{w})|} \right) \\ &\leq \frac{(1+\epsilon|\nabla\phi_{\mu\lambda t}(\tilde{w})|)^N}{1-\epsilon|\nabla\phi_{\mu\lambda t}(\tilde{w})|}\bar{\alpha} \leq \frac{(1+\epsilon\gamma\mu\lambda)^N}{1-\epsilon\gamma\mu\lambda}\bar{\alpha} < \alpha, \end{aligned} \tag{4.33}$$

provided γ is small enough, thanks to (4.32). If $\bar{\beta} = 0$, we can omit the second inequality in (4.33) and still get $G(0) = G(\beta) < \alpha$. Since $\alpha = G(\beta)$, we have reached a contradiction in either case.

To prove claim (4.32) let $0 < \delta_0 < \mu/2$, be a parameter to be chosen and let $\hat{v}$ be the p-harmonic function in $\Omega \cap B(0,1) \setminus \bar{B}(e_n/8, \rho)$ with continuous boundary values, $\hat{v} \equiv u$ on $\partial[\Omega \cap B(0,1)]$ and $\hat{v} = (1-\delta_0\lambda\epsilon)u$ on $\partial B(e_n/8,\rho)$. We note from Theorem 1 in [Li] that $\hat{v}$ has a $C^{1,\xi}$-extension to the closure of $B(e_n/8, 2\rho) \setminus B(e_n/8,\rho)$ for some small $\xi > 0$ with

$$|\nabla\hat{v}(x)| \leq cu(e_n/8)\rho^{-1} \leq c^2 u(x) \tag{4.34}$$

on $B(e_n/8,2\rho) \setminus \bar{B}(e_n/8,\rho)$, for $c = c(p,n)$ large enough. Next observe from the maximum principle for p-harmonic functions that

$$1/2 \leq 1 - \epsilon\delta_0\lambda \; \leq \; \frac{\hat{v}}{u} \leq 1 \tag{4.35}$$

in $\Omega \cap B(0,1) \setminus B(e_n/8,\rho)$. From (4.35), (4.4), and Lemma 2.5 with $u, \hat{v}$ playing the role of $\hat{v}, \hat{u}$, respectively, we deduce that if $\delta_0 = \delta_0(p,n,M) > 0$ is small enough, then for some $c = c(p,n,M) \geq 1$,

$$c^{-1}\frac{\hat{v}(x)}{d(x,\partial\Omega)} \leq |\nabla\hat{v}(x)| \leq c\frac{\hat{v}(x)}{d(x,\partial\Omega)} \tag{4.36}$$

whenever $x \in \Omega \cap B(0,9/10) \setminus B(e_n/8,2\rho)$. From (4.21) and the definition of Γ, we see that we can also choose δ_0 so small that

$$\tilde{v}_t \leq \hat{v} \tag{4.37}$$

on $\partial(D^+(\tilde{v}_t) \cap B(0,9/10))$. From Lemma 3.5 and the maximum principle for p-harmonic functions, we find that this inequality also holds in $D^+(\tilde{v}_t) \cap B(0,9/10)$. With δ_0 now fixed, let $\hat{L}$ be the elliptic operator in (1.11), with u', v' replaced by $u, \hat{v}$ and put $e = u - \hat{v}$. From (4.4), (4.34)-(4.36), and (1.12), (1.13), we conclude that $\hat{L}$ is uniformly elliptic in $\Omega \cap [\bar{B}(e_n/8,3\rho) \setminus B(e_n/8,\rho)]$ with bounded measurable coefficients. Constants are proportional to $u(e_n/8)^{p-2}$. Also, it follows from these displays that if $x \in B(0,9/10) \setminus B(e_n/8,3\rho)$, then $\hat{L}$ is uniformly elliptic with bounded coefficients in $B(x, d(x,\partial\Omega)/2)$. Constants are proportional to $(u(x)/d(x,\partial\Omega))^{p-2}$. We assert that

$$\lambda\epsilon/c \leq e/u \;\text{ on } \partial B(e_n/8,2\rho) \tag{4.38}$$

provided $c = c(p,n)$ is large enough. To prove (4.38) we compare boundary values of e, u, and argue as in the proof of Lemma 2.13. We omit the details. From (4.4), (4.34) - (4.38), Harnack's inequality for $e, u, \hat{v}$, and Theorem 2.22, we conclude that if $\tilde{c} = \tilde{c}(p, n, M) \geq 1$ is large enough, then

$$\frac{\epsilon\lambda}{\tilde{c}} \leq \frac{e}{u} \tag{4.39}$$

in $\Omega \cap B(\tilde{w}, 1/\tilde{c})$. Using (4.39), (4.37) we see that

$$\tilde{v}_t(y) \leq \hat{v}(y) \leq (1 - \epsilon\lambda/\tilde{c})u(y) \text{ whenever } y \in B(\hat{w}, \hat{\rho}) \cap B(\tilde{w}, 1/\tilde{c}). \tag{4.40}$$

Clearly, (4.40), (4.29), (4.31), imply (4.32). In view of our earlier remarks, the proof of Lemma 4.3 is now complete. □

Proof of Theorem 1. Theorem 1 follows from Lemma 4.3 and an iteration argument, as discussed in section 1 below Step 2. □

5 Regularity in a one-phase free boundary problem

Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and let $w \in \partial\Omega$, $0 < r < r_0$. To start with, recall that the definition of BMO was given below (2.7). If f is a vector valued function, $f = (f_1, .., f_n)$, then we set $f_\Delta = (f_{1,\Delta}, .., f_{n,\Delta})$ and we let the BMO-norm of f be defined as in (2.8) with $|f - f_\Delta|^2 = \langle f - f_\Delta, f - f_\Delta \rangle$. Moreover, we say that f is of vanishing mean oscillation on $\Delta(w, r) = \partial\Omega \cap B(w, r)$, written $f \in VMO(\Delta(w, r))$, provided for each $\epsilon > 0$ there is a $\delta > 0$ such that (2.8) holds with A replaced by ϵ whenever $0 < s < \min(\delta, r)$ and $x \in \Delta(w, r)$. Note that if $\hat{u}, \Omega, p, r, w$, are as in Theorem 2.9, then $\nabla\hat{u}$ exists as a non-tangential limit, σ almost everywhere on $\Delta(w, 4r)$, and $\log|\nabla\hat{u}| \in BMO(\Delta(w, r))$.

In this section we use Theorem 1 to prove the following theorem.

Theorem 5.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap \bar{B}(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Moreover, assume that $\log|\nabla u| \in VMO(\Delta(w, r))$. Then the outer unit normal to $\Delta(w, r)$ is in $VMO(\Delta(w, r/2))$.*

We let n denote the outer unit normal to $\partial\Omega$. To begin the proof of Theorem 5.1 we first outline the argument used in the proof of Theorem 3 in [LN2]. In particular, we define

$$\eta = \lim_{\tilde{r} \to 0} \sup_{\tilde{w} \in \Delta(w, r/2)} \|n\|_{BMO(\Delta(\tilde{w}, \tilde{r}))} \tag{5.2}$$

and we note that to prove Theorem 5.1 it is enough to prove that $\eta = 0$. To do this we argue by contradiction and assume that (5.2) holds for some $\eta > 0$. This assumption implies that there exist a sequence of points $\{w_j\}$, $w_j \in \Delta(w, r/2)$, and a sequence of scales $\{r_j\}$, $r_j \to 0$, such that $\|n\|_{BMO(\Delta(w_j, r_j))} \to \eta$ as $j \to \infty$. To establish a contradiction we then use a blow-up argument. In particular, let u be as in the statement of Theorem 5.1 and extend u to $B(w, 4r)$ by putting $u = 0$ in $B(w, 4r) \setminus \Omega$. For $\{w_j\}$, $\{r_j\}$ as above we define $\Omega_j = \{r_j^{-1}(x - w_j) : \ x \in \Omega\}$ and

$$u_j(z) = \lambda_j u(w_j + r_j z) \text{ whenever } z \in \Omega_j \tag{5.3}$$

where $\{\lambda_j\}$ is an appropriate sequence of real numbers defined as in [LN2, (4.21)]. It also follows from [LN2, section 4] that a subsequence of $\{\Omega_j\}$, converges to Ω_∞, in the Hausdorff distance sense, where Ω_∞ is an unbounded Lipschitz graph domain with Lipschitz constant bounded by M. Moreover, by the choice of the sequence $\{\lambda_j\}$ it follows that a subsequence of $\{u_j\}$ converges uniformly on compact subsets of $\mathbf{R}^n$ to u_∞, a positive p-harmonic function in Ω_∞, which is Hölder continuous in $\mathbf{R}^n$ with $u_\infty \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\infty$. Also,

$$\int\limits_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla \phi \rangle dx = - \int\limits_{\partial \Omega_\infty} \phi d\sigma_\infty \tag{5.4}$$

whenever $\phi \in C_0^\infty(\mathbf{R}^n)$ and

$$c^{-1} \leq |\nabla u_\infty(z)| \leq 1 \text{ whenever } z \in \Omega_\infty. \tag{5.5}$$

In (5.4), σ_∞ is surface measure on $\partial\Omega_\infty$ and c is a constant, $1 \leq c < \infty$, depending only on p, n and M. The proof of Theorem 5.1 now boils down to proving that (5.4) and (5.5) imply Ω_∞ is a halfspace. Indeed once this is shown, we can get a contradiction to our assumption that $\eta > 0$, by arguing as in [LN2, (4.42)].

In [LN2] we proved Ω_∞ is a halfspace using (5.4), (5.5) and a theorem of Alt, Caffarelli and Friedman [ACF]. In order to apply this theorem we needed a smallness assumption on the Lipschitz constant M of Ω. Here we show Theorem 1 implies this conclusion without any smallness assumption on M. Thus we prove,

Lemma 5.6. *Let $\Omega_\infty \subset \mathbf{R}^n$ be constructed as above and assume that (5.4), (5.5) hold. Then Ω_∞ is a half space.*

Proof. We intend to first show that u_∞ is a weak solution in $\mathbf{R}^n$, in the sense of Definition 1.4, to the free boundary problem in (1.3) with $u_\infty^- \equiv 0$ and $G(s) = 1 + s, s \in [0, \infty)$. To this end, assume $w \in F(u_\infty)$ and that there exists a ball $B(\hat{w}, \hat{\rho})$, $\hat{w} \in \mathbf{R}^n \setminus \partial\Omega_\infty$ and $\hat{\rho} > 0$, such that $w \in \partial B(\hat{w}, \hat{\rho})$. Let P be the plane through w with normal $\nu = (\hat{w} - w)/|\hat{w} - w|$. We claim that P is a tangent plane to Ω_∞ at w in the usual sense. That is given $\epsilon > 0$ there exists $\hat{r}(\epsilon) > 0$ such that

$$h(P \cap B(w, r), \partial\Omega_\infty \cap B(w, r)) \leq \epsilon r \tag{5.7}$$

whenever $0 < r \leq \hat{r}(\epsilon)$, where the Hausdorff distance, $h(\cdot, \cdot)$ between two sets $E, F \subset \mathbf{R}^n$ is defined by

$$h(E, F) = \max\left(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\}\right).$$

Once (5.7) is proved we can show that

$$\begin{aligned} &(i) \quad \text{if } B(\hat{w}, \hat{\rho}) \subset \Omega_\infty \text{ then } u_\infty^+(x) = \langle x - w, \nu \rangle + o(|x - w|) \text{ in } \Omega_\infty \\ &(ii) \quad \text{if } B(\hat{w}, \hat{\rho}) \subset \mathbf{R}^n \setminus \Omega_\infty \text{ then } u_\infty^+(x) = \langle w - x, \nu \rangle + o(|x - w|) \text{ in } \Omega_\infty. \end{aligned} \tag{5.8}$$

Indeed, to prove (5.8), we assume that $w = 0, \nu = e_n$, and $\hat{\rho} = 1$. This assumption is permissible since linear functions and the p-Laplacian are invariant under rotations, translations, and dilations. Then $\hat{w} = e_n$ and either $B(e_n, 1) \subset \Omega_\infty$ or $B(e_n, 1) \subset \mathbf{R}^n \setminus \bar{\Omega}_\infty$. In the proof that

(5.7) implies (5.8) we also assume that $B(e_n,1)\subset\Omega_\infty$, since the other possibility, $B(e_n,1)\subset \mathbf{R}^n\setminus\bar\Omega_\infty$, is handled similarly. Let $\{r_j\}$ be a sequence of positive numbers tending to 0 and let $\hat u_j(z)=u_\infty(r_jz)/r_j$ whenever $z\in\mathbf{R}^n$. Let $\hat\Omega_j=\{z: r_jz\in\Omega_\infty\}$ be the corresponding blow-up regions. Then $\hat u_j$ is p-harmonic in $\hat\Omega_j$ and Hölder continuous in $\mathbf{R}^n$ with $\hat u_j\equiv 0$ on $\mathbf{R}^n\setminus\hat\Omega_j$. Moreover, (5.5) is valid for each j with u_∞ replaced by $\hat u_j$. Using these facts, assumption (5.7), and Lemmas 2.1, 2.2 we see that a subsequence of $\{\hat u_j\}$, denoted $\{u_j'\}$, converges uniformly on compact subsets of $\mathbf{R}^n$, as $j\to\infty$, to a Hölder continuous function u_∞'. Moreover, u_∞' is a nonnegative p-harmonic function in $H=\{x: x_n>0\}$ with $u_\infty'\equiv 0$ on $\mathbf{R}^n\setminus H$.

Let $\{\Omega_j'\}$ be the subsequence of $\{\hat\Omega_j\}$ corresponding to $\{u_j'\}$. From (5.7) we see that $\Omega_j'\cap B(0,R)$ converges to $H\cap B(0,R)$ whenever $R>0$, in the sense of Hausdorff distance as $j\to\infty$. Finally we note that $\nabla u_j'\to\nabla u_\infty'$ uniformly on compact subsets of H and hence

$$c^{-1}\le|\nabla u_\infty'|\le 1 \tag{5.9}$$

where c is the constant in (5.5). Next we apply Theorem 2.3 with

$$\Omega=H, \hat u(x)=u_\infty'(x), \text{ and } \hat v(x)=x_n.$$

Letting $r\to\infty$ in Theorem 2.3 it follows that

$$u_\infty'(x)=lx_n \tag{5.10}$$

for some nonnegative l. From (5.9) and the above discussion we conclude that

$$c^{-1}\le l\le 1. \tag{5.11}$$

Next using (5.4) we see that if σ_j' is surface area on Ω_j', σ surface area on H, and $\phi\ge 0\in C_0^\infty(\mathbf{R}^n)$, then

$$\begin{aligned}\int\limits_{\partial\{u_j'>0\}}\phi\,d\sigma_j' &= -\int\limits_{\mathbf{R}^n}|\nabla u_j'|^{p-2}\langle\nabla u_j',\nabla\phi\rangle dx\\ &\to -\int\limits_{\mathbf{R}^n}|\nabla u_\infty'|^{p-2}\langle\nabla u_\infty',\nabla\phi\rangle dx = l^{p-1}\int\limits_{\{x_n=0\}}\phi\,d\sigma\end{aligned} \tag{5.12}$$

as $j\to\infty$. Moreover, using the divergence theorem we find that

$$\int\limits_{\partial\{u_j'>0\}}\phi\,d\sigma_j'\ge-\int\limits_{\{u_j'>0\}}\nabla\cdot(\phi e_n)dx\to-\int\limits_{\{u_\infty'>0\}}\nabla\cdot(\phi e_n)dx=\int\limits_{\{x_n=0\}}\phi\,d\sigma \tag{5.13}$$

as $j\to\infty$. Combining (5.12), (5.13) we obtain first that $l\ge 1$ and thereupon from (5.11) that $l=1$. Thus any blowup sequence of u_∞, relative to zero, tends to x_n^+ uniformly on compact subsets of $\mathbf{R}^n$, and the corresponding gradients tend uniformly to e_n on compact subsets of H. This conclusion is easily seen to imply (5.8). Hence (5.7) implies (5.8)(i).

Proof of (5.7). The proof of (5.7) is by contradiction. We continue under the assumption that $w = 0, \nu = \hat{w} = e_n$, and $\hat{\rho} = 1$. First suppose that

$$B(e_n, 1) \subset \Omega_\infty. \tag{5.14}$$

If (5.7) is false, then there exists a sequence $\{s_m\}$ of positive numbers and $\delta > 0$ with $\lim_{m \longrightarrow \infty} s_m = 0$ and the property that

$$\Omega_\infty \cap \partial B(0, s_m) \cap \{x : x_n \leq -\delta s_m\} \neq \emptyset \tag{5.15}$$

for each m. To get a contradiction we show that (5.15) leads to

$$\limsup_{t \longrightarrow 0} t^{-1} u_\infty(t e_n) = \infty \tag{5.16}$$

which in view of the mean value theorem from elementary calculus, contradicts (5.5). For this purpose let f be the p-harmonic function in $B(e_n, 1) \setminus \bar{B}(e_n, 1/2)$ with continuous boundary values,

$$f \equiv 0 \text{ on } \partial B(e_n, 1) \text{ and } f \equiv \min_{\bar{B}(e_n, 1/2)} u_\infty \text{ on } \partial B(e_n, 1/2).$$

Now f can be written explicitly in the form,

$$f(x) = \begin{cases} A|x - e_n|^{(p-n)/(p-1)} + B \text{ when } p \neq n, \\ -A \log|x - e_n| + B \text{ when } p = n, \end{cases}$$

where A, B are constants. Doing this we see that

$$\lim_{t \longrightarrow 0} t^{-1} f(t e_n) > 0. \tag{5.17}$$

From the maximum principle for p-harmonic functions we also have

$$u_\infty \geq f \text{ in } B(e_n, 1) \setminus \bar{B}(e_n, 1/2). \tag{5.18}$$

Next we show that if $0 < s < 1/4$, and $u_\infty \geq kf$ in $\bar{B}(0, s) \cap B(e_n, 1)$, for some $k \geq 1$, then there exists $\xi = \xi(p, n, M, \delta) > 0$ and $s', 0 < s' < s/2$, such that

$$u_\infty \geq (1 + \xi) k f \text{ in } \bar{B}(0, s') \cap B(e_n, 1). \tag{5.19}$$

Clearly (5.17) - (5.19) and an iterative argument yield (5.16).

To prove (5.19) we observe from a direct calculation or Lemma 2.5 in [LN] that

$$|\nabla f(x)| \approx f(x)/(1 - |x - e_n|) \text{ when } x \in B(e_n, 1) \setminus \bar{B}(e_n, 1/2), \tag{5.20}$$

where proportionality constants depend only on p, n. Also, we observe from (5.15) and Lipschitzness of $\partial \Omega_\infty$ that if m_0 is large enough, then there exists a sequence of points $\{t_l\}_{m_0}^\infty$ in $\Omega_\infty \cap \{x : x_n = 0\}$ and $\eta = \eta(p, n, M, \delta) > 0$ such that for $l \geq m_0$,

$$\eta s_l \leq |t_l| \leq \eta^{-1} s_l \text{ and } d(t_l, \partial \Omega_\infty) \geq \eta |t_l|. \tag{5.21}$$

Choose $t_m \in \{t_l\}_{m_0}^{\infty}$ such that $\eta^{-1}|t_m| \leq s/100$. If $\rho = d(t_m, \partial\Omega_\infty)$, then from (5.21), Lemma 2.1 for u_∞ and Harnack's inequality we deduce for some $C = C(p, n, M, \delta) \geq 1$ that

$$C u_\infty(t_m) \geq \max_{\bar{B}(0,4|t_m|)} u_\infty. \tag{5.22}$$

From (5.22), the assumption that $kf \leq u_\infty$, Lemma 2.1 for kf, and the fact that t_m lies in the tangent plane to $B(e_n, 1)$ through 0, we see there exists $\lambda = \lambda(p, n, M, \delta), 0 < \lambda \leq 10^{-2}$, and $m_1 \geq m_0$ such that if $m \geq m_1$ and $t'_m = t_m + 3\lambda\rho e_n$, then

$$B(t'_m, 2\rho\lambda) \subset B(e_n, 1) \text{ and } (1+\lambda)kf \leq u_\infty \text{ on } \bar{B}(t'_m, \rho\lambda). \tag{5.23}$$

Let $\tilde{f}$ be the p-harmonic function in $G = B(0, 4|t_m|) \cap B(e_n, 1) \setminus \bar{B}(t'_m, \rho\lambda)$ with continuous boundary values $\tilde{f} = kf$ on $\partial[B(e_n, 1) \cap B(0, 4|t_m|)]$ while $\tilde{f} = (1+\mu)kf$ on $\partial B(t'_m, \rho\lambda)$. If $0 < \mu \leq \lambda$ and μ is small enough, depending on p, n, M, δ, then we can use (5.20), Theorem 2.22, and Harnack's inequality for $\tilde{f} - kf, kf$ as in the proof of (4.40), in order to deduce the existence of $\tau > 0, \bar{c} \geq 1$, with

$$(1 + \tau\mu)kf \leq \tilde{f} \tag{5.24}$$

in $B(e_n, 1) \cap \bar{B}(0, |t_m|/\bar{c})$ where $\tau = \tau(p, n, M, \delta), 0 < \tau < 1$, and $\bar{c} = \bar{c}(p, n, M) \geq 1$. Moreover, using the maximum principle for p-harmonic functions we see from (5.23) that

$$\tilde{f} \leq u_\infty \text{ in } G. \tag{5.25}$$

Combining (5.24), (5.25), we get (5.19) with $\xi = \tau\mu$ and $s' = |t_m|/\bar{c}$. As mentioned earlier, (5.19) leads to a contradiction. Hence (5.7) is true when (5.14) holds.

If $B(e_n, 1) \subset \mathbf{R}^n \setminus \bar{\Omega}_\infty$ we proceed similarly. That is, if (5.7) is false, then there exists a sequence $\{s_m\}$ of positive numbers and $\delta > 0$ with $\lim\limits_{m\to\infty} s_m = 0$ and the property that

$$\mathbf{R}^n \setminus \bar{\Omega}_\infty \cap \partial B(0, s_m) \cap \{x : x_n \leq -\delta s_m\} \neq \emptyset \tag{5.26}$$

for each m. To get a contradiction we show that (5.26) leads to

$$\liminf_{t\to 0} t^{-1} \max_{B(0,t)} u_\infty = 0 \tag{5.27}$$

which in view of Lipschitzness of Ω_∞ and the mean value theorem from elementary calculus, contradicts (5.5). For this purpose let g be the p-harmonic function in $B(0, 2) \setminus \bar{B}(e_n, 1)$ with continuous boundary values,

$$g \equiv 0 \text{ on } \partial B(e_n, 1) \text{ and } g \equiv \max_{\bar{B}(0,2)} u_\infty \text{ on } \partial B(0, 2).$$

Then $u_\infty \leq g$ in $B(0, 2) \setminus \bar{B}(e_n, 1)$ and

$$\limsup_{t\to 0} t^{-1} \max_{B(0,t)\setminus\bar{B}(e_n,1)} g < \infty. \tag{5.28}$$

(5.28) can be proved for example by comparison with functions of the type used to define f above (5.17). In analogy with (5.19) we show that if $0 < s < 1/4$, and $u_\infty \leq kg$ in $\Omega_\infty \cap B(0, s)$, for some $0 < k \leq 1$, then there exists $\xi' = \xi'(p, n, M, \delta) > 0$ and $s', 0 < s' < s/2$, such that

$$u_\infty \leq (1 - \xi')kg \text{ in } \Omega_\infty \cap B(0, s'). \tag{5.29}$$

To prove (5.29) we argue as in (5.21) to get $t_m \in (\mathbf{R}^n \setminus \bar{\Omega}_\infty) \cap \{x : x_n = 0\}$ and $\eta' = \eta'(p, n, M, \delta) > 0$, with $|t_m| \leq s\eta'/100$ while $\rho' = d(t_m, \partial\Omega_\infty)/8 \geq \eta'|t_m|$. From Lemma 2.1 and Theorem 2.4 we may also suppose $|t_m|$ is so small that

$$|\nabla g(x)| \approx g(x)/d(x, \partial B(e_n, 1)) \text{ when } x \in B(0, 4|t_m|) \setminus \bar{B}(e_n, 1). \tag{5.30}$$

If $t'_m = t_m - 3\rho' e_n$, then

$$B(t'_m, 2\rho') \subset \mathbf{R}^n \setminus \bar{\Omega}_\infty \text{ and } C'g \geq \max_{B(0,4|t_m|)\setminus B(e_n,1)} g \text{ on } B(t'_m, \rho'). \tag{5.31}$$

Here $C' = C'(p, n, \delta) \geq 1$. Let $\tilde{g}$ be the p-harmonic function in $G = B(0, 4|t_m|) \setminus [\bar{B}(e_n, 1) \cup \bar{B}(t'_m, \rho')]$ with continuous boundary values $\tilde{g} = kg$ on $\partial[B(0, 4|t_m|) \setminus \bar{B}(e_n, 1)]$ while $\tilde{g} = (1 - \mu')kg$ on $\partial B(t'_m, \rho')$. Using (5.30), (5.31), and arguing as in the proof of (5.24), we get

$$\tilde{g} \leq (1 - \tau'\mu')kg \tag{5.32}$$

in $B(0, |t_m|/c')$ where τ', μ', c', depend only on p, n, M, δ. Since $u_\infty \leq \tilde{g}$ on ∂G we see from the boundary maximum principle for p-harmonic functions, that (5.32) holds with $\tilde{g}$ replaced by u_∞. Thus (5.29) is true with $s' = |t_m|/c', \xi' = \tau'\mu'$. From (5.29), (5.28), and iteration we get (5.27) which leads to a contradiction. Hence (5.7) is true in all cases. From (5.7) and our earlier remarks we now get (5.8). Thus u_∞ satisfies the tangent ball condition in Definition 1.4.

To complete the proof of Lemma 5.6 we assume, as we may, that $\partial\Omega_\infty = \{(x', x_n) : x_n > \psi_\infty(x')\}$, where $\psi_\infty : \mathbf{R}^{n-1} \to \mathbf{R}$ is Lipschitz with Lipschitz norm $\leq M$, since otherwise we change coordinates and use invariance of the p-Laplacian under rotations. Thanks to (5.8), we can now apply Theorem 1 with $u(x) = u_\infty(Rx)/R, x \in B(0, 2)$, and with $\partial\Omega \cap B(0, 2) = \{(x', \psi_\infty(Rx')/R)\} \cap B(0, 2)$. Doing this, we find that

$$\sup_{\{x' :\ (x', \psi_\infty(x')) \in B(0, R/8)\}} |\nabla\psi_\infty(x') - \nabla\psi_\infty(0)| \leq c|x'|^\sigma R^{-\sigma} \tag{5.33}$$

where $c = c(p, n, M)$. Letting $R \to \infty$ we see that $\nabla\psi_\infty \equiv \nabla\psi_\infty(0)$. Thus ψ_∞ is linear and consequently Ω_∞ is a halfspace. This completes the proof of Lemma 5.6. In view of the discussion above Lemma 5.6, we also conclude the validity of Theorem 5.1. $\square$

6 Concluding Remarks

We note that with minor modifications in the proof of Theorem 1 one can get, see the end of [C1] and [W], the following generalization of Theorem 1.

Theorem 2. *Let $D \subset \mathbf{R}^n$ be a bounded domain, assume that $u \in C(\bar{D})$ and assume that u is a solution in D, for some $1 < p < \infty$, to the problem in (1.3) in the sense of Definition 1.4 with $G(\cdot)$ replaced by $G(\cdot, w, \nu)$ where $G : [0, \infty) \times \mathbf{R}^n \times \mathbf{S}^{n-1} \to (0, \infty)$. Assume that the function G satisfies the following conditions.*

- (i) $\log G(s, w, \nu)$ *is Lipschitz in w and ν with Lipschitz constant independent of s.*
- (ii) $G(s, w, \nu)$ *is, for fixed w and ν, strictly increasing in s and $s^{-N}G(s, w, \nu)$ is decreasing in s on $(0, \infty)$ for some $N > 0$ which is independent of w and ν.*

Then the statement and conclusion of Theorem 1 remain true.

References

[ACF] H.W. Alt, L. Caffarelli and A. Friedman, *A free boundary problem for quasilinear elliptic equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), no.1, 1-44.

[C1] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. Part I, Lipschitz free boundaries are* $C^{1,\alpha}$, Revista Math. Iberoamericana **3** (1987), 139-162.

[C2] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz,* Comm. Pure Appl. Math. **42** (1989), no. 1, 55–78.

[C3] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on* X, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), no. 4, 583–602.

[CC] X. Cabre and L. Caffarelli, *Fully non-linear elliptic equations*, American Mathematical Society Colloquium Publications, **43**, American Math Soc, Providence, RI, 1995.

[CFMS] L. Caffarelli, E. Fabes, S. Mortola, S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (4) (1981) 621-640.

[CFS] M.C. Cerutti, F. Ferrari, S. Salsa, *Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are* $C^{1,\gamma}$, Arch. Ration. Mech. Anal. 171 (2004), 329-448.

[F] M. Feldman, *Regularity for two nonisotropic two phase problems with Lipschitz free boundaries*, Diff. Int. Eqns. 10 (1997), 227-251.

[F1] M. Feldman, *Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations*, Indiana Univ. Math. J. **50** (2001), no. 3, 1171-1200.

[Fe] F. Ferrari, *Two- phase problems for a class of fully nonlinear elliptic operators, Lipschitz free boundaries are* $C^{1,\gamma}$, Amer. J. Math. 128 (2006), no.3, 541-571.

[FS] F. Ferrari and S. Salsa, *Regularity of the free boundary in two-phase problems for linear elliptic operators*, Adv. Math. 214 (2007), 288 - 322.

[FS1] F. Ferrari and S. Salsa, *Subsolutions of elliptic operators in divergence form and applications to two - phase free boundary problems*, Bound. Value. Probl. (2007), Art. ID 57049, 21pp.

[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations,* Dover Publications, 2006.

[HL] S. Hofmann and J. Lewis, *The Dirichlet problem for parabolic operators with singular drift term,* Mem. Amer. Math. Soc. **151** (2001), no. 719, 1- 113.

[KP] C. Kenig and J. Pipher, *The Dirichlet problem for elliptic operators with drift term,* Publ. Mat. **45** (2001), no. 1, 199-217.

[L] J. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations,* Indiana Univ. Math. J. **32** (6) (1983), 849-858.

[Li] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations,* Non-Linear Anal. **12** (1988), no. 11, 1203-1219.

[LN] J. Lewis and K. Nyström, *Boundary behaviour for p-harmonic functions in Lipschitz and starlike Lipschitz ring domains,* Ann. Sc. École Norm. Sup. (4) **40** (2007), no. 4, 765-813.

[LN1] J. Lewis and K. Nyström, *Boundary behaviour and the Martin boundary problem for p-harmonic functions in Lipschitz domains,* to appear in Annals of Mathematics.

[LN2] J. Lewis and K. Nyström, *Regularity and free boundary regularity for the p-Laplacian in Lipschitz and C^1-domains,* Ann. Acad. Sci. Fenn. **33** (2008), 523-548.

[LN3] J. Lewis and K. Nyström, *New results on p-harmonic functions,* to appear in Pure and Applied Math Quarterly.

[LN4] J. Lewis and K. Nyström, *Boundary behaviour of p-harmonic functions in domains beyond Lipschitz domains,* Advances in the Calculus of Variations **1** (2008), 133-170.

[LN5] J. Lewis, N. Lundström and K. Nyström, *Boundary Harnack inequalities for operators of p-Laplace type in Reifenberg flat domains,* to appear in Perspectives in PDE, Harmonic Analysis, and Applications, Proceedings of Symposia in Pure Mathematics, Dorina and Marius Mitrea, editors.

[LSW] W. Littman, G. Stampacchia, and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients,* Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 43-77.

[W] P. Wang, *Regularity of free boundaries of two-phase problems for fully non-linear elliptic equations of second order. Part 1: Lipschitz free boundaries are $C^{1,\alpha}$*, Communications on Pure and Applied Mathematics. **53** (2000), 799-810.