# $L^2$ Solvability and Representation by Caloric Layer Potentials in Time-Varying Domains

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**ABSTRACT:** We consider boundary value problems for the heat equation in time varying graph domains of the form $\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > A(x,t)\}$, obtaining solvability of the Dirichlet and Neumann problems when the data lies in $L^2(\partial\Omega)$. We also prove optimal regularity estimates for solutions to the Dirichlet problem when the data lies in a parabolic Sobolev space of functions having a tangential (spatial) gradient, and 1/2 of a time derivative in $L^2(\partial\Omega)$. Furthermore, we obtain representations of our solutions as caloric layer potentials. We prove these results for functions $A(x,t)$ satisfying a minimal regularity condition which is essentially sharp from the point of view of the related singular integral theory. We construct counter examples which show that our results are in the nature of " best possible. "

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**0. Background and Notation.** A longstanding problem concerning solvability of the Dirichlet problem for Laplace's equation in a Lipschitz domain was resolved by B. Dahlberg [D1], who showed that in such domains harmonic measure, $d\omega$, and surface measure, $d\sigma$, are mutually absolutely continuous, and furthermore, that the Dirichlet problem is solvable with data in $L^2(d\sigma)$ (and consequently with data in $L^p, 2-\epsilon < p < \infty$). R. Hunt proposed the problem of finding an analogue of Dahlberg's result for the heat equation in domains whose boundaries are given locally as graphs of functions $A(x,t)$ which are Lipschitz in the space variable. It was conjectured at one time that $A$ should be $\mathrm{Lip}_{\frac{1}{2}}$ in the time variable, but subsequent counterexamples of Kaufmann and Wu [KW] showed that this condition does not suffice. Motivated in part by work of Strichartz [Stz] on BMO Sobolev spaces, and in part by work of M. Murray [Mu], Lewis and Murray [LM], made significant progress toward a solution of Hunt's question, by establishing mutual absolute continuity of caloric measure and a certain parabolic analogue of surface measure in the case that $A$ has $\frac{1}{2}$ of a time derivative in $BMO(\mathbb{R}^n)$ on rectangles, a condition only slightly stronger than $\mathrm{Lip}_{\frac{1}{2}}$. Furthermore these authors obtained solvability of the Dirichlet problem with data in $L^p$, for $p$ sufficiently large, but unspecified. The regularity condition which Lewis and Murray imposed upon $A(x,t)$ (or, to be more precise, an equivalent formulation of it) was shown by the first named author to be necessary and sufficient for $L^2$ boundedness of the first parabolic Calderón commutator, thus further clarifying the connection between the results of [LM] and those of [D1]. Still, by analogy to [D1], it remained an open problem to treat the case of boundary value problems with $L^2$ data in the parabolic setting. It is this issue of $L^2$ solvability that we address here.

To be more specific in this paper we study the Dirichlet and Neumann problems for the heat equation in non cylindrical (i.e. time-varying) graph domains. We treat each of these problems in the case that the data belongs to $L^2$ with respect to a certain projective Lebesgue measure. We also consider regularity estimates for solutions of the Dirichlet problem when the data belongs to a parabolic Sobolev space having a full spatial derivative and one half of a time derivative in $L^2$. Existence of our solutions will be obtained by using the method of layer potentials. In addition we shall give an alternate, simpler proof of recent results of the first author [H2] concerning " smoothing operators of Calderòn type, " including the caloric single layer potential.

We shall study these problems in graph domains of the form

$$\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > A(x,t)\,\} \tag{0.1}$$

where $n \geq 2$ and $A(x,t)$ is Lipschitz in the space variable, uniformly in time, i.e.,

$$|A(x,t) - A(y,t)| \leq \beta_0\,|x-y|,\; x, y \in \mathbb{R}^{n-1},\; t \in \mathbb{R}, \tag{0.2}$$

and where $A(x,t)$ satisfies a certain half order smoothness condition in the time variable. To describe this condition we follow Fabes and Riviere [FR1] and define a half-order time derivative by

$$\mathbb{D}_n A(x,t) = \left(\frac{\tau}{\|(\xi,\tau)\|}\hat{A}(\xi,\tau)\right)^{\check{}}(x,t) \tag{0.3}$$

where $\hat{}$ and $\check{}$ denote respectively the Fourier and inverse Fourier transforms on $\mathbb{R}^n$, and $\xi, \tau$ denote, respectively, the space and time variables on the Fourier transform side. Also $\|z\|$ denotes the parabolic " norm " of $z$. We recall that this " norm " satisfies the non-isotropic dilation invariance property $\|(\delta x, \delta^2 t)\| \equiv \delta\|(x,t)\|$. Indeed, $\|(x,t)\|$ is defined as the unique positive solution $\rho$ of the equation

$$\sum_{i=1}^{n-1} \frac{x_i^2}{\rho^2} + \frac{t^2}{\rho^4} = 1. \tag{0.4}$$

The half order smoothness condition in the time variable which we impose upon $A$ is that $\mathbb{D}_n A \in$ (parabolic) BMO. We recall that parabolic BMO is the space of all locally integrable functions modulo constants satisfying

$$\|b\|_* \equiv \sup_B \frac{1}{|B|} \int_B |b(z) - m_B b| \, dz \, < \, \infty. \tag{0.5}$$

Here, $z = (x,t)$ and $B$ denotes the parabolic ball

$$B \equiv B_r(z_0) \equiv \{z \in \mathbb{R}^n : \|z - z_0\| < r\} \tag{0.6}$$

where $|B|$ denotes the Lesbegue $n$ measure of $B$ and

$$m_B b \equiv \frac{1}{|B|} \int_B b(z) dz.$$

We note that $|B_r(z_0)| \equiv c r^d$ where $c$ is a constant and $d = n+1$ is the homogeneous dimension of $\mathbb{R}^n$ endowed with the metric induced by $\| \cdot \|$. We observe that $\mathbb{R}^n$ so endowed is a space of homogeneous type in the sense of Coifman and Weiss [CW]. Indeed, there is a polar decomposition

$$\begin{aligned} z &\equiv (x,t) \equiv \, (\rho\theta_1, \ldots, \rho\theta_{n-1}, \rho^2\theta_n), \\ dz &\equiv dxdt \equiv \rho^{d-1}(1+\theta_n^2) d\rho \, d\theta \end{aligned} \tag{0.7}$$

where $\theta = (\theta_1, \ldots, \theta_n)$, $|\theta| = 1$, and $d\theta$ denotes surface area on the unit sphere.

Throughout this paper $L^p(\mathbb{R}^{n-1}), 1 < p < \infty$, denotes, as usual, the space of $p$ th power integrable functions $f$ on $\mathbb{R}^{n-1}$ with norm, $\|f\|_p$ . To explain the significance of the conditions which we have imposed upon $A$, we recall a result of the first author [H1], which states that

$$\left\| \left[ \sqrt{\Delta - \frac{\partial}{\partial t}} \, , \, A \right] \right\|_{op} \approx \, \|\nabla_x A\|_\infty + \|\mathbb{D}_n A\|_*,$$

where $\approx$ means the two quantities are bounded by constant multiples of each other. Moreover, $\| \cdot \|$ denotes the operator norm on $L^2(\mathbb{R}^{n-1})$, and

$$\nabla_x \equiv \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right). \tag{0.8}$$

Finally $\left[\sqrt{\Delta - \frac{\partial}{\partial t}}\,,\,A\right]$ denotes the commutator with multplication by $A$ of the square root of the heat operator in $\mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}$ defined by

$$\left(\sqrt{\Delta - \partial/\partial t}\, f\right)^{\widehat{}}(\xi, \tau) \equiv c\sqrt{|\xi|^2 - i\tau}\, \hat{f}(\xi, \tau).$$

Since this commutator is the parabolic analogue of the first Calderón commutator, the condition

$$\|A\|_{\text{comm}} \equiv \|\nabla_x A\|_\infty + \|\mathbb{D}_n A\|_* \leq \beta < \infty \tag{0.9}$$

is, at least from the standpoint of singular integral theory, the parabolic analogue of the Lipschitz condition which has proved to be of fundamental importance in elliptic theory ( see e.g. Calderón [Ca1], [Ca2], Coifman, McIntosh, and Meyer [CMM], Dahlberg [D1], Jerison and Kenig [JK1], Kenig [K], Verchota [V], and Dahlberg and Kenig [DK 1, 2], to name just a few . In [H1] it is shown that (0.9) implies the parabolic Lipschitz condition:

$$|A(x,t) - A(y,s)| \leq c\beta\, \|(x,t) - (y,s)\| \approx c\beta\, (|x-y| + |t-s|^{1/2}). \tag{0.10}$$

Thus in analogy to the elliptic case, it is natural to conjecture that (0.9) defines essentially the minimal amount of regularity needed in a graph domain

$$\Omega = \{(x_0, x, t) : x_0 > A(x,t),\ (x,t) \in \mathbb{R}^n\} \tag{0.11}$$

to obtain solvability of our boundary value problems with data in $L^2(\partial\Omega)$, as well as establish the mutual absolute continuity of parabolic measure and a certain " surface measure " (all terms will be defined below). Indeed, as mentioned above, it follows from the work of Kaufman and Wu [KW] (see also [LS]) that there exist domains $\subset \mathbb{R}^2$ whose boundaries are graphs of functions $\mathrm{Lip}_{1/2}$ in the time variable (this condition is only slightly weaker than ours) and such that the corresponding caloric measure fails to be absolutely continuous with respect to the " surface measure " on $\partial\Omega$

mentioned above. By "surface measure " on $\partial\Omega$ we mean the measure

$$d\sigma_t(Q)dt \tag{0.12}$$

where for fixed $t > 0$, $d\sigma_t$ denotes true $n-1$ dimensional surface measure on the boundary of the cross-section

$$\Omega_t \equiv \{(x_0, x, t) \in I\!R \times I\!R^{n-1} \times \{t\} : x_0 > A(x,t)\,\}.$$

Define $L^p(\partial\Omega), 1 < p < \infty$, to be equivalence classes of functions which are $p$ th power integrable and measurable with respect to the " surface measure " defined in (0.12). We shall also denote the norm in $L^p(\partial\Omega)$ by $\|\cdot\|_p$ when there is no chance for confusion.

We now state our boundary value problems. To solve the Dirichlet problem, for given $g \in L^p(\partial\Omega), p$ fixed, $1 < p < \infty$, we seek a function $u$ such that

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = 0 \text{ in } \Omega, \\ \\ u = g \text{ a.e on } \partial\Omega. \end{cases} \tag{0.13}$$

In (0.13) a.e means almost everywhere with respect to the measure in (0.12). For fixed $t, 1 < t < \infty$, let $n_t$ denote the outer unit normal to $\partial\Omega_t$ considered as a subset of $I\!R \times I\!R^{n-1}$. To solve the Neumann problem for given $g \in L^p(\partial\Omega)$ we seek a function $u$ such that

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = 0, \\ \\ \frac{\partial u}{\partial n_t} = g, \text{ a.e on } \partial\Omega_t, \text{ for } -\infty < t < \infty. \end{cases} \tag{0.14}$$

In the case of cylindrical domains(i.e. $A(x,t) \equiv A(x)$) the Dirichlet problem (0.13) was solved by Fabes and Salsa [FS], in the optimal range $p > 2 - \epsilon$ where $\epsilon = \epsilon(\|\nabla_x A\|_\infty)$ by using the relationship between caloric measure and Green's function for the heat (also adjoint heat) equation. The Neumann problem (0.14) was solved by Brown [BR1, the case $p = 2$ ] and [BR2, $1 < p \leq 2 + \epsilon$] using the method of layer potentials. Brown also obtained (again in the cylinder case) optimal regularity estimates for solutions to the Dirichlet problem (0.13), when the boundary data $g \in L^p_{1,\frac{1}{2}}$, [BR1, $p = 2$], [BR2, $1 < p \leq 2 + \epsilon$]. Following Fabes and Jodeit [FJ] we define the parabolic Sobolev space $L^p_{1,\frac{1}{2}}(\partial\Omega)$ as follows: Let $\pi : \partial\Omega \to \mathbb{R}^n$ be the projection $\pi(A(x,t),x,t) = (x,t)$ and set $\tilde{f} \equiv f \circ \pi^{-1}$. Then $L^p_{1,\frac{1}{2}}(\partial\Omega)$ consists of equivalence classes of functions $f$ with distributional derivatives in $x$ satisfying $||f||_{L^p_{1,\frac{1}{2}}(\partial\Omega)} < \infty$, where

$$||f||_{L^p_{1,\frac{1}{2}}(\partial\Omega)} \equiv ||\tilde{f}||_{L^p_{1,\frac{1}{2}}(\mathbb{R}^n)} \equiv \|\mathbb{D}\tilde{f}\|_p. \tag{0.15}$$

Here,

$$\begin{aligned} \left(\mathbb{D}\tilde{f}\right)\hat{}\,(\xi,\tau) &\equiv \|\xi,\tau\|\,\hat{f}(\xi,\tau), \\ i.e.,\ \tilde{f} \ &= \ \mathbb{D}^{-1}\phi,\ \phi \in L^p(\mathbb{R}^n), \end{aligned} \tag{0.16}$$

where $\mathbb{D}^{-1}$ is a parabolic Riesz potential. If $p = 2$ in (0.15), then from Plancherel's theorem we see that

$$\|\mathbb{D}\tilde{f}\|_2 \approx \|D^t_{1/2}\tilde{f}\|_2 + \|\nabla_x\tilde{f}\|_2 \tag{0.17 }$$

where $D^t_{1/2}$ denotes the one dimensional 1/2 fractional derivative of $f$ in the time variable. Recall that if $0 < \alpha \leq 2$, then for $g \in C^\infty_0(\mathbb{R})$ the one dimensional fractional differentiation operators $D_\alpha$ are defined by

$$\widehat{(D_\alpha g)}(\tau) \equiv |\tau|^\alpha\,\hat{g}(\tau).$$

It is well known that if $0 < \alpha < 1$, then

$$D_\alpha g(s) \equiv c\int_{\mathbb{R}} \frac{g(s) - g(\tau)}{|s - t|^{1+\alpha}}\,d\tau$$

whenever $s \in \mathbb{R}$ and $I_\alpha = \ c\,D_\alpha^{-1}$ where $I_\alpha(s) = |s|^{\alpha-1}$, $s \in \mathbb{R}$, is the one dimensional Riesz transform of order $\alpha$. If $h \in C^\infty_0(\mathbb{R}^n)$, then by $D^t_\alpha h : \mathbb{R}^n \to \mathbb{R}$ we mean the function $D_\alpha h(x,\cdot)$ defined a.e. for each fixed $x \in \mathbb{R}^{n-1}$.

In the case of time - varying graph domains much less is known about the above boundary value problems. As outlined in the beginning, recently, in [LM, ch 3, Thm 2] it is shown that if $A$ satisfies (0.2) and

$$\|D^t_{1/2} A\|_* \leq \beta_1 < \infty, \tag{0.18}$$

then parabolic measure on $\partial\Omega$ evaluated at certain points in $\Omega$ and the measure in (0.12) are $A_\infty$ weights with respect to each other (see section 2 for a definition of $A_\infty$) with constants depending only on $n$ and $\|\nabla_x A\|_\infty + \|D^t_{1/2} A\|_*$ . Using this fact and the relationship between parabolic measure and Green's function for the heat - adjoint heat equations, it is easily seen that the Dirichlet problem (0.13) has a solution for large $p$, say $p \geq p_0$, where $p_0$ depends on $\|\nabla_x A\|_\infty + \|D^t_{1/2} A\|_*$. In fact these authors show [LM, ch 3, Thm 2] that if (0.2) is weakened to $\nabla_x A \in$ parabolic BMO (componentwise), then these measures remain $A_\infty$ weights with respect to each other which is a parabolic analogue of a result of Jerison and Kenig [JK2, Thm 10.1] for " BMO $_1$ " domains.

To illustrate the relationship of the present work to that of [LM], we note that in section 8 we shall show

$$\|D^t_{1/2} A\|_* + \|\nabla_x A\|_\infty \approx \|\mathbb{D}_n A\|_* + \|\nabla_x A\|_\infty . \tag{0.19}$$

Thus the above measures are also $A_\infty$ weights with respect to each other when (0.9) holds. To further illustrate the connection between this paper and [LM], it shall also be important to know that the right and left hand sides of (0.19) are equivalent in the small in the following sense : Given $\epsilon > 0, 0 < \epsilon < 1$, and $\gamma, 0 < \gamma < \infty$, there exists $\delta = \delta(\epsilon, \gamma) > 0$ such that if $\|\nabla_x A\|_\infty \leq \gamma < \infty$, then

$$\min\left\{\|D^t_{1/2} A\|_*, \, \|\mathbb{D}_n A\|_*\right\} \leq \delta \implies \max\left\{\|D^t_{1/2} A\|_*, \, \|\mathbb{D}_n A\|_*\right\} \leq \epsilon. \tag{0.20}$$

This fact will be proved in section 8.

In view of the results for cylinders, the work of [LM] still leaves several questions unanswered in time - varying domains: For example, if $A$ satisfies (0.9),

(a) Can the Dirichlet problem (0.13) be solved for $p \geq 2 - \epsilon$?

(b) Can the Neumann problem be solved for any $p$? If so is it solvable for $1 < p \leq 2 + \epsilon$?

(c) If solutions exist to the Dirichlet and Neumann problems for some $p$, can one obtain a representation for the solutions in terms of layer potentials?

The method of layer potentials used in [LM, see Thms 5, 6] and (0.20) shows that the answer to all these questions is yes (in fact for $1 < p < \infty$) provided $\beta$ in (0.9) is small enough (see Theorems 1.13 - 1.15 in section 1 for an exact statement of these results).

In this paper we answer the above questions when $p = 2$, $0 < \|\nabla_x A\|_\infty < \infty$, provided

$$\|\mathbb{D}_n A\|_* \leq \epsilon_0 \tag{0.21}$$

and $\epsilon_0 = \epsilon_0(\|\nabla_x A\|_\infty) > 0$ is sufficiently small. In view of (0.20), it would be equivalent to take $\|D^t_{1/2} A\|_* \leq \delta_0$. Thus our results will remove the smallness assumption on $\|\nabla_x A\|_\infty$ when $p = 2$ in the work of [LM]. To do this will require a large amount of effort on our part. A precise statement of our results is given in section 1.

We close this section by introducing some more notation which will be used throughout this paper. In (0.8) we defined $\nabla_x$, the gradient in the $x$ variable when $x \in \mathbb{R}^{n-1}$. The gradient in $X = (x_0, x)$ will be denoted by

$$\nabla = \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}} \right).$$

In sections 2-4, we shall often denote points in $\mathbb{R}^n$ by $z = (x, t)$ and the usual inner product in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$ by $\langle \cdot, \cdot \rangle$. In these sections we shall also frequently find it convenient to use the notational convention $T(z, v)$, $z, v \in \mathbb{R}^n$ (or simply $T(z)$ in the convolution case), to denote the kernel of an operator $T$. Throughout this paper, $\alpha$, will denote the $n$ dimensional multi-index $\alpha \equiv (1, \ldots, 1, 2)$ so that if $z = (x, t)$, then

$$\lambda^{\alpha} z \equiv (\lambda x,\ \lambda^2 t)$$

$$\lambda^{-\alpha} z \equiv (\tfrac{x}{\lambda}, \tfrac{t}{\lambda^2}).$$

We define a parabolic approximate identity(which will be fixed throughout this paper) as follows: let $P(z) \in C_0^{\infty}(B_1(0))$ and with $d = n + 1$ set

$$P_{\lambda}(z) \ \equiv\ \lambda^{-d} P(\lambda^{-\alpha} z). \tag{0.22}$$

Next let $P_{\lambda} f$ be the convolution operator

$$P_{\lambda} f(z) \ \equiv\ \int_{\mathbb{R}^n} P_{\lambda}(z - v) f(v)\, dv.$$

In addition we take $P(z)$ to be an even non-negative function, with $\int_{\mathbb{R}^n} P(z)\, dz \ \equiv\ 1$.

We shall also use two other notational conventions: first $c$ will denote a positive constant, not necessarily the same at each occurence, but depending only on the dimension and other harmless factors such as our choice of $P(z)$. In general $c_{\beta,\mu,\nu}$ denotes a positive constant depending only on $\beta, \mu, \nu$, and the above harmless factors, not necessarily the same at each occurence. Second $Q_{\lambda}$ will denote a " generic " approximation to the zero operator, not necessarily the same at each occurence, but chosen from a finite stock of such operators depending only on our original choice of $P_{\lambda}$. That is, $Q_{\lambda}$ will denote the operator of convolution with a generic kernel of the form

$$Q_{\lambda}(z) \equiv \lambda^{-d} Q(\lambda^{-\alpha}\, z\,), \tag{0.23}$$

where $Q \in C_0^{\infty}(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} Q(z)\, dz \equiv 0$. Similarly, $\tilde{Q}_{\lambda}$ will denote a generic approximation to the zero operator whose kernel may not have compact support, but at least satisfies the conditions

$$\begin{aligned} |\tilde{Q}_{\lambda}(z)| \ &\leq\ \frac{c\,\lambda}{(\lambda + \|z\|)^{d+1}} \\ |\,\tilde{Q}_{\lambda}(z) \ -\ \tilde{Q}_{\lambda}(v)\,| \ &\leq\ \frac{c\,\|z - v\|^{\delta}}{(\lambda + \|z\|)^{d+\delta}}\ , \end{aligned} \tag{0.24}$$

where the latter estimate holds for some $\delta \in (0,1]$, whenever $2\|z-v\| \le \|z\|$. Similarly, $Q_\lambda^{(0)}$ will denote an approximation to the zero operator whose kernel in addition to (0.24) satisfies for $1 \le j \le n-1$, the moment condition

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} x_j \, Q_\lambda^{(0)}(x,t)\, dx\, dt \;=\; 0\,. \tag{0.25}$$

Finally, we shall often refer to the inequality $ab \le \frac{1}{2}\epsilon a^2 + \frac{b^2}{2\epsilon}$ as Cauchy's inequality with $\epsilon$ 's.

**1. Statement of Results.** We begin this section with a description of the layer potentials which we shall use to represent our solutions. Given $(X,t) \in \mathbb{R}^{n+1}$ let

$$W(X,t) \;=\; (4\pi t)^{-n/2} \,\exp\left\{\frac{-|X|^2}{4t}\right\} \chi_{(0,\infty)}\,(t)$$

denote the usual Gaussian in $\mathbb{R}^{n+1}$. Next for given $\epsilon > 0$, let $W_\epsilon(X,t) = W(X,t)$ when both $|X| > \epsilon$, $t > \epsilon^2$. Otherwise, $W_\epsilon \equiv 0$. Given $f \in L^p(\partial\Omega), 1 < p < \infty$, we define the single and double layer potentials respectively of $f$ by

$$\begin{aligned}\mathcal{S}f(X,t) \;&\equiv\; \int_{-\infty}^{t} \int_{\partial\Omega_s} W(X-Q,t-s)\, f(Q,s)\, d\sigma_s(Q)\, ds \\ \mathcal{D}f(X,t) &\equiv \int_{-\infty}^{t} \int_{\partial\Omega_s} \frac{\partial}{\partial n_s} W(X-Q,t-s)\, f(Q,s) d\sigma_s(Q) ds,\end{aligned} \tag{1.1}$$

where, as in (0.14), $n_s = n_s(Q,s)$, $(Q,s) \in \partial\Omega,$ is the outer unit normal to $\partial\Omega_s$ considered as a subset of $\mathbb{R}^n$ and $\frac{\partial}{\partial n_s}$ denotes differentiation at $Q$ in the direction of $n_s$. Next for $(P,t) \in \partial\Omega$ and

$\epsilon > 0$, define $K_\epsilon, K^*_\epsilon$, and associated maximal operators $\bar{K}, \bar{K^*}$ by

$$
\begin{aligned}
K_\epsilon f(P,t) &\equiv \int_{-\infty}^{t} \int_{\partial\Omega_s} \frac{\partial}{\partial n_s} W_\epsilon(P-Q,t-s)\, f(Q,s)\, d\sigma_s(Q)\, ds \\
K^*_\epsilon f(P,t) &\equiv \int_{-\infty}^{t} \int_{\partial\Omega_s} \frac{\partial}{\partial n_t} W_\epsilon(P-Q,t-s)\, f(Q,s)\, d\sigma_s(Q)\, ds \\
\bar{K} f(P,t) &\equiv \sup_{\epsilon>0} |K_\epsilon f|(P,t) \\
\bar{K^*} f(P,t) &\equiv \sup_{\epsilon>0} |K^*_\epsilon f|(P,t).
\end{aligned}
\tag{1.2}
$$

In (1.2), $n_t = n_t(P,t)$. Set

$$
\begin{aligned}
Kf(P,t) &= \lim_{\epsilon\longrightarrow 0} K_\epsilon f(P,t), \\
K^* f(P,t) &= \lim_{\epsilon\longrightarrow 0} K^*_\epsilon f(P,t), \\
S_b f(P,t) &\equiv \int_{-\infty}^{t} \int_{\partial\Omega_s} W(P-Q,t-s)\, f(Q,s)\, d\sigma_s(Q) ds,
\end{aligned}
\tag{1.3}
$$

whenever $(P,t) \in \partial\Omega$, and these expressions make sense. $Kf$ and $S_b f$ are called the boundary single and double layer potentials of $f$, respectively. It is easily seen from Sobolev type estimates that $S_b f(P,t)$ exists for a.e $(P,t)$ with respect to the surface measure defined in (0.12).

The first key ingredient in the method of layer potentials is to obtain estimates for singular integral operators similar to $K, K^*$. The following results are of fundamental importance to us. The first result is due to Lewis and Murray [LM, chs 1, 2] (see also [H2, Thm 2] for a simpler and more direct proof).

**Theorem 1.4** *Let $A$ satisfy (0.9) and suppose $f \in L^p(\partial\Omega)$, for some $p, 1 < p < \infty$. Then*

$$
\max\left\{ \|\bar{K^*} f\|_p \,,\, \|\bar{K} f\|_p \right\} \le c_{p,\beta}\, \|f\|_p
$$

*where $c_{p,\beta} \to 0$ as $\beta \to 0$. Moreover*

$$
\begin{aligned}
Kf(P,t) &= \lim_{\epsilon\longrightarrow 0} K_\epsilon f(P,t), \\
K^* f(P,t) &= \lim_{\epsilon\longrightarrow 0} K^*_\epsilon f(P,t),
\end{aligned}
$$

*for almost every $(P,t) \in \partial\Omega$ with respect to the measure defined in (0.12).*

We remark that Theorem 1.4 was proved in [LM] under the assumption that (0.10) holds and A satisfies a certain Carleson measure condition. However, these assumptions are equivalent to (0.9) ( as well as (0.2), (0.18) ). The next result is due to the first author [H2, Thm 4].

**Theorem 1.5** *If $A$ satisfies (0.9) and $f \in L^p(\partial\Omega)$ for some $p, 1 < p < \infty$, then*

$$\|S_b(f)\|_{L^p_{1,1/2}(\partial\Omega)} \leq c_{p,\beta}\, \|f\|_p.$$

**Remarks**:

1) The main issue in Theorem 1.5 was the $L^p$ boundedness of $I\!\!D_n(S_b f \circ \pi^{-1})$. The boundedness of spatial derivatives of $S_b f \circ \pi^{-1}$ was at least implicit in [LM].

2) Neither of Theorems 1.4, 1.5, requires the smallness assumption, (0.21) - we shall only need this extra assumption when we discuss invertibility of the layer potentials.

Theorems 1.4 and 1.5 are a consequence of a more general pair of results of the first author [H2, Thms 1,3] for operators of " Calderón type ," as can be seen for example by writing the single and double layer potentials in graph coordinates and then using the method of Coifman, David, and Meyer [CDM]. To be more specific, let $H$ and $J$ denote kernels which satisfy the homogeneity properties

$$\begin{aligned} H(\delta x, \delta^2 t) &\equiv \delta^{-d}\, H(x,t), \\ J(\delta x, \delta^2 t) &\equiv \delta^{-d+1}\, J(x,t), \end{aligned} \tag{1.6}$$

where $d = n+1$ and $(x,t) \in I\!\!R^n$. We also assume that $H, J$ are sufficiently smooth away from the origin, i.e, $H, J \in C^m(I\!\!R^n \setminus \{0\})$, for some large $m$. With this notation, let $E$ denote either sine or cosine, and define singular integrals of Calderón type by

$$\begin{aligned} T_A\, f(z) &\equiv \ \mathrm{pv} \int_{\mathbb{R}^n} H(z-v)\, E\left(\frac{A(z)-A(v)}{\|z-v\|}\right) f(v)\, dv \\ T_{A,B}\, f(z) &\equiv \ \mathrm{pv} \int_{\mathbb{R}^n} H(z-v)\, E\left(\frac{A(z)-A(v)}{\|z-v\|}\right) \frac{B(z)-B(v)}{\|z-v\|}\, f(v)\, dv, \end{aligned} \tag{1.7}$$

when $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Here pv means the limit as $\epsilon \to 0$ of truncated kernels similar to those in Theorem 1.4. Set

$$\begin{aligned} S_A\, f(z) &\equiv \int_{\mathbb{R}^n} J(z-v)\, E\left(\frac{A(z)-A(v)}{\|z-v\|}\right) f(v)\, dv \\ U_{A,B}\, f(z) &\equiv \int_{\mathbb{R}^n} J(z-v)\, E\left(\frac{A(z)-A(v)}{\|z-v\|}\right) \frac{B(z)-B(v)}{\|z-v\|}\, f(v)\, dv. \end{aligned} \tag{1.8}$$

We have the following results of the first author mentioned above [H2, Thms 1, 3].

**Theorem 1.9** *Let $\|A\|_{\mathrm{comm}}, \|B\|_{\mathrm{comm}} < \infty$ and $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. If $E =$ cosine, and $H(x,t)$ is odd in $x$, or if $E =$ sine, and $H(x,t)$ is even in $x$, then for $H$ sufficiently smooth away from the origin and for some large positive $N$, we have*

$$\|T_A f\|_p \ \le\ c_{p,H}\, (1 + \|A\|_{\mathrm{comm}})^N\, \|f\|_p\,.$$

*Similarly, if the parity of $H(x,t)$ is the same in $x$ as that of $E$, then*

$$\|\, T_{A,B}\, f\,\| \ \le\ c_{p,H}\, \|B\|_{\mathrm{comm}} (\, 1 + \|A\|_{\mathrm{comm}}\,)^N\, \|f\|_p\,.$$

**Theorem 1.10** *Let $\|A\|_{\mathrm{comm}}, \|B\|_{\mathrm{comm}} < \infty$ and $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. Suppose that $J$ is sufficiently smooth away from the origin. If $J(x,t)$ has the same parity in $x$ as does $E$, then for some large positive $N$, we have*

$$\|S_A f\|_{L^p_{1,1/2}} \ \le\ c_{p,J}\, (1 + \|A\|_{\mathrm{comm}})^N\, \|f\|_p\,.$$

*Similarly if $J(x,t)$ has opposite parity in $x$ to that of $E$, then*

$$\|U_{A,B}\, f\,\|_{L^p_{1,1/2}} \;\le\; c_{p,J}\,\|B\|_{\mathrm{comm}}\,(1+\|A\|_{\mathrm{comm}})^N\,\|f\|_p\,.$$

**Remarks**

1) Using the method of [CDM], one can immediately replace the trigonometric function $E$ by any sufficiently smooth function defined on $\mathbb{R}$ with the same parity as $E$. One can also treat layer potentials via this method.

2) In section 3, we shall give a new and simpler proof of Theorem 1.10. The original proof in [H2] was made more difficult by the fact that operators like $\mathbb{D}_n S_A$ are in general " rough operators " which need not map constants into BMO. Our proof in the present paper will be an easy consequence of estimates for " square functions of Calderón type " which we also establish in section 3. Such square function estimates in the special case that $S_A f$ is the single layer potential of $f$ will also be crucial to our proof of the invertibility of the layer potentials.

3) Theorem 3 in [H2] is stated for $A_2$ weights but implies our Theorem 1.10 (see the remarks after Theorem 3 in [H2]).

To continue our discussion of layer potentials, for given $a>0$ let $(P,t) = (p_0,p,t) \in \partial\Omega$ and let $\tilde\Gamma(P,t) = \tilde\Gamma_a(P,t)$ be the parabolic cone :

$$\tilde\Gamma(P,t) \;\equiv\; \{\,(q_0,q,s)\in\Omega : \|(p-q,t-s)\| \;<\; a|q_0 - A(p,t)|\,\} \tag{1.11}$$

and if $h$ is a function defined on $\Omega$ define the nontangential maximal function $\tilde N_* h : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde N_*\, h(P,t) \;\equiv\; \sup_{(Q,s)\in\tilde\Gamma} |h|(Q,s). \tag{1.12}$$

If $(P,t) \in \partial\Omega$, then by $\lim_{(Q,s)\longrightarrow(P,t)} h(Q,s)$ we mean the limit as $(Q,s)\to(P,t)$ in $\tilde{\Gamma}(P,t)$. With this notation we note that the second key ingredient in solving the Dirichlet problem by the method of layer potentials is to obtain estimates for $\tilde{N}_* u$ where $u = \mathcal{D}f, f \in L^p(\partial\Omega)$, as well as show that $u$ has nontangential limits a.e equal to $(-\frac{1}{2}I + K)f$. The third and perhaps most difficult ingredient to obtain is to show that a certain integral equation on $\partial\Omega$ has a solution.

Regarding this method, we have the following theorem (see [LM, ch 3, Thm 5]).

**Theorem 1.13** *Let $A$ satisfy (0.9) for some $\beta$ and $g \in L^p(\partial\Omega), 1 < p < \infty$. If $\beta = \beta(a,p,n)$ is small enough there exists $f \in L^p(\partial\Omega)$ such that $\|f\|_p \approx \|g\|_p$ and such that $\mathcal{D}f$ is the unique solution to the heat equation in $\Omega$ for which (A) and (B) hold*

$$\lim_{(Q,s)\longrightarrow(P,t)} \mathcal{D}f(Q,s) \;=\; g(P,t) \tag{A}$$

*for a.e $(P,t) \in \partial\Omega$ with respect to the surface measure defined in (0.12)*

$$\|\tilde{N}_*(\mathcal{D}f)\,\|_p \;\leq\; c_{p,\beta}\,\|g\|_p\,. \tag{B}$$

*$f$ is the solution to the integral equation.*

$$-\tfrac{1}{2}f(P,t) + Kf(P,t) \;\equiv\; g(P,t)$$

*for almost every $(P,t) \in \partial\Omega$.*

The layer potential method can also be used to solve the Neumann problem for $\Omega$. The following result is stated but not proved in [LM, ch 3, Thm 6].

**Theorem 1.14** *Let $A$ satisfy (0.9) for some $\beta$ and $g \in L^p(\partial\Omega), 1 < p < \infty$. If $\beta = \beta(a,p,n)$ is small enough there exists $f \in L^p(\Omega)$ such that $\|f\|_p \approx \|g\|_p$ and such that $\mathcal{S}f + c$ are all solutions to the heat equation in $\Omega$ for which (A) and (B) hold.*

$$\lim_{(Q,s)\longrightarrow(P,t)} \langle \nabla \mathcal{S}f(Q,s),\, n_t(P,t)\,\rangle \;=\; g(P,t) \tag{ A}$$

*for a.e $(P,t) \in \partial\Omega$ with respect to the surface measure defined in (0.12)*

$$\left\| \tilde{N}_* \left( \frac{\partial \mathcal{S}f}{\partial x_i} \right) \right\|_p \;\leq\; c_{p,\beta}\, \|g\|_p\,,\ \textit{for}\ 0 \leq i \leq n-1. \tag{B}$$

*$g$ is the solution to the integral equation.*

$$\tfrac{1}{2} f(P,t) + K^* f(P,t) \;\equiv\; g(P,t)$$

*for almost every $(P,t) \in \partial\Omega$. Modulo constants, this solution is unique among those for which $\tilde{N}_*(\nabla u) \in L^p(\partial\Omega)$.*

Using Theorem 1.10 and applying the method of layer potentials as in the proof of Theorem 1.13 one can also solve the following boundary value problem for functions in $L^p_{1,1/2}(\partial\Omega)$.

**Theorem 1.15** *Let $A$ satisfy (0.9) for some $\beta$ and $g \in L^p_{1,1/2}(\partial\Omega), 1 < p < \infty$. If $\beta = \beta(a,p,n)$ is small enough there exists $c_{p,\beta}$ and $f \in L^p(\partial\Omega)$ such that $\|f\|_p \;\leq\; c_{p,\beta}\, \|g\|_{L^p_{1,1/2}(\partial\Omega)}$ and such that $\mathcal{S}f + c$ are all solutions to the heat equation in $\Omega$ with nontangential limits at a.e every point of $\partial\Omega$ for which (A) and (B) hold :*

$$\mathcal{S}f \;\equiv\; g \;\; in\ L^2_{1,1/2}(\partial\Omega), \tag{ A}$$

$$\sum_{i=0}^{n-1} \|\tilde{N}_* \left( \frac{\partial \mathcal{S}f}{\partial x_i} \right) \|_p \;\leq\; c_{p,\beta} \|f\|_{L^p_{1,1/2}}. \tag{B}$$

*Modulo constants, the solution is unique among those for which $\tilde{N}_*(\nabla u) \in L^p(\partial\Omega)$.*

**Remarks**

1) Theorems 1.13 - 1.15 are, after deleting the smallness assumption on $\beta$, a precise statement of the boundary value problems for the heat equation discussed in section 0 (see (0.13), (0.14)).

2) In Theorem 1.13, the nontangential maximal function result, can be obtained by estimating $\tilde{N}_*(\mathcal{D}f)$ in terms of $\bar{K}f$ (see (1.2)) and a certain Hardy Littlewood maximal function of $f$ defined relative to $\|\cdot\|$ (see [LM,ch 3, sec 2]). A similar estimate with $\bar{K}^*$ in (1.2) replacing $\bar{K}$ can be made for the nontangential maximal functions in Theorems 1.14 and 1.15 (see Lemma 2.14 of section 2 for estimates of the nontangential maximal function of $D^t_{1/2}(S_b f \circ \pi^{-1})$ ).

3) Theorem 1.4 and the fact that $c_{p,\beta}{\to}0$, as $\beta{\to}0$, imply that the Neumann series $\sum_{i=0}^{\infty}(2K)^i\, g$, converges absolutely in $L^p(\partial\Omega)$ for small enough $\beta$. Thus in this case it is easy to get a solution to the integral equation in Theorem 1.13. A similar statement holds for the integral equation in Theorem 1.14.

In this paper we prove

**Theorem 1.16** *Theorems 1.13 - 1.15 remain valid for arbitrary $\beta_0 < \infty$, and for sufficiently small $\epsilon_0$ when $p = 2$ and $A$ satisfies (0.2) and (0.21).*

**Remarks**

1) In view of (0.20), we could replace the smallness of $\|I\!\!D_n A\|_*$ by that of $\|D^t_{1/2}A\|_*$.

2) As in section 0, we note that Theorem 1.16 is much more difficult to prove than Theorems 1.13 - 1.15, because we now must use another method (other than convergence of the Neumann series) to show that the integral equations in these theorems have a solution.

3) Theorem 1.16 is equivalent to solving the above boundary value problems for solutions to $k^2\Delta u - u_t = 0$, in $\Omega$ whenever $\|A\|_{\text{comm}} < \infty$, as long as the " conductivity coefficient " $k^2$ is large enough. This equivalence is easily shown using the mapping $t \rightarrow t/k^2$.

To show that the smallness assumption on $\|\mathbb{D}_n A\|_*$ in Theorem 1.16 cannot be removed, we shall prove the following theorem.

**Theorem 1.17** *Given $p, 1 < p < \infty$, there exists $\beta = \beta(p) < \infty$ and $A : \mathbb{R} \rightarrow \mathbb{R}\,(A \leq 0)$, such that $\|D^t_{1/2} A\|_* \leq \beta$ and the adjoint Green's function $G$ of the domain*

$$D = \{(x,t) : A(t) < x < 2, -4 < t < 4\}$$

*with pole at (1, 1) for the adjoint heat equation, satisfies*

$$\int_0^{1/2} \left|\frac{\partial G}{\partial x}\right|^p (A(t), t)\, dt = +\infty.$$

Theorem 1.17 will be used to prove the following corollaries.

**Corollary 1.18** *Given $p, 1 < p < \infty$, there exists $\beta = \beta(p) < \infty$ and $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\|A\|_{\text{comm}} \leq \beta$ and the Radon-Nikodym derivative of parabolic measure at $(1, 0, \ldots, 0, 1)$ in $\Omega$ (with respect to the surface measure defined in (0.12)) is not locally in $L^p(\partial\Omega)$.*

**Corollary 1.19** *Given $p, 1 < p < \infty$, there exists $\beta = \beta(p) < \infty$ such that the Dirichlet, Neumann, and related Sobolev boundary value problems cannot be solved in the sense of Theorems 1.13-1.15 for some $A$ with $\|A\|_{\text{comm}} \leq \beta$.*

**Remarks:**

1) Corollary 1.18 shows that the results of [LM, ch 3] are essentially best possible.

2) Corollary 1.19 with $p = 2$ implies that the smallness assumption on $\| I\!\!D_n A\|_*$ in Theorem 1.16 cannot be removed.

To outline our strategy for proving Theorem 1.16, suppose $f \in L^2(\partial\Omega)$ and $\tilde{f} = f \circ \pi^{-1} \in C_0^\infty(\Omega)$ with $u = Sf$. Set $u^+ \equiv u|_\Omega$, $u^- \equiv u|_{I\!\!R^n \setminus \bar{\Omega}}$. First we discuss the case when $A(x,t) \equiv A(x)$ and $\tilde{f}(x,t) \equiv \tilde{f}(x)$, so that $u$ is a solution to Laplace's equation in $I\!\!R^n \setminus \partial\Omega$. Verchota (see section 0 for references) observed that in order to show $\|g\|_2 \leq \|(\frac{1}{2}I + K^*)g\|_2$ it suffices by the triangle inequality to show $\|(\frac{1}{2}I - K^*)g\|_2 \leq c_\beta \|(\frac{1}{2}I + K^*)g\|_2$, and this inequality can be written as

$$\int_{\partial\Omega_1} (u^-_{n_1})^2 \, d\sigma_1 \leq c_\beta \int_{\partial\Omega_1} (u^+_{n_1})^2 \, d\sigma_1. \tag{1.20}$$

He obtained the above inequality by using a certain Rellich inequality to show that both of the above integrals are $\approx$ equal to the $L^2(\partial\Omega)$ norm squared of the tangential derivatives of $u$. Since $u^+, u^-$, have the same tangential derivatives, he then gets (1.20). Brown generalized (1.20) to Lipschitz cylinders by showing that

$$\|f\|_2 \approx \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} \tag{1.21}$$

and

$$\|f\|_2 \approx \|(\tfrac{1}{2}I + K^*)f\|_2 \tag{1.22}$$

where all constants depend on $\beta$. His argument again uses a Rellich inequality, but he also makes masterful use of the Fourier transform in the $t$ variable.

To prove (1.21), (1.22) in our situation, we use a non Fourier transform version of Brown's argument more akin to Shen [Sh]. Unfortunately, though, in our noncylindrical domains, we do not get the excellent cancellation which occurs in the cylindrical case. In short in applying the method of the above authors we arrive at error terms involving square functions. We then show that each error term can be estimated by $c_{\epsilon_0} \|f\|_2^2$, where $c_{\epsilon_0} \to 0$ as $\epsilon_0 \to 0$. Finally we obtain (1.21) and (1.22) (see (5.2), (5.3)) for small $\|\mathbb{D}_n A\|_*$ in section 6. (1.21), (1.22), Theorems 1.9, 1.10, and a continuity method (see [K, p 150]) imply that the integral equations in Theorems 1.13, 1.14 have a solution.

The rest of the paper is organized as follows. In section 2, we discuss certain Carleson measures involving $A$ and also the behaviour of layer potentials on the boundary of our graph domains. In section 3, we prove our estimates for " square functions of Calderòn type " which will be used to estimate the error terms mentioned above and also to give a new and simpler proof of Theorem 1.10. In section 4, we give analogous square function estimates in the special case that our " Calderòn - type " operators arise as caloric layer potentials. In this section, we also sketch another proof of this special case, due to Dahlberg, Kenig, Pipher, and Verchota (oral communication). In section 5 we begin the proof of existence in Theorem 1.16. Using a Rellich type argument we reduce the proof of existence in this theorem to a " main lemma " which we then prove in section 6. In section 7 we use our apriori estimates, (1.21) and (1.22), to establish uniqueness in our boundary value problems. In section 8 we prove (0.19) and (0.20). In section 9 we prove Theorem 1.17 and Corollaries 1.18 - 1.19.

Finally we remark that in order to keep this paper at a manageable length we have not included local versions of our results. In a future paper we plan to discuss local versions of our results and also obtain analogues of Brown's work in our situation for the $L^p$ Dirichlet problem (2 - $\epsilon \leq p < \infty$) and the $L^p$ Neumann problems ($1 \leq p \leq 2 + \epsilon$).

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**2. Carleson Measures and Boundary Behavior of Layer Potentials.** We begin by introducing the Dahlberg - Kenig - Stein change of variable mentioned in the acknowledgements of section 1. We use the notation in section 0. Let $\lambda > 0$ and put

$$x_0 = \lambda + P_{\gamma\lambda} A(x,t), \tag{2.1}$$

where $P_\lambda$ denotes the parabolic approximate identity defined in (0.22) and where $\gamma$ is a fixed small positive number (depending on $\|A\|_{\text{comm}}$ ) which is chosen to make our estimates work out. In particular, we first suppose $\gamma$ is so small that if $z = (x,t) \in \mathbb{R}^n$, then

$$1/2 \,\le\, 1 + \tfrac{\partial P_{\gamma\lambda} A(z)}{\partial \lambda} \,\le 3/2. \tag{2.2}$$

We can always do this since by (0.10)

$$|A(z) - A(v)| \,\le\, c\|A\|_{\text{comm}}\, \|z - v\|, \text{ whenever } z, v \in \mathbb{R}^n. \tag{2.3}$$

From (2.2) and $\lim\limits_{\lambda \longrightarrow 0} P_{\gamma\lambda} A(x,t) \,=\, A(x,t),$ we see that the change of variable (2.1) defines a 1-1 mapping of our domain $\Omega$ onto the upper half-space $\mathbb{R}^{n+1}_+ = \{(\lambda, x, t) : \lambda > 0, (x,t) \in \mathbb{R}^n\}$. We denote by $\rho$ the " lifting " of $\mathbb{R}^{n+1}_+$ onto $\Omega$ defined by

$$\rho(\lambda, x, t) \,\equiv\, (\lambda + P_{\gamma\lambda} A(x,t), x, t). \tag{2.4}$$

Define $\rho(0, x, t) \equiv (A(x,t), x, t)$ and note that $\rho$ is continuous on the closure of $\mathbb{R}^{n+1}_+$ . For a function $g$ on $\mathbb{R}^{n+1}_+$, and fixed $a \ge 1$, we define analogous to (1.12) the non - tangential maximal function of $g$, denoted $N_* g$, by

$$N_* g(x,t) \,\equiv\, \sup_{\lambda > 0} \{\, |g|(\lambda, y, s) : (y,s) \in B_{a\lambda}(x,t) \,\}$$

where $B_{a\lambda}(x,t)$ is defined as in (0.6) with $r = a\lambda$ and $z = (x,t)$. Let $\Gamma_a(x,t)$ denote the parabolic cone $\{(\lambda, y, s) : \lambda > 0, \text{ and } (y,s) \in B_{a\lambda}(x,t)\,\}$, and note from geometry, (2.3), and the definition of $\tilde{\Gamma}$ in (1.11) that $\tilde{\Gamma}_{\bar{a}}(\rho(0,x,t)) \subset \rho(\Gamma_a(x,t))$ for sufficiently small $\bar{a}$ depending on $a$, $\|A\|_{\text{comm}}$. Given $a \ge 1$, let $\lim\limits_{(\lambda,y,s) \longrightarrow (0,x,t)}$ mean the limit as $(\lambda, y, s) \rightarrow (0, x, t)$ in $\Gamma_a(x,t)$.

Let $\mu$ be a positive Borel measure on $\mathbb{R}^{n+1}_+$ and $d = n+1$. Then $\mu$ is said to be a Carleson measure with respect to $\|\cdot\|$ if

$$\mu\,(B_r(z) \times (0,r)) \,\le c_\mu\, r^d$$

for some positive $c_\mu < \infty$ independent of $z \in \mathbb{R}^n$ and $r > 0$. Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative Lebesgue measurable function and put

$$\tilde{\omega}(E) = \int_E \omega \, dz, \; E \text{ Borel } \subset \mathbb{R}^n.$$

Given $p, 1 < p < \infty$, recall that $\omega$ is said to be an $A_p$ weight on $\mathbb{R}^n$ with respect to $\| \cdot \|$ (written $\omega \in A_p$), provided that for some $c^*_{\omega,p}, 0 < c^*_{\omega,p} < \infty$, and all balls $B = B_r(z) \subset \mathbb{R}^n$, we have

$$\left( \int_B \omega \, dz \right) \left( \int_B \omega^{-1/(p-1)} \, dz \right)^{(p-1)} \leq c^*_{\omega,p} \, |B|^p \, . \tag{2.5}$$

We note that $A_p \subset A_q$ when $q > p$. We say that $\omega$ is an $A_\infty$ weight ( $\omega \in A_\infty$ ) if $\omega \in A_p$ for some $p$. For several other equivalent definitions of $A_\infty$ see [ GR, ch 4]. For $1 < p < \infty$, we let $L^p_\omega = L^p_\omega(\mathbb{R}^n)$ denote the space of Lebesgue measurable functions $g$ with $\|g\|_{p,\omega} < \infty$, where

$$\|g\|^p_{p,\omega} = \int_{\mathbb{R}^n} |g|^p \, \omega \, dz \, .$$

In the sequel, unless otherwise stated, we write $c_{\omega,p}$ for a constant depending only on $p, n$, and $c^*_{\omega,p}$ (the $A_p$ constant). A similar interpretation applies to any constant which has $\omega$ as a subscript. Next let $\tilde{Q}_\lambda$ be the approximation to the zero operator defined in section 0 satisfying the conditions in (0.24). If $g \in$ parabolic BMO and $\omega \in A_\infty$, we note that the measure $\nu$ defined by

$$d\nu(\lambda, z) = ( \tilde{Q}_\lambda g )^2 \, (z) \, \omega(z) \, \lambda^{-1} \, dz d\lambda$$

is a weighted Carleson measure in the sense that whenever $B = B_r(z) \subset \mathbb{R}^n$,

$$\nu(B \times (0, r)) \leq c_{\omega,p} \, \tilde{\omega}(B) \, \|g\|^2_* \, . \tag{2.6}$$

The unweighted version of this inequality ($\omega \equiv 1$) is due to Fefferman and Stein. For general $\omega$, see [H2, sec 6, Lem 1]. Next let $g, \nu$, be as in (2.6) and $\omega \in A_p$ for some $p, 1 < p < \infty$. If $h : \mathbb{R}^{n+1}_+ \to \mathbb{R}$

is continuous, then

$$\int_{\mathbb{R}^{n+1}_+} |h|^p \, d\nu \;\leq\; c_{\omega,p} \, \|g\|_*^2 \int_{\mathbb{R}^n} (N_* h)^p \, \omega \, dz, \tag{2.7}$$

The unweighted version of this inequality is well known and due to Carleson. For the weighted verion see [GR, p 470]. To estimate the error terms mentioned in section 1, we shall need the following lemma.

**Lemma 2.8** *Let $\sigma, \theta$ be nonnegative integers and $\phi = (\phi_1, \ldots, \phi_{n-1})$, a multi-index, with $l = \sigma + |\phi| + \theta$. If $\omega \in A_\infty$ and $A$ satisfies (0.9) for some $\beta < \infty$, then the measure $\nu$ defined at $(\lambda, x, t)$ by*

$$d\nu = \left( \frac{\partial^l P_{\gamma\lambda} A}{\partial \lambda^\sigma \partial x^\phi \partial t^\theta} \right)^2 \omega \, \lambda^{(2l+2\theta-3)} \, dx dt d\lambda$$

*is a weighted Carleson measure whenever either $\sigma + \theta \geq 1$ or $|\phi| \geq 2$, with*

$$(a) \quad \nu(B_r(z) \times (0,r)) \;\leq\; c_{l,\omega} \, \omega(B_r(z)) \, \gamma^{(2-2|\phi|-4\theta)} \, b^2 \, (1+\beta)^2,$$

*where $b = \|\mathbb{D}_n A\|_*$ when $\theta \geq 1$ and $b = 1$, if $\theta = 0$. Moreover if $l \geq 1$, then*

$$(b) \quad \left\| \frac{\partial^l P_{\gamma\lambda} A}{\partial \lambda^\sigma \partial x^\phi \, \partial t^\theta} \right\|_\infty \leq c_l \, \gamma^{(1-|\phi|-2\theta)} \, \lambda^{(1-l-\theta)} \, b \, (1+\beta)$$

*while if either $\sigma + \theta \geq 1$ or $|\phi| \geq 2$, then*

$$(c) \quad \lim_{(\lambda,y,s) \longrightarrow (0,x,t)} \left( \lambda^{(l+\theta-1)} \, \frac{\partial^l P_{\gamma\lambda} A}{\partial \lambda^\sigma \partial y^\phi \, \partial t^\theta} \right) = 0, \quad \text{for a.e } (x,t) \in \mathbb{R}^n.$$

**Proof**: We prove $(a)$ and $(b)$ of Lemma 2.7 only for $(+)$ $\theta = 1, \sigma = |\phi| = 0$, and $(++)$ $\sigma = 1, |\phi| = \theta = 0$, since the other cases are similar to these two cases. In case $(+)$ we note from the definition of $I\!D_n, I\!D$, in (0.3), (0.4), and (0.16) that

$$
\begin{aligned}
I\!D &\equiv \sum_{j=1}^{n} R_j I\!D_j \text{ where} \\
I\!D_j &= \tfrac{\partial}{\partial x_j} \text{ for } 1 \le j \le n-1, \\
(R_j)\hat{}(\xi,\tau) &\equiv i\xi_j/\|(\xi,\tau)\| \text{ for } 1 \le j \le n-1, \\
(R_n)\hat{}(\xi,\tau) &= \tau/\|(\xi,\tau)\|^2 .
\end{aligned} \tag{2.9}
$$

In the above display, $i = \sqrt{-1}$ and $R_j, 1 \le j \le n$, are the parabolic version of Riesz transforms. That is, each $R_j$ has average zero on spheres about the origin (with respect to the weight $1+\theta_n^2$ defined in (0.7)), and if $z=(x,t)$, then $R_j(\lambda^\alpha z) = \lambda^{-d} R_j(z)$. Furthermore, each $R_j, 1 \le j \le n$, has a Calderón - Zygmund kernel, in the sense that

$$
\begin{aligned}
&|R_j(z)| \le c\|z\|^{-d} \\
&|\, R_j(z) - R_j(v)\,| \le c\|z-v\|/\|z\|^{d+1} \text{ for } \|z-v\| \le \|z\|/2 .
\end{aligned} \tag{2.10}
$$

For the above properties of $R_j, 1 \le j \le n$, see [FR1]. Using these properties, it follows from Calderón - Zygmund theory that $R_j$ is a bounded operator on $L^p(I\!R^n), 1 < p < \infty$, with norm $\le c$. This fact, (2.10), and an easy argument often called Peetre's Lemma imply that $R_j, 1 \le j \le n$, is a bounded operator on parabolic BMO with norm $\le c$. From this discussion and (2.9) we deduce as in [FR1] that at $z=(x,t)$

$$
\begin{aligned}
\tfrac{\partial P_{\gamma\lambda} A}{\partial t} &= -i(I\!D P_{\gamma\lambda}) * I\!D_n A \\
&= -i\sum_{j=1}^{n} (I\!D_j P_{\gamma\lambda}) * (R_j I\!D_n A) \\
&= (\gamma\lambda)^{-1} \sum_{j=1}^{n} \tilde{Q}_{j,\gamma\lambda} * (R_j I\!D_n A)
\end{aligned} \tag{2.11}
$$

where each $\tilde{Q}_{j,\lambda}$, $1 \le j \le n$, satisfies (0.24). From (2.6), (2.11), we conclude that $(a)$ is valid in case (+). $(b)$ follows from (0.24), (2.11), and a well known argument of Fefferman and Stein. In case (++) we observe that $\frac{\partial P_{\gamma\lambda}}{\partial \lambda} = \lambda^{-1} Q^{(0)}_{\gamma\lambda}$, where $Q^{(0)}_{\lambda}$ denotes (as in section 0) a kernel satisfying in addition to (0.24) the moment condition (0.25). We now apply Lemma 1 of [H2, section 2] which states that at $z = (x, t)$,

$$\lambda^{-1} I\!D^{-1} Q_\lambda^{(0)} \equiv \tilde{Q}_\lambda \tag{2.12}$$

where $\tilde{Q}_\lambda$ satisfies (0.24). From the above observations and (2.12) we find

$$\begin{aligned} \tfrac{\partial P_{\gamma\lambda} A}{\partial\lambda} &= (I\!D^{-1} \tfrac{\partial P_{\gamma\lambda}}{\partial\lambda}) * (I\!D A) \\ &= \gamma \tilde{Q}_{\gamma\lambda} * (\sum_{j=1}^{n} R_j I\!D_j A). \end{aligned} \tag{2.13}$$

We conclude from (2.13) and (2.6) that $(a)$ is valid in case $(++)$. $(b)$ follows from (2.13) and the above mentioned argument of Fefferman and Stein. Thus $(a), (b)$, are true in Lemma 2.8.

To prove $(c)$ we use $(a)$ and $(b)$ . Indeed from $(a)$ with $\omega = 1$ and Fubini's theorem we see that

$$\int_0^1 \left( \frac{\partial^l P_{\gamma\lambda} A}{\partial\lambda^\sigma \partial x^\phi \partial t^\theta} \right)^2 \lambda^{(2l+2\theta-3)} \, d\lambda < \infty,$$

for a.e. $(x, t)$. This inequality and $(b)$ with $(\sigma, \phi, \theta)$ replaced by $(\sigma + 1, \phi, \theta)$ imply that the limit as $(\lambda, x, t) \to (0, x, t)$ of the function in $(c)$, vanishes for a.e. $(x, t) \in I\!R^n$. Existence of this limit, $(b)$ with $(\sigma, \phi, \theta)$ replaced by $(\sigma, \phi, \theta + 1), (\sigma, \tilde{\phi}, \theta)$ where $|\tilde{\phi}| = |\phi| + 1$, Egoroff's theorem, and a point of density type argument imply that the nontangential limit in $(c)$ exists a.e for any $a \geq 1$. □

We note that Lemma 2.8 $(a)$ does not say anything about the case when $\phi = 1, \sigma = \theta = 0$. In fact in this case the corresponding measure need not be Carleson, which is unfortunate, since otherwise many of our arguments could be substantially simplified. To continue our discussion of layer potentials, we state the following lemma.

**Lemma 2.14** *Let $f \in L^2(\partial\Omega)$ and let $u = \mathcal{S}f, u' = \mathcal{D}f$ denote the single and double layer potentials* *of $f$ in $\Omega$ defined as in (1.1). If $A$ satisfies (0.9) ( i.e $\|A\|_{\mathrm{comm}} \leq \beta$ ) and $a \geq 1$ is fixed, then*

$$(i) \qquad \| N_*(u' \circ \rho) \|_2 \leq c_{a,\beta} \|f\|_2,$$

$$(ii) \qquad \| N_*(u_{x_i} \circ \rho) \|_2 \leq c_{a,\beta} \|f\|_2 \ \textit{for } 0 \leq i \leq n-1,$$

$$(iii) \qquad \| N_* \left( H D^t_{1/2}(u \circ \rho) \right) \|_2 \leq c_{a,\beta} \, (1 + \gamma^{-1} \, \| I\!\!D_n A \|_*) \, \|f\|_2,$$

*where $H$ denotes the one dimensional Hilbert transform acting in the $t$ variable ($x$ fixed) and $D^t_{1/2}$ is defined following (0.17).*

**Proof** From the geometric observation on mappings of parabolic cones we see that $(i)$ and $(ii)$ are similar to $(B)$ of Theorems 1.13 and 1.14. As pointed out in section 1, the proof of these inequalities are essentially given in [LM, ch 3, sec 2] where the nontangential maximal functions are estimated in terms of the maximal operators $\bar{K}f, \bar{K}^*f$ and a certain Hardy Littlewood maximal function of $f$. We omit the details. It remains to prove $(iii)$. To do this given $(x,t) \in I\!\!R^n$ observe for fixed $a \geq 1, K \geq 2$, $\lambda > 0$, $(\tilde{x}, \tilde{t}) \in B_{a\lambda}(x,t)$ that

$$\begin{aligned} HD^t_{1/2}\,(u \circ \rho)(\lambda, \tilde{x}, \tilde{t}) \;&=\; \lim_{\epsilon \longrightarrow 0} \int_{\{\epsilon < |s-\tilde{t}| < 1/\epsilon\}} \frac{\mathrm{sgn}\ (\tilde{t}-s)}{|\tilde{t}-s|^{3/2}}\ (u \circ \rho)(\lambda, \tilde{x}, s)\, ds \\ &= \lim_{\epsilon \longrightarrow 0} \int_{\{\epsilon < |s-\tilde{t}| \leq (Ka\lambda)^2\}} \ldots + \lim_{\epsilon \longrightarrow 0} \int_{\{(Ka\lambda)^2 < |s-\tilde{t}| < 1/\epsilon\}} \ldots \\ &= g_1(\lambda, \tilde{x}, \tilde{t}) + g_2(\lambda, \tilde{x}, \tilde{t})\,. \end{aligned} \tag{2.15}$$

As a first step in estimating $g_1$ we note that if

$$g_3(\lambda, \tilde{x}, \tilde{t}) = \sup_{\{\tau : |\,\tau - \tilde{t}\,| \leq (2Ka\lambda)^2\}} |\,(u \circ \rho)_\tau\,(\lambda, \tilde{x}, \tau)\,|,$$

then

$$|g_1|(\lambda, \tilde{x}, \tilde{t}) \;\leq\; g_3(\lambda, \tilde{x}, \tilde{t}) \int_{\{|s-\tilde{t}| \leq (2aK\lambda)^2\}} |s - \tilde{t}|^{-1/2}\, ds \;\leq caK\lambda g_3(\lambda, \tilde{x}, \tilde{t}). \tag{2.16}$$

Next if $|\tau - \tilde{t}| \leq (2Ka\lambda)^2$, then at $(\lambda, \tilde{x}, \tau)$ we observe that

$$(u \circ \rho)_\tau \;=\; u_\tau \circ \rho \;+\; (u_{\tilde{x}_0} \circ \rho)(\tfrac{\partial P_{\gamma\lambda} A}{\partial \tau})\,. \tag{2.17}$$

From $(b)$ of Lemma 2.8 with $\theta = 1, |\phi| = \sigma = 0$, we see for $(\lambda, \tilde{x}, \tau)$, as above that

$$|(u_{\tilde{x}_0} \circ \rho)(\tfrac{\partial P_{\gamma\lambda} A}{\partial \tau})(\lambda, \tilde{x}, \tau)| \leq c_{a,K,\beta} (\gamma \lambda)^{-1} \|I\!\!D_n A\|_* N_{**}(u_{\tilde{x}_0} \circ \rho)(x,t). \tag{2.18}$$

In order to avoid confusion in (2.18) we have let $N_{**}$ be the nontangential maximal function defined relative to $\Gamma_{c_1 K a}(x,t)$ where $c_1$ is so large that $(\lambda, \tilde{x}, \tau) \in \Gamma_{c_1 K a}(x,t)$.

We note that $u,\ u_{x_i},\ 0 \leq i \leq n-1$, are solutions to the heat equation in $\Omega$. Using this fact and standard estimates for the heat equation in caloric balls we deduce for given $b = (b_1, \ldots, b_n) \in I\!\!R^n$ and for $|\tau - \tilde{t}| \leq (2aK\lambda)^2$, that at $(\lambda, \tilde{x}, \tau)$

$$\begin{aligned}
|\, u_\tau \circ \rho\,|^2 \;&\leq\; c \sum_{i=0}^{n-1} |\, u_{x_i x_i} \circ \rho\,|^2 \\
&\leq\; c_\beta\, \lambda^{-n-4} \int_{B_{\lambda/2}(\tilde{x},\tau)} \int_{\lambda/2}^{3\lambda/2} \sum_{i=0}^{n-1} [\,(u_{x_i} \circ \rho) - b_i\,]^2\, (\tilde{\lambda}, y, s) d\tilde{\lambda} dy ds \qquad (2.19)\\
&\leq\; c_{a,K,\beta}\, \lambda^{-2} \sum_{i=0}^{n-1} \,[N_{**}(u_{x_i} \circ \rho - b_i)]^2\ (x,t)
\end{aligned}$$

provided $c_1$ is chosen still larger if necessary so that the above integrals are integrated over subsets of $\Gamma_{c_1 K a}(x,t)$. We use (2.18) and (2.19) with $b = 0$ in (2.17) to obtain an estimate for $g_3(\lambda, \tilde{x}, \tilde{t})$. Putting this estimate in (2.16), we get

$$N_* g_1(x,t) \;\leq\; c_{a,K,\beta}\, (1 + \gamma^{-1} \|I\!\!D_n A\|_*) \sum_{i=0}^{n-1} N_{**}(u_{x_i} \circ \rho)(x,t) \tag{2.20}$$

and so from (ii),

$$\|N_* g_1\|_2 \;\leq\; c_{a,K,\beta}\, (1 + \gamma^{-1} \|I\!\!D_n A\|_*)\, \|f\|_2. \tag{2.21}$$

As for $g_2$ set

$$g_4(\lambda, \bar{x}, \bar{t}) = \lim_{\epsilon \longrightarrow 0} \int_{\{(Ka\lambda)^2 < |s - \bar{t}| < 1/\epsilon\}} \frac{\text{sgn}\ (\bar{t} - s)}{|\bar{t} - s|^{3/2}}\ (u \circ \rho)(0, \bar{x}, s)\, ds$$

and note from Sobolev type estimates that $g_4(\lambda, \bar{x}, \bar{t})$ is well defined for a.e $(\bar{x}, \bar{t}) \in \mathbb{R}^n$. Arguing as in (2.16) and using $|x - \tilde{x}| + |t - \tilde{t}|^{1/2} \leq c\, a\lambda$, we get

$$| g_2(\lambda, \tilde{x}, \tilde{t}) - g_4(\lambda, x, t) | \leq | g_2(\lambda, \tilde{x}, t) - g_2(\lambda, x, t)| + | g_2(\lambda, \tilde{x}, t) - g_2(\lambda, \tilde{x}, \tilde{t})|$$

$$+ | g_2(\lambda, x, t) - g_4(\lambda, x, t) |$$

$$\leq \int_{\{(Ka\lambda)^2 < |s-t|\}} \frac{| u\circ\rho(\lambda,\tilde{x},s) - u\circ\rho(\lambda,x,s) |}{|t-s|^{3/2}} ds + \int_{\{(Ka\lambda)^2 < |\xi|\}} \frac{| u\circ\rho(\lambda,\tilde{x},\xi+t) - u\circ\rho(\lambda,\tilde{x},\xi+\tilde{\tau}) |}{|\xi|^{3/2}} d\xi$$

$$+ \int_{\{(Ka\lambda)^2 < |s-t|\}} \frac{| u\circ\rho(\lambda,x,s) - u\circ\rho(0,x,s) |}{|t-s|^{3/2}} ds \tag{2.22}$$

$$\leq c_{a,\beta}\, \lambda \sum_{i=0}^{n-1} \int_{\{(Ka\lambda)^2 < |s-t|\}} \frac{N_*((u\circ\rho)_{x_i})(x,s)}{|t-s|^{3/2}} ds + c_{a,\beta}\, \lambda^2 \int_{\{(Ka\lambda)^2 < |s-\tilde{t}|\}} \frac{M_n((u\circ\rho)_s)(\tilde{x},s)}{|\tilde{t}-s|^{3/2}} ds$$

$$\leq c_{a,\beta}\, K^{-1} \left( \sum_{i=0}^{n-1} M_n(N_*(u\circ\rho)_{x_i})(x,t) + \lambda\, M_n^{(2)}((u\circ\rho)_{\tilde{t}})(\lambda, \tilde{x}, \tilde{t}) \right).$$

where $M_n$, denotes the one dimensional Hardy Littlewood maximal function in the $t$ variable, while the other variables are held constant and $M_n^{(2)} = M_n \circ M_n$. The last line in the above inequality was obtained as in [S1, Thm 1, p 62]. From (2.17)-(2.19) with $b = 0$ and $K = c$ we deduce for $c$ large enough and $\xi \in \mathbb{R}$ that

$$\lambda\, (u\circ\rho)_{\tilde{t}}(\lambda, \tilde{x}, \tilde{t} + \xi) \leq c_{a,\beta}\, (1 + \gamma^{-1}\|\mathbb{D}_n A\|_*) \left( \sum_{i=0}^{n-1} \bar{N}_*((u_{x_i} \circ \rho))(x, t+\xi) \right) \tag{2.23 a}$$

where $\bar{N}_*$ is defined relative to $\Gamma_{ca}$. Also as in (2.17) we see for any point in $\mathbb{R}^{n+1}_+$ and $0 \leq i \leq n-1$ that

$$(u\circ\rho)_{x_i} = u_{x_i} \circ \rho + (u_{x_0} \circ \rho)\left(\frac{\partial P_{\gamma\lambda} A}{\partial x_i}\right) \tag{2.23 b}$$

Putting (2.23) into (2.22) and using Lemma 2.8 $(b)$ with $|\phi| = 1, \theta = 0 = \sigma$, we get

$$N_* g_2(x,t) \leq \sup_{\lambda > 0} |g_4(\lambda, x, t)| + K^{-1} c_{a,\beta}\, (1 + \gamma^{-1}\|\mathbb{D}_n A\|_*) \sum_{i=0}^{n-1} M_n^{(2)}(\bar{N}_*(u_{x_i} \circ \rho))(x,t). \tag{2.24}$$

Let $\psi(x,t) = \sup_{\lambda>0} |g_4(\lambda, x, t)|$ whenever $(x,t) \in \mathbb{R}^n$. Then from $(ii)$ of Lemma 2.14, (2.24), and the

Hardy-Littlewood maximal theorem, we see that

$$\|N_* g_2\|_2 \le K^{-1} c_{a,\beta} (1 + \gamma^{-1} \|I\!\!D_n A\|_*) \|f\|_2 + \|\psi\|_2. \tag{2.25}$$

From (2.25),(2.21), with $K = 2$, and (2.15) we see that in order to complete the proof of $(iii)$ of Lemma 2.14 it suffices to show that $\|\psi\|_2 \le c_{a,\beta} \|f\|_2$. To do this we note that $u \circ \rho(0,x,t) = S_b f(A(x,t),x,t)$. Moreover from Theorem 1.5, (0.16), (0.17), and the discussion following (0.17), we find for fixed $x \in I\!\!R^{n-1}$ that

$$S_b f(A(x,t),x,t) = cI_{1/2}(D^t_{1/2} S_b f)(A(x,t),x,t) = cI_{1/2} h(x,t),$$

where $h \in L^2(I\!\!R^n)$ and $\|h\|_2 \le c_\beta \|f\|_2$. Thus

$$\psi(x,t) = c \sup_{\epsilon > 0} | \tilde{V}_\epsilon h(x,t) | = c \tilde{V}_* h(x,t), \tag{2.26}$$

where $V_\epsilon$ is defined on functions $k \in L^2(I\!\!R)$ by

$$V_\epsilon k(t) = \int_{\{|s-t| > \epsilon\}} \frac{\text{sgn } (t-s) I_{1/2} k(s)}{|s-t|^{3/2}} ds$$

and $\tilde{V}_\epsilon h(x,t) = V_\epsilon h(x,\cdot)$ evaluated at $t$. We conclude from (2.26), this discussion, and Fubini's theorem that the proof of $(iii)$ will be complete once we prove (*) of the following lemma.

**Lemma 2.27** *If $k \in L^2(I\!\!R)$ and $V_* k = \sup_{\epsilon > 0} |V_\epsilon k|$, then*

$$\|V_* k\|_2 \le c \|k\|_2 \tag{*}$$

*and*

$$\lim_{\epsilon \to 0} V_\epsilon k(t) = cHk(t) \tag{**}$$

*for almost every $t \in I\!\!R$ with respect to one dimensional Lebesgue measure.*

**Proof:** We believe that Lemma 2.27 is well known although we have not been able to find a proof of (*) in the literature. Therefore we shall sketch the proof of this lemma. We note that $V_\epsilon k = L_\epsilon * k$ where $L_\epsilon(t) = \epsilon^{-1}L(t/\epsilon)$ and

$$\begin{aligned}\hat{L}(\tau) &= 2i\,|\tau|^{-1/2}\int_1^\infty \tfrac{\sin(\tau s)}{|s|^{3/2}}\,ds \\ &= b\ \mathrm{sgn}\ \tau\ +\ 2i|\tau|^{-1/2}\textstyle\int_0^1 \tfrac{\sin(\tau s)}{|s|^{3/2}}\,ds \\ &= b\ \mathrm{sgn}\ \tau\ +\ \hat{J}(\tau),\end{aligned} \tag{2.28}$$

where $b$ is a complex constant and $\hat{}$ denotes the Fourier transform on $I\!R$. Next observe that $\hat{J} \in C^\infty(I\!R \setminus \{0\})$ with

$$|d\hat{J}/d\tau^l| \ \leq\ c_l \min\{\,|\tau|^{1/2-l}, |\tau|^{-l}\,\} \text{ for } l \geq 0.$$

Using this observation and the fact that the Fourier - inverse Fourier transforms turn derivatives into powers, it is easily seen that $J_\epsilon$ is a standard Calderón - Zygmund kernel in the sense that (2.10) holds with $R_j, d, \|\cdot\|$ replaced by $J_\epsilon, 1, |\cdot|$. Thus $(J_\epsilon)$ is uniformly bounded on $L^p(I\!R), 1 < p < \infty$. Now by (2.28), $L_\epsilon$ is a linear combination of the Hilbert transform (H) and $J_\epsilon$, so the same statement also applies to $V_\epsilon$.

To show that $V_* k \in L^2(I\!R)$ requires a deeper analysis based on Cotlar's inequality(see [T, p 291, Lem 6.1]). To outline the proof, from (2.28) we see that $(L_\epsilon)$ tends weakly in $L^2(I\!R)$ to $\alpha H$ for some complex $\alpha$. We claim that if $t_0 \in I\!R$ and $k \in L^2(I\!R)$, then

$$L_* k(t_0)\ \leq\ cM(Hk)(t_0) + cM^2 k(t_0) \tag{+}$$

where $M$ denotes the Hardy-Littlewood maximal function on intervals, $M^2 = M \circ M$, and $H$ is the Hilbert transform. To prove (+) one first shows that given $\epsilon > 0$,

$$|L_\epsilon k(t_0) - \alpha Hk(t_0)|\ \leq c|Hk_1(t_0)| + cM^2 k(t_0),$$

where $k = k_1 + k_2$ and $k = k_1$ in $\{t : |t - t_0| \leq 4\epsilon\}$ while $k_1$ vanishes outside of this set. Next one proves that

$$|Hk_1(t_0)| \leq c(|Hk(t_0)| + |Hk(t)| + |Hk_1(t)| + Mk(t_0)),$$

whenever $|t - t_0| \leq \epsilon$. Integrating the above inequality over a certain subset of $\{t : |t - t_0| \leq \epsilon\}$ and using the fact that $H$ is of weak type ( 1, 1), we conclude first that

$$|Hk_1(t_0)| \leq c|M(Hk)(t_0)| + cMk(t_0)$$

and thereupon that (+) is true. Taking the supremum over all such $\epsilon$ we get our claim. (*) in Lemma 2.27 follows from (+) and properties of $H, M$. (**) is a consequence of ($*$) and the fact that the limit exists for smooth functions. □

From our earlier remarks we now have proved $(iii)$ and so Lemma 2.14 is valid . □

We note that (**) was not needed in the proof of Lemma 2.14, but it will be useful in the proof of the next lemma.

To set the stage for the next lemma recall that $n_t(P,t)$ as defined in (0.14) is the outer normal to $\partial\Omega_t$ considered as a subset of $\mathbb{R}^n$ at $(P,t) \in \partial\Omega_t$ which exists almost everywhere with respect to the measure in (0.14). Let $\gamma_j = \gamma_j(P,t), 1 \leq j \leq n-1$, be the curve through $(P,t)$ which is the image under $\rho$ of the line $l_j$ through $\rho^{-1}(P,t)$ that is parallel to the $j$ th coordinate axis. We parametrize $\gamma_j$ so that the positive direction on this curve corresponds to the direction of increasing $x_j$ on $l_j$. Let $T_j, 1 \leq j \leq n-1$, be the unit tangent vector to $\gamma_j$ at $(P,t)$ whose tip points in the positive direction along $\gamma_j$. Clearly, $T_j$ exists for a.e $(P,t) \in \partial\Omega$ and is orthogonal to $n_t$. Also $\{T_j\}_1^{n-1}$ is a basis for the tangent space at $(P,t)$. Finally let $K, K^*$ be as in (1.2). With this notation we have

**Lemma 2.29** *Let $u, u', f, a$ be as in Lemma 2.14. Then for fixed $a > 0, 1 \leq j \leq n-1$, and almost* *every* $(x,t) \in \mathbb{R}^n$ *with respect to* $\sigma$ *in (0.12) we have*

$$(\alpha) \qquad \lim_{(\lambda,y,s)\longrightarrow(0,x,t)} u' \circ \rho(\lambda,y,s) \;=\; (-\tfrac{1}{2}f + Kf)\circ\rho(0,x,t),$$

$$(\beta) \qquad \lim_{(\lambda,y,s)\longrightarrow(0,x,t)} \langle\, \nabla u \circ \rho(\lambda,y,s),\, n_t \circ \rho(0,x,t)\,\rangle \;=\; (\tfrac{1}{2}f + K^*f)\circ\rho(0,x,t),$$

$$(\gamma) \qquad \lim_{(\lambda,y,s)\longrightarrow(0,x,t)} \langle \nabla u \circ \rho(\lambda,y,s), T_j \circ \rho(0,x,t)\,\rangle \;=\; [u\circ\rho(0,x,t)]_{x_j},$$

$$(\delta) \qquad \lim_{(\lambda,y,s)\longrightarrow(0,x,t)} HD^t_{1/2}\,(u\circ\rho)(\lambda,y,s) \;=\; HD^t_{1/2}(u\circ\rho)(0,x,t),$$

*where the limits are taken in the non tangential sense.*

In Lemma 2.29, $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^n$ and the partial derivatives of the righthand side in $(\gamma)$ are meant in the distributional sense. The proof of $(\alpha)$ and $(\beta)$ again follow from our geometric observation on mappings of parabolic cones and (A) of Theorems 1.13 and 1.14. To prove $(\gamma)$ one first shows as in [LM, ch3, sec 2] that for fixed $a > 0$

$$\lim_{(\lambda,y,s)\longrightarrow(0,x,t)} \nabla u \circ \rho(\lambda,y,s) \text{ exists componentwise and nontangentially for a.e } (x,t). \tag{2.30}$$

Thus the lefthand side of $(\gamma)$ in Lemma 2.29 makes sense for a.e $(x,t)$. To see this limit equals the righthand side of $(\gamma)$ observe that $(x,t)\to u\circ\rho(\epsilon,x,t)$ has distributional derivatives in $x_j$ for $1 \le j \le n-1$. Letting $\epsilon\to 0$, using (2.30) and $(ii)$ of Lemma (2.14), as well as dominated convergence we see that $u\circ\rho(0,x,t)$ has distributional derivatives in $x$ which agree with those obtained from using the chain rule in a naive way. Thus the righthand side of $(\gamma)$ makes sense a.e. Using these observations and writing the lefthand side of $(\gamma)$ in graph coordinates we see that $(\gamma)$ is valid.

It remains to prove $(\delta)$. To do so we retrace the proof of $(iii)$ of Lemma 2.14. Recall that $HD^t_{1/2}\,(u\circ\rho) \;=\; g_1 + g_2$ (see (2.15)) and $g_1$ was estimated in terms of $\lambda g_3$ (see (2.16)). We claim that

$$\lim_{(\lambda,\tilde{x},\tilde{t})\longrightarrow(0,x,t)} |g_1(\lambda,\tilde{x},\tilde{t})| \;=\; \lim_{(\lambda,\tilde{x},\tilde{t})\longrightarrow(0,x,t)} \lambda\, g_3(\lambda,\tilde{x},\tilde{t}) \;=\; 0. \tag{2.31}$$

From (2.16) we see that it suffices to prove this inequality for $\lambda g_3$ and with $\limsup$ replacing $\lim$. To estimate $\lambda g_3$ note from (c) of Lemma 2.8 with $\theta = 1, |\phi| = \sigma = o$, and $(ii)$ of Lemma (2.14) that for a.e $(x,t)$

$$\lim_{(\lambda,\tilde{x},\tau)\longrightarrow(0,x,t)} \lambda \,|(u_{\tilde{x}_0} \circ \rho)(\tfrac{\partial P_{\gamma\lambda} A}{\partial \tau})|\,(\lambda,\tilde{x},\tau) \;=\; 0. \tag{2.32}$$

where the limit is taken in $\Gamma_{c_1 K a}(x,t)$ and $c_1$ is defined in the sentence after (2.18). Using (2.30) and arguing as in (2.19) with $b = \lim\limits_{(\lambda,y,s)\longrightarrow(0,x,t)} (\nabla u \circ \rho)(\lambda, y, s)$ we deduce for a.e $(x,t)$ that

$$\begin{aligned} \limsup_{(\lambda,\tilde{x},\tau)\longrightarrow(0,x,t)} \lambda^2 \,|\, u_\tau \circ \rho\,|^2(\lambda,\tilde{x},\tau) \;&\leq\; c \limsup_{(\lambda,\tilde{x},\tau)\longrightarrow(0,x,t)} \lambda^2 \sum_{i=0}^{n-1} |\, u_{x_i x_i} \circ \rho\,|^2(\lambda,\tilde{x},\tau) \\ &\leq \limsup_{(\lambda,\tilde{x},\tau)\longrightarrow(0,x,t)} c_\beta\, \lambda^{-n-2} \int_{B_{\lambda/2}(\tilde{x},\tau)} \int_{\lambda/2}^{3\lambda/2} |(\nabla u \circ \rho)(\tilde{\lambda}, y, s) - b|^2 \, d\tilde{\lambda} dy ds \\ &= 0. \end{aligned} \tag{2.33}$$

From (2.32) and (2.33) we find in view of (2.17) that (2.31) holds.

Next as in (2.24) we see that

$$\limsup_{(\lambda,\tilde{x},\tilde{t})\longrightarrow(0,x,t)} |\, g_2(\lambda,\tilde{x},\tilde{t}) - g_4(\lambda,x,t)\,| \;\leq\; c^+ K^{-1} \sum_{i=0}^{n-1} M_n^{(2)}(\bar{N}_*(u_{x_i} \circ \rho))(x,t) \tag{2.34}$$

where $c^+$ depends only on $a, \beta, n,$ and $\gamma$. In (2.34) the $\limsup$ is taken through points in $\Gamma_a(x,t)$. Also from the note above (2.26) and Lemma 2.27 (**) we get

$$\lim_{\lambda\longrightarrow 0} g_4(\lambda,x,t) \;=\; HD_{1/2}(u \circ \rho)(0,x,t) \tag{2.35}$$

Hence from (2.35) and (2.34) we have

$$\limsup_{(\lambda,\tilde{x},\tilde{t})\longrightarrow(0,x,t)} |g_2(\lambda,\tilde{x},\tilde{t}) \;-\; HD^t_{1/2}(u \circ \rho)(0,x,t)| \;\leq c^+ K^{-1} \sum_{i=0}^{n-1} M_n^{(2)}(\bar{N}_*(u_{x_i} \circ \rho))(x,t) \tag{2.36}$$

From (2.36) and (2.31) we find that

$$\limsup_{(\lambda,\tilde{x},\tilde{t})\longrightarrow(0,x,t)} |\, HD_{1/2}(u \circ \rho)(\lambda,\tilde{x},\tilde{t}) \;-\; HD_{1/2}(u \circ \rho)(0.x,t)\,| \leq\; c^+ K^{-1} \sum_{i=0}^{n-1} M_n^{(2)}(\bar{N}_*(u_{x_i} \circ \rho))(x,t) \tag{2.37}$$

where again the limit is relative to $\Gamma_a(x,t)$. Since $K$ in (2.37) is arbitrary and the sum in (2.37) is finite for a.e $(x,t)$, we conclude that $(\delta)$ is true. The proof of Lemma 2.29 is now complete. □

Let $\rho_-(\lambda,x,t) = \rho(-\lambda,x,t)$ when $(\lambda,x,t) \in \mathbb{R}^{n+1}_- = \{(\lambda,x,t) : \lambda < 0 \text{ and } (x,t) \in \mathbb{R}^n\}$. We note that $\rho_-$ maps $\mathbb{R}^{n+1}_-$ onto $\mathbb{R}^{n+1} \setminus \Omega$ and $\rho_-$ extends continuouly to the closure of $\mathbb{R}^{n+1}_-$ by putting $\rho_- = \rho$ on the boundary of this domain. Given $a > 0$, let $\Gamma_-(x,t) = \Gamma_{-,a}(x,t) = \{(\lambda,y,s) : \lambda < o \text{ and } (y,s) \in B_{a|\lambda|}(x,t)\}$. Given a function $g$ defined on $\mathbb{R}^{n+1}_-$ let $N_* g(x,t)$ denote the nontangential maximal function of $g$ defined relative to $\Gamma_-(x,t)$ and let $\lim\limits_{(\lambda,y,s)\longrightarrow(0,x,t)} g(\lambda,y,s)$ be the limit in $\Gamma_-(x,t)$. For use in section 5 we now state analogues of Lemmas 2.14 and 2.29 for $u \circ \rho_-$.

**Lemma 2.38** *Let $A, a, f$ be as in Lemma 2.14 and let $u^- = \mathcal{S}f$, be the single layer potential of $f$ in $\mathbb{R}^{n+1} \setminus \bar{\Omega}$. Then $(ii) - (iii)$ of Lemma 2.14 are valid with $u, \rho$ replaced by $u^-, \rho_-$.*

**Lemma 2.39** *Let $u^-, f, a, A,$ be as in Lemma 2.37 and let $n_t, T_j$, $1 \leq j \leq n-1$, be as in Lemma 2.29. Then $(\gamma), (\delta)$ remain true with $u, \rho$ replaced by $u^-, \rho_-$. Moreover, for a.e $(x,t)$*

$$\lim_{(\lambda,y,s)\longrightarrow(0,x,t)} \langle \nabla u^- \circ \rho_-(\lambda,y,s),\, n_t \circ \rho_-(0,x,t) \rangle = (-\tfrac{1}{2} f + K^* f) \circ \rho_-(0,x,t).$$

The proofs of Lemmas 2.38 and 2.39 are essentially the same as the proofs of Lemmas 2.14 and 2.29, so we omit the details. Comparing Lemmas 2.29 and 2.39 we see that $u, u^-$ have the same "tangential derivatives " in the space variable and the same 1/2 derivatives in time at points of $\partial\Omega$. Also the " outer normal derivatives " of these functions at a point $(P,t) \in \partial\Omega_t$ differ by $2K^* f(P,t)$. This jump relation is one of the key facts which will allow us to prove Theorem 1.16 in sections 5-7.

## 3. Square Functions of "Calderón - type " and a Simple Proof of Theorem 1.10.

In this section we prove the square function estimates mentioned in section 1 which are in the spirit of those considered by David, Journé, and Semmes [DJS, sec 11]. However we shall use a different method of proof. Our method of proof is based on an idea of P. Jones [JnsP], who gave a proof of the deep result of Coifman, MacIntosh, and Meyer [CMM] concerning the $L^2$ boundedness of the Cauchy integral operator along a Lipschitz curve, by viewing the Lipschitz curve as (locally) a pertubation of an approximating line and then controlling the resulting error terms by a certain Carleson measure estimate. In this context see also the work of Fang [Fng], and the monograph of Christ [Ch]. We note that an important antecedent of Jones' ideas is contained in the work of Dorronsoro [Do]. We shall apply our square function estimates to get an alternative proof of Theorem 1.10.

To this end let $K_\lambda(z,v)$, $(\lambda,z),(\lambda,v)\in \mathbb{R}^{n+1}_+$ be a family of real valued kernels satisfying

$$|K_\lambda(z,v)| \le c_K \frac{\lambda}{(\lambda+\|z-v\|)^{d+1}}\,. \tag{3.1}$$

$$|K_\lambda(z,v)-K_\lambda(z,\tilde v)| \le c_K\,\|v-\tilde v\|\,\min\left(\frac{1}{\lambda^d\,\|z-v\|}\,,\,\frac{\lambda}{\|z-v\|^{d+2}}\right) \tag{3.2}$$

whenever $2\|v-\tilde v\|\le\|z-v\|$. Let $\omega$ be an $A_2$ weight and $f\in L^2_\omega(\mathbb{R}^n)$ (see (2.5)). Put

$$K_\lambda f(z)=\int_{\mathbb{R}^n} K_\lambda(z,v)\,f(v)dv,\ z\in\mathbb{R}^n.$$

Using the $A_2$ condition and making estimates in terms of the maximal function of $f^2$ with respect to $\tilde\omega$ it is easily seen for fixed $\lambda>0$ that the integral defining $K_\lambda f(z)$ is absolutely convergent for a.e $z=(x,t)\in\mathbb{R}^n$ with respect to Lebesgue $n$ measure. We shall need the following " orthogonality " theorem. This result is perhaps well known, but for completeness we sketch the proof.

**Theorem 3.3** *Let* $(K_\lambda)$ *satisfy (3.1), (3.2) and let* $\omega, f$ *be as above. If* $K_\lambda 1 \equiv 0$ *for each* $\lambda > 0$, *then*

$$\int_{\mathbb{R}_+^{n+1}} (K_\lambda f)^2(z)\,\omega(z)\,\frac{dzd\lambda}{\lambda} \;\le\; c_{K,\omega}\,\|f\|_{2,\omega}^2\,.$$

In Theorem 3.3, $c_{K,\omega}$ denotes a constant depending only on $K, d$, and the $A_2$ constant in (2.5), which is the same convention we used in section 2. Theorem 3.3 is stated in [Ch, p 69, Thm 20] for $\omega = 1$ ( see also [CJ2] ) under slightly weaker hypotheses.

**Proof:** Let $\tilde{Q}_\lambda$ be an approximation to the zero operator (see (0.23)) which satisfies (0.24) and the condition that

$$\int_0^\infty \tilde{Q}_\lambda * \tilde{Q}_\lambda\,\frac{d\lambda}{\lambda} \;=\; \text{identity}$$

as a convolution operator on $L^2(\mathbb{R}^n)$. Here we are using our notational convention: the two $\tilde{Q}_\lambda$ operators need not be the same.

Thus we can write $f = \int_0^\infty \tilde{Q}_\delta^2 f \frac{d\delta}{\delta}$, and use the fact that $K_\lambda$, $\tilde{Q}_\delta$ kill constants and satisfy "standard estimates " to get the bound

$$|K_\lambda \tilde{Q}_\delta \tilde{Q}_\delta f|(z) \;\le\; c_K\,\min\left(\frac{\delta}{\lambda},\frac{\lambda}{\delta}\right)\,M(\,\tilde{Q}_\delta f\,)(z), \tag{3.4}$$

where $M$ is the maximal function defined relative to balls in $\|\cdot\|$. Using (3.4), Hardy's inequality, Muckenhoupt's theorem for $A_2$ weights (see [GR, ch 4]), and weighted Littlewood-Paley theory we find

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} (K_\lambda f)^2(z)\,\omega(z)\,\frac{dzd\lambda}{\lambda} \;&\le\; c_K \int_{\mathbb{R}_+^{n+1}} \left(M(\tilde{Q}_\lambda f)\right)^2(z)\,\omega(z)\,\frac{dzd\lambda}{\lambda} \\
&\le c_{K,\omega} \int_{\mathbb{R}_+^{n+1}} (\tilde{Q}_\lambda f)^2(z)\,\omega(z)\,\frac{dzd\lambda}{\lambda} \;\le\; c_{K,\omega}\,\|f\|_{2,\omega}^2\,.
\end{aligned} \tag{3.5}$$

This completes the sketch of Theorem 3.3. □

Next let $H \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfy the homogeneity condition

$$H(\delta^\alpha z) = \delta^{-d-1} H(z) \text{ for } z = (x,t),\; d = n+1, \tag{3.6}$$

and assume that $F \in C^1(\mathbb{R})$ with

$$\begin{aligned}
|F(r)| \;&\le\; c_F\,\frac{1}{1+|r|^{d+1}} \\
|F'(r)| \;&\le\; c_F\,\frac{1}{1+|r|^{d+2}}
\end{aligned} \tag{3.7}$$

whenever $r \in \mathbb{R}$ . For $f, \omega$ as above define a square function $G$ of " Calderón type " by setting

$$R_\lambda f(z) \;\equiv\; \lambda \int_{\mathbb{R}^n} H(z-v)\,F\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v)\,dv, \tag{3.8}$$

$$Gf(z) = \left(\int_0^\infty |\,R_\lambda f(z)\,|^2\,\frac{d\lambda}{\lambda}\right)^{1/2}. \tag{3.9}$$

We now prove the following theorem.

**Theorem 3.10** *Suppose that for $H, F$ as above we have either $F$ is odd and $H(x,t)$ is odd in $x$ for each fixed $t$; or else that $F$ is even, $H(x,t)$ is even in $x$ for each fixed $t$, and also that $\int_{\mathbb{R}} F(r)dr = 0$. If $\|A\|_{\text{comm}} \le \beta < \infty$, and $\omega, f$ are as above, then there exists a positive integer $N$ depending only on $d$ such that*

$$\|Gf\|_{2,\omega} \;\le\; c_{F,H,\omega}\,(1+\beta)^N\,\|f\|_{2,\omega}.$$

**Proof:** As in section 0 let $P \in C_0^\infty(B_1(0))$ be an even function with $\int_{\mathbb{R}^n} P_\lambda(z)dz \equiv 1$ and let $P_\lambda f$ be the convolution operator defined in (0.22). Put

$$Q_\lambda^* f(z) \equiv \lambda \int_{\mathbb{R}^n} H(z-v)\, F\left(\frac{\langle \nabla_{z'} P_\lambda A(z), z'-v' \rangle + \lambda}{\|z-v\|}\right) f(v)\, dv,$$

where $z' = x, v' = y$ if $z = (x,t)$ and $v = (y,s)$. Then

$$\begin{aligned} Gf(z) \, &\leq \left( \int_0^\infty |\, (R_\lambda - Q_\lambda^*) f(z)\, |^2 \frac{d\lambda}{\lambda} \right)^{1/2} + \left( \int_0^\infty |Q_\lambda^* f(z)|^2 \frac{d\lambda}{\lambda} \right)^{1/2} \\ &= G_1 f(z) + G_2 f(z). \end{aligned} \tag{3.11}$$

We set $V_\lambda \equiv R_\lambda - Q_\lambda^*$ and observe from (3.6), (3.7) and a ball park estimate that the kernel $V_\lambda(z,v)$ of $V_\lambda$ satisfies

$$|V_\lambda(z,v)| \leq c(1+\beta)^{d+2} \frac{\lambda}{(\,\lambda + \|z-v\|\,)^{d+2}}\, |A(z) - A(v) - \langle \nabla_{z'} P_\lambda A(z),\, z'-v' \rangle\, | \tag{3.12}$$

where $c$ depends on $F, H, d$. Using (3.12) and (0.10) we deduce that $V_\lambda$ satisfies (3.1) with $K$ replaced by $V$ and $c_K$ replaced by $c(1+\beta)^{d+3}$. Also by the same argument we see that the kernel of $Q_\lambda^*$ satisfies (3.1) with $Q^* = K$ and the same constants as $V$. Moreover, since $H \in C^1(\mathbb{R}^n \setminus \{0\})$ we find in addition from (3.6), (3.7), (0.10), that the kernels of $V_\lambda$, $Q_\lambda^*$ satisfy (3.2) with the same constants as in (3.1).

First we consider $G_1$ in (3.11). This term will be handled in the spirit of [JnsP]. From the above discussion we see that if $K_\lambda = V_\lambda - (V_\lambda 1)\, P_\lambda$, then we may apply Theorem 3.10 to $K_\lambda f$ since $K_\lambda 1 \equiv 0$ for each $\lambda > 0$. Thus to show

$$\|G_1 f\|_{2,\omega} \leq c_{F,H,\omega}\, (1+\beta)^N\, \|f\|_{2,\omega} \tag{3.13}$$

we need only prove

$$\int_0^\infty \int_{\mathbb{R}^n} (V_\lambda 1\, P_\lambda f)^2\, \omega\, \frac{dz d\lambda}{\lambda} \leq c_{F,H,\omega}\, (1+\beta)^{2N}\, \|f\|_{2,\omega}$$

To prove this inequality we note from simple estimates involving the maximal function and Muckenhoupt's theorem it follows that

$$\|N_*(P_\lambda f)\|_{2,\omega} \;\le\; c_\omega \|f\|_{2,\omega},$$

where $N_*(P_\lambda f)(z)$ is the nontangential maximal function of $P_\lambda f(z)$ defined relative to $\Gamma_1(z)$ as in section 2. From this note and (2.7) we deduce that in order to prove the above inequality it suffices to show that

$$d\nu(\lambda, z) \;=\; [V_\lambda 1(z)]^2\,\omega(z)\,dzd\lambda/\lambda \tag{3.14}$$

is a weighted Carleson measure with norm comparable to the constants in Theorem 3.10. For this purpose let $z_0 \in I\!\!R^n, r > 0$, and let $\chi$, $\chi^*$ denote the characteristic functions of $B_{10r}(z_0)$, $I\!\!R^n \setminus B_{10r}(z_0)$, respectively. Using (3.1) for $V_\lambda$ we deduce first that

$$\int_0^r \int_{B_r(z_0)} (V_\lambda \chi^*)^2(z)\,\omega(z)\,\frac{dzd\lambda}{\lambda} \;\le\; c\,(1+\beta)^{2d+6}\,\omega(B_r(z_0))$$

and thereupon that it suffices to prove that $\nu$ in (3.14) is a Carleson measure with 1 replaced by $\chi$. Next we put $\tilde{A}(z) = \psi(\|z - z_0\|)\,(A(z) - A(z_0)),\; z \in I\!\!R^n,$ where $\psi \in C_0^\infty(-20r, 20r)$ is an even function with $\psi \equiv 1$ on $[-15r, 15r]$. Then $V_\lambda\,\chi(z)$ is unchanged for $z \in B_{10r}(z_0)$, $0 < \lambda < r$, if we replace $A$ in its definition by $\tilde{A}$. Moreover from [H2, sec 6, Lem 2] we have

$$\begin{aligned} &(i) \quad \|\tilde{A}\|_{\mathrm{comm}} \;\le\; c\,\|A\|_{\mathrm{comm}} \\ &(ii) \quad \text{For } 1 < p < \infty,\; \|\, I\!\!D\tilde{A}\,\|_p^p \;\le\; c_p\,\beta^p\,r^d. \end{aligned} \tag{3.15}$$

Using (3.12), Schwarz's inequality, and the change of variable $\lambda \to \lambda/2^l$ we obtain, for $N$ large enough that

$$\begin{aligned} &(1+\beta)^{-2N} \int_0^r \int_{B_r(z_0)} (V_\lambda\chi)^2(z)\,\omega(z)\frac{dzd\lambda}{\lambda} \\ &\le c\sum_{l=0}^{\infty} 2^{-l} \int_0^\infty \int_{I\!\!R^n} \lambda^{-d-2}\left[\int_{B_\lambda(z)} |\tilde{A}(z) - \tilde{A}(v) - \langle \nabla_{z'} P_{2^{-l}\lambda}\tilde{A}(z), z' - v'\,\rangle|^2 dv\right]\omega(z)\frac{dzd\lambda}{\lambda} \\ &\le\; c_\omega\,\beta^2\,\omega(B_r(z_0)), \end{aligned} \tag{3.16}$$

where the last inequality follows from (3.15) and an argument involving Plancherel's Theorem in the case $\omega \equiv 1$ ( see [H2, sec 5] for more details) or else the argument of [H2, sec 6, Lem 3] in the weighted case. Thus (3.13) is true.

To prove (3.13) with $G_1$ replaced by $G_2$ we note from (3.1), (3.2) for $Q_\lambda^*$ and Theorem 3.3 that it is enough to show that $Q_\lambda^* 1 \equiv 0$. To do this we introduce parabolic polar coordinates as in (0.7) to get

$$Q_\lambda^* 1(z) = \lambda \int_S \left( \int_0^\infty F(\langle \vec{a}, \sigma' \rangle + \lambda/\rho) \, d\rho/\rho^2 \right) H(\sigma) \, \Phi(\sigma) \, d\sigma,$$

where $\vec{a} = \nabla_{z'} P_\lambda A(z)$, $\Phi(\sigma) = (1 + \sigma_n^2)$, and $\sigma = (\sigma', \sigma_n) \in S =$ the unit sphere in $\mathbb{R}^n$. We change variables in the above integral by $\rho \to \lambda\rho$, then $r = 1/\rho$, then $r \to r - \langle \vec{a}, \sigma' \rangle$, to obtain

$$Q_\lambda^* 1(z) \; = \; \int_S \left( \int_{\langle \vec{a}, \sigma' \rangle}^\infty F(r) \, dr \right) H(\sigma) \Phi(\sigma) d\sigma \; = \; 0$$

since our hypotheses in Theorem 3.10 guarantee that this last expression is zero. Indeed $\int_{\langle \vec{a}, \sigma' \rangle}^\infty F(r) dr$ is a function of $\sigma'$ having opposite parity to $H(\sigma)\Phi(\sigma)$, for each fixed non-zero $\vec{a}$. The case $\vec{a} = 0$ is much simpler : if $H$ is odd in $\sigma'$, then clearly $\int_S H(\sigma)\Phi(\sigma) \, d\sigma \; = \; 0$, and if $F$ is even with $\int_{-\infty}^\infty F(r) dr \; = \; 0$, then $\int_0^\infty F(r) dr = 0$ . Thus (3.13) also holds for $G_2$. From (3.13) for $G_1, G_2$ we find in view of (3.11) that Theorem 3.10 is true. $\square$

Theorem 3.10 is easily generalized. Indeed, let $H, F$, be as in (3.6), (3.7), and let $B : \mathbb{R}^n \to \mathbb{R}$ with $\|B\|_{\text{comm}} \le \beta_0 < \infty$. Let $A$ be as in Theorem 3.10 and put

$$\tilde{R}_\lambda f(z) \; \equiv \; \lambda \int_{\mathbb{R}^n} H(z - v) \, \frac{B(z) - B(v)}{\|z - v\|} \, F\left( \frac{A(z) - A(v) + \lambda}{\|z - v\|} \right) f(v) \, dv, \tag{3.17}$$

$$\tilde{G} f(z) = \left( \int_0^\infty | \, \tilde{R}_\lambda f(z) \, |^2 \, \frac{d\lambda}{\lambda} \right)^{1/2} \tag{3.18}$$

We then have

**Theorem 3.19** *Let $H, F, A, B$ be as above and suppose that either $F$ is odd and $H(x,t)$ is even in $x$ for each fixed $t$; or else that $F$ is even, $H(x,t)$ is odd in $x$ for each fixed $t$, and also that $\int_{\mathbb{R}} F(r)dr = 0$. If $f, \omega$ are as in Theorem 3.10, then there exists a positive integer $N$ depending only on $d$ such that*

$$\|\tilde{G}f\|_{2,\omega} \leq c_{F,H,\omega}\, \beta_0\, (1+\beta)^N\, \|f\|_{2,\omega}.$$

**Proof:** We shall be brief, since the ideas are now familiar. Put

$$U_\lambda f(z) \equiv \lambda \int_{\mathbb{R}^n} H(z-v)\, \frac{\langle \nabla_{z'} P_\lambda B(z), z'-v' \rangle}{\|z-v\|}\, F\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v)\, dv.$$

Then as in (3.11)

$$\begin{aligned} \tilde{G}f(z) &\leq \left(\int_0^\infty |\,(\tilde{R}_\lambda - U_\lambda) f(z)\,|^2 \frac{d\lambda}{\lambda}\right)^{1/2} + \left(\int_0^\infty |U_\lambda f(z)|^2 \frac{d\lambda}{\lambda}\right)^{1/2} \\ &= \tilde{G}_1 f(z) + \tilde{G}_2 f(z). \end{aligned} \tag{3.20}$$

If $\tilde{V}_\lambda = \tilde{R}_\lambda - U_\lambda$, then as in (3.12) we deduce

$$|\tilde{V}_\lambda(z,v)| \leq c(1+\beta)^{d+2} \min\left(\frac{\lambda}{\|z-v\|^{d+2}}, \frac{1}{\lambda^d \|z-v\|}\right) |B(z)-B(v)-\langle \nabla_{z'} P_\lambda B(z),\, z'-v' \rangle|$$

where $c$ depends on $F, H, d$. Using this inequality in place of (3.12) we can now repeat the argument following (3.12) through (3.16) to get that (3.13) holds with $G_1$ replaced by $\tilde{G}_1$ and constants as in Theorem 3.19. As for $\tilde{G}_2$ we note from (0.10) that the kernel of $U_\lambda$ can be written as a sum of $L^\infty$ functions (the components of $\nabla_{z'} P_\lambda B(z)$) times operators whose kernels satisfy the hypotheses of Theorem 3.10. Thus (3.13) holds with $G_1$ replaced by $\tilde{G}_2$ and constants as in Theorem 3.19. In view of (3.20) we conclude that Theorem 3.19 is valid. □

In our applications the square functions defined in (3.8)-(3.9) and (3.17)-(3.18) model the second derivatives of the single layer potential lifted by $\rho$ in (2.4) to $\mathbb{R}^{n+1}_+$. We shall also need a model for higher order derivatives. We refrain from proving the most general result of this type as it would lead us too far astray from the purposes of this paper. Suppose $L \in C^1(I\!R^n \setminus \{0\})$ with

$$L(\delta^\alpha z) = \delta^{-d-2} L(z), z \in I\!R^n, \tag{3.21}$$

and let $E \in C^1(I\!R)$ with

$$\begin{aligned} |E(r)| &\leq c_E \frac{1}{1+|r|^{d+2}} \\ |E'(r)| &\leq c_E \frac{1}{1+|r|^{d+3}} \end{aligned} \tag{3.22}$$

whenever $r \in I\!R$. Suppose that either $E$ is even with $\int_{I\!R} E(r)dr = 0$ and $L(x,t)$ is odd in $x$ for each fixed $t$; or else that $E$ is odd, with $\int_{I\!R} rE(r)dr = 0$, and $L(x,t)$ is even in $x$ for each fixed $t$. Next assume that $\tilde{L} \in C^1(I\!R^n \setminus \{0\})$ satisfies (3.21) and $\tilde{E} \in C^1(I\!R)$ satisfies (3.22). Suppose that either $\tilde{E}$ is even with $\int_{I\!R} \tilde{E}(r)dr = 0$ while $\tilde{L}(x,t)$ is even in $x$ for each fixed $t$; or else that $\tilde{E}$ is odd with $\int_{I\!R} r\tilde{E}(r)dr = 0$, while $\tilde{L}(x,t)$ is odd in $x$ for each fixed $t$. We set

$$\begin{aligned} T_\lambda f(z) &\equiv \lambda^2 \int_{I\!R^n} L(z-v)\, E\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v)\, dv, \\ \tilde{T}_\lambda f(z) &\equiv \lambda^2 \int_{I\!R^n} \tilde{L}(z-v)\, \frac{B(z)-B(v)}{\|z-v\|}\, \tilde{E}\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right) f(v) dv, \end{aligned} \tag{3.23}$$

where $A, B$ are as in Theorem 3.19 and

$$\begin{aligned} g(f)(z) &= \left(\int_0^\infty |\, T_\lambda f(z)\,|^2 \frac{d\lambda}{\lambda}\right)^{1/2} \\ \tilde{g}(f)(z) &= \left(\int_0^\infty |\, \tilde{T}_\lambda f(z)\,|^2 \frac{d\lambda}{\lambda}\right)^{1/2} \end{aligned} \tag{3.24}$$

With this notation we have

**Theorem 3.25** *Let $E, L, \tilde{E}, \tilde{L}, g, \tilde{g}, A, B$, be as above. Then there exists a positive integer $N = N(d)$ such that if $f, \omega$ are as in Theorem 3.10, we have*

$$\|g(f)\|_{2,\omega} + \beta_0^{-1} \|\tilde{g}(f)\|_{2,\omega} \leq c(1+\beta)^N \|f\|_{2,\omega}.$$

*where* $c$ *depends on* $\omega, E, L, \tilde{E}, \tilde{L},$ *and* $d$.

**Proof:** We prove Theorem 3.25 only for $g$ since the bound for $\tilde{g}$ can be deduced from the one for $g$ in the same way that Theorem 3.10 implied Theorem 3.19. We shall be very brief since the reader should now be well versed in our method of proof. Using (3.21), (3.22) in place of (3.6), (3.7) and arguing exactly as in the proof of Theorem 3.10, we reduce matters to demonstrating that $q_\lambda 1 = 0$, for each $\lambda > 0$, where

$$q_\lambda(z, v) = \lambda^2 \, L(z - v) \, E\left( \frac{\langle \nabla_{z'} P_\lambda A(z), z' - v' \rangle + \lambda}{\| z - v \|} \right).$$

To do this we again pass to parabolic polar coordinates and write

$$q_\lambda 1(z) = \lambda^2 \int_S \left( \int_0^\infty E(\langle \vec{a}, \sigma' \rangle + \lambda/\rho) \, \frac{d\rho}{\rho^3} \right) \, L(\sigma) \Phi(\sigma) d\sigma,$$

where $\vec{a} = \nabla_{z'} P_\lambda A(z)$, as previously. Again we change variables by $\rho \to \lambda\rho$, then $r = 1/\rho$, then $r \to r - \langle \vec{a}, \sigma' \rangle$. We obtain

$$q_\lambda 1(z) = \int_S \left( \int_{\langle \vec{a}, \sigma' \rangle}^\infty r E(r) dr \right) \, L(\sigma) \Phi(\sigma) d\sigma$$

$$- \int_S \left( \int_{\langle \vec{a}, \sigma' \rangle}^\infty E(r) dr \right) \, \langle \vec{a}, \sigma' \rangle \, L(\sigma) \Phi(\sigma) d\sigma = 0,$$

as follows from our parity assumptions on $E, L$. This concludes the proof of Theorem 3.25. $\square$

**Alternative Proof of Theorem 1.10 :** Next we shall use Theorems 3.10, 3.19, 3.25, and Theorem 1.9 to give an alternate proof of Theorem 1.10 (i.e essentially Theorem 3 of [H2]). As mentioned in section 1, our reduction of the proof of Theorem 1.10 to square function estimates which we have proved in the previous theorems, will be in the spirit of some recent work of Li, MacIntosh, and

Semmes [LiMS, sec 4]. In this vein see also Mitrea [Mi]. To begin, we consider first the operator $S = S_A$ of Theorem 1.10. For specificity, we consider

$$Sf(z) \equiv \int_{\mathbb{R}^n} J(z-v) \cos\left(\frac{A(z)-A(v)}{\|z-v\|}\right) f(v)dv,$$

where

$$\begin{aligned} &(a) \quad J(x,t) \text{ is even in } x, \text{ for each fixed } t, \\ &(b) \quad J(\lambda^\alpha z) \equiv \lambda^{1-d} J(z), z \in \mathbb{R}^n, \\ &(c) \quad J \in C_0^N(\mathbb{R}^n \setminus \{0\}), \text{ for some large } N. \end{aligned} \tag{3.26}$$

Then from (0.15) we need to show for some large $N$ that

$$\|\mathbb{D} Sf\|_p \leq c_{J,p} (1+\beta)^N \|f\|_p. \tag{3.27}$$

whenever $f \in L^p(\mathbb{R}^n)$, and $1 < p < \infty$. To prove (3.27) recall from (2.9) that $\mathbb{D} \equiv \sum_{j=1}^{n-1} R_j \mathbb{D}_j$ where $R_j$ are parabolic Riesz transforms which are bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$, and $\mathbb{D}_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq n-1$. Since $\nabla_x A \in L^\infty(\mathbb{R}^n)$ it follows from a truncation type argument and Theorem 1.9 that we can differentiate inside the integral defining $Sf$ to deduce that $\mathbb{D}_j S$, $1 \leq j \leq n-1$, is a bounded function times a Calderón-Zygmund operator falling under the scope of Theorem 1.9. Thus to prove (3.27) it suffices to prove this inequality with $\mathbb{D}$ replaced by $\mathbb{D}_n = i\mathbb{D}^{-1}\frac{\partial}{\partial t}$, where $\mathbb{D}_n$ is defined as in (0.3). In fact if $\omega$ is an $A_2$ weight and $f \in L^2_\omega(\mathbb{R}^n)$, we shall show that

$$\|\mathbb{D}_n Sf\|_{2,\omega} \leq c_{J,\omega} (1+\beta)^N \|f\|_{2,\omega}. \tag{3.28}$$

Once (3.28) is proved, it then follows from extrapolation (see [GR, ch 4, thm 5.19]) that (3.27) is true.

To make our arguments rigorous, observe that since $|A(z)-A(v)| \leq c\|A\|_{\text{comm}} \|z-v\|$ (see (0.10)), we can replace the cosine in the definition of $Sf$ by $\Lambda$ where $\Lambda(r) = \phi(r)\cos(r)$ and $\phi \in C_0^\infty(\mathbb{R})$ is an even function with $\phi \equiv 1$ on $[-c\beta, c\beta]$. Clearly we can also choose $\phi$ so that $\int_{-\infty}^{\infty} \Lambda(r)dr = 0$.

We make the a priori assumption that $f \in C_0^\infty(\mathbb{R}^n), A \in C^\infty(\mathbb{R}^n)$, and that $J$ has been smoothly truncated so that it is supported on a parabolic annulus. These assumptions allow us to easily justify repeated differentiations and integrations by parts in the argument which follows. In the rest of the proof we shall systematically suppress the truncation, so as not to tire the reader with routine details. This means that we shall be ignoring certain error terms which arise as a result of the truncation, but these are not difficult to handle. Of course, our estimates will not have any quantitative dependence upon our a priori assumptions.

Under these assumptions we first use a construction of Kenig-Stein to write $Sf(z) = \lim_{\lambda \longrightarrow 0} S_\lambda f(z)$, where

$$S_\lambda f(z) \equiv \int_{\mathbb{R}^n} J(z-v)\, \Lambda \left( \frac{P_{\gamma\lambda} A(z) + \lambda - A(v)}{\|z-v\|} \right) f(v)\, dv,\ z \in \mathbb{R}^n,$$

and $P_{\gamma\lambda}$ is as in (0.22), while $\gamma \geq [c(1+\beta)]^{-1}$ is the largest number $\leq 1$ such that (2.2) is true. Next let $g \in C_0^\infty(\mathbb{R}^n)$ with $\|g\|_{2,1/\omega} = 1$ and observe that

$$\|\mathbb{D}_n Sf\|_{2,\omega} = \sup \left| \int_{\mathbb{R}^n} \mathbb{D}_n Sf\ g dz \right|,$$

where the supremum is taken over all such $g$. Moreover,

$$\begin{aligned} -\int_{\mathbb{R}^n} \mathbb{D}_n Sf\ g dz &= \int_0^\infty \int_{\mathbb{R}^n} \tfrac{\partial}{\partial\lambda} (\mathbb{D}_n S_\lambda f\ P_\lambda g)\, dz d\lambda \\ &= \int_0^\infty \int_{\mathbb{R}^n} \mathbb{D}_n \tfrac{\partial}{\partial\lambda} S_\lambda f\ \ P_\lambda g\, dz d\lambda + \int_0^\infty \int_{\mathbb{R}^n} \mathbb{D}_n S_\lambda f\ \ \tfrac{\partial}{\partial\lambda} P_\lambda g\, dz d\lambda \\ &= I + II. \end{aligned} \tag{3.29}$$

We recall from (2.12) and the observation directly above (2.12) that $\frac{\partial P_\lambda}{\partial\lambda} = \mathbb{D}\tilde{Q}_\lambda$ where $\tilde{Q}$ is an approximation to the zero operator which satisfies (0.24). Thus since $\mathbb{D}_n = i\mathbb{D}^{-1}\frac{\partial}{\partial t}$, we have

$$|II| = \left| \int_0^\infty \int_{\mathbb{R}^n} \tfrac{\partial}{\partial t} S_\lambda f\, \tilde{Q}_\lambda g\, dz d\lambda. \right|$$

We note from $\|g\|_{2,1/\omega} = 1$, and weighted Littlewood-Paley theory that

$$\int_0^\infty \int_{\mathbb{R}^n} (\tilde{Q}_\lambda g)^2 \, (1/\omega) \, dz \frac{d\lambda}{\lambda} \le c_\omega \, .$$

Using this note and Schwarz's inequality we deduce

$$|II|^2 \le c_\omega \int_0^\infty \int_{\mathbb{R}^n} |\tfrac{\partial}{\partial t} S_\lambda f|^2 \, \omega \, \lambda dz d\lambda. \tag{3.30}$$

Now let

$$w(x_0, z) \equiv \int_{\mathbb{R}^n} J(z - v) \, \Lambda \left( \frac{x_0 - A(v)}{\|z - v\|} \right) f(v) dv$$

and let $\rho(\lambda, z) = (\lambda + P_{\gamma\lambda} A(z), z)$ be as defined in (2.4). Since $w \circ \rho(\lambda, z) = S_\lambda f(z)$, we have for $z = (x, t)$

$$\begin{aligned} \tfrac{\partial}{\partial t} S_\lambda f(z) &= \tfrac{\partial}{\partial t} (w \circ \rho)(\lambda, x, t) \\ &= w_t \circ \rho(\lambda, x, t) + w_{x_0} \circ \rho(\lambda, x, t) \, \tfrac{\partial}{\partial t} P_{\gamma\lambda} A(x, t). \end{aligned} \tag{3.31}$$

To handle the contribution of $w_t \circ \rho$ to the integral in (3.30) we use the change of variable

$$\tilde{\lambda} \equiv \lambda + P_{\gamma\lambda} A(z) - A(z), \tag{3.32}$$

which defines a mapping $(\lambda, z) \rightarrow (\tilde{\lambda}, z)$ of $\mathbb{R}_+^{n+1}$ with Jacobian

$$1 + \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A(z) = \eta(\lambda, z).$$

Since $|\tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A(z)| \le \frac{1}{2}$ (see (2.2)) and $\lim\limits_{\lambda \to 0} P_{\gamma\lambda} A = A$, we deduce first that $1/2 \le \eta \le 3/2$ and thereupon that the above mapping is 1-1 and onto $\mathbb{R}_+^{n+1}$. Differentiating under the integral defining $w$ with respect to $t$ and then changing variables as in (3.32) we find at $z$ that

$$\tilde{\lambda} \, w_t \circ \rho = R_{\tilde{\lambda},1} f + R_{\tilde{\lambda},2} f,$$

where $R_{\lambda,i}$ is defined as in (3.8) with $H_i, F_i$ replacing $H, F$ for $i = 1, 2$. Here at $z$, $H_1 = \frac{\partial J}{\partial t}$, $H_2 = -J \frac{\partial \|z\|}{\partial t} / \|z\|$, and at $r$, $F_1 = \Lambda$, $F_2 = r\Lambda'$. Let $G^i$ be the square functions defined relative to $R_{\lambda,i}$ as in (3.9) for $i = 1, 2$. Then from the above equality and Theorem 3.10, we get

$$\int_0^\infty \int_{\mathbb{R}^n} (w_t \circ \rho)^2 \, \omega \, \lambda dz d\lambda \le c \sum_{i=1}^{2} \|G^i f\|_{2,\omega}^2 \le c_{J,\omega} (1+\beta)^{2N} \|f\|_{2,\omega}^2, \tag{3.33}$$

which is the desired estimate for the contribution of $w_t \circ \rho$ to the integral in (3.30).

To handle the contribution of the second term in (3.31) to this integral, we shall briefly discuss Theorem 1.9 and some of its consequences. To begin this discussion, we note that Theorem 1.9 extends to $A_2$ weights with $T_A, T_{A,B}$ replaced by the corresponding maximal operators, $\bar{T}_A$, $\bar{T}_{A,B}$ . To be more specific, $\bar{T}_A, \bar{T}_{A,B}$ are defined by

$$\bar{T}_A f = \sup_{\epsilon > 0} |T_{A,\epsilon} f|, \;\; \bar{T}_{A,B} f = \sup_{\epsilon > 0} |T_{A,B,\epsilon} f|,$$

where $T_{A,\epsilon}$ denotes the operator whose kernel is

$$T_{A,\epsilon}(z,v) = \begin{cases} T_A(z,v) \text{ if } \|z-v\| \ge \epsilon, \\ 0 \text{ if } \|z-v\| < \epsilon \end{cases}$$

and $T_{A,B,\epsilon}$ is defined similarly. If $\omega$ is as above, $\|A\|_{\text{comm}} \le \beta$, $\|B\|_{\text{comm}} \le \beta_0$, and $f \in L_{2,\omega}(\mathbb{R}^n)$, then Theorem 1.9 extends to

$$\|\bar{T}_A f\|_{2,\omega} + \beta_0^{-1} \|\bar{T}_{A,B} f\|_{2,\omega} \le c_{H,\omega} (1+\beta)^N \|f\|_{2,\omega}. \tag{3.34}$$

By [H2], (3.34) holds with $T_\epsilon$ in place of $\bar{T}$, with a bound uniform in $\epsilon$. To get $\bar{T}$ in (3.34) one first shows that

$$\lim_{\epsilon \longrightarrow 0} T_{A,\epsilon} f(z), \;\text{ and }\; \lim_{\epsilon \longrightarrow 0} T_{A,B,\epsilon} f(z),$$

exist weakly in $L^2(\mathbb{R}^n)$. Second, from (3.34) for $T_A$, $T_{A,B}$, existence of the above limits, the fact that the kernels of these operators satisfy " standard estimates," and an argument of Cotlar (see [T, p 291, Lem 6.1] or [DK]), one obtains (3.34).

Next for $w$ as above we note that (3.34) implies

$$\|N_*(w_{x_0}\circ\rho)\|_{2,\omega} \le c_{J,\omega}\,(1+\beta)^N\,\|f\|_{2,\omega}. \tag{3.35}$$

In fact using (3.26) it is easily shown that

$$N_*(w_{x_0}\circ\rho)(z) \le c_{J,\omega}\,(Mf+\bar{T}_A f)(z) \tag{3.36}$$

where $T_A$ is defined relative to the sine and $H(z) = -J(z)/\|z\|$. Squaring both sides of (3.36), integrating with respect to $\omega$, and using (3.34), as well as Muckenhoupt's Theorem, we obtain (3.35). This ends our discussion of Theorem 1.9.

From (3.35), (2.7), and Lemma 2.8 with $\theta = 1, \sigma = |\phi| = 0$, we deduce that

$$\int_0^\infty\int_{\mathbb{R}^n}(w_{x_0}\circ\rho)^2\,(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)^2\,\omega\,\lambda dz d\lambda \le c_{J,\omega}\beta^2\,(1+\beta)^{2N}\,\|f\|^2_{2,\omega}. \tag{3.37}$$

Combining (3.37), (3.33), we find in view of (3.30), (3.31) that

$$|II| \le c_{J,\omega}\beta\,(1+\beta)^N\,\|f\|_{2,\omega} \tag{3.38}$$

We now turn to $I$ in (3.29). We integrate by parts in the integral defining $I$ to get

$$\begin{aligned} -I &= \int_0^\infty\int_{\mathbb{R}^n} D_n\tfrac{\partial^2}{\partial\lambda^2}S_\lambda f\;\; P_\lambda g\,\lambda dz d\lambda \\ &\qquad + \int_0^\infty\int_{\mathbb{R}^n} D_n\tfrac{\partial}{\partial\lambda}S_\lambda f\;\;\tfrac{\partial}{\partial\lambda}P_\lambda g\,\lambda dz d\lambda \\ &\qquad = I_1 + I_2. \end{aligned} \tag{3.39}$$

Arguing as in the proof of (3.30) we find

$$\begin{aligned} |I_2|^2 &= \left|\int_0^\infty\int_{\mathbb{R}^n}\tfrac{\partial}{\partial t}\tfrac{\partial}{\partial\lambda}S_\lambda f\;\tilde{Q}_\lambda g\,\lambda dz d\lambda\right|^2 \\ &\le c_\omega\int_{\mathbb{R}^n}|\tfrac{\partial^2}{\partial t\,\partial\lambda}S_\lambda f|^2\,\omega\,\lambda^3\,dz d\lambda \end{aligned} \tag{3.40}$$

Again

$$\begin{aligned}\tfrac{\partial^2}{\partial t\partial\lambda} S_\lambda f &= \tfrac{\partial^2}{\partial t\partial\lambda} w\circ\rho = \tfrac{\partial}{\partial t}([w_{x_0}\circ\rho]\,[1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A]) \\ &= (w_{x_0t}\circ\rho)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A) + (w_{x_0x_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A) \\ &\quad + (w_{x_0}\circ\rho)(\tfrac{\partial^2}{\partial t\partial\lambda}P_{\gamma\lambda}A) = \Lambda_1+\Lambda_2+\Lambda_3\,.\end{aligned} \tag{3.41}$$

Since $|\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A|\le 1/2$, we have $\Lambda_1\le 2|w_{x_0t}\circ\rho|$. Observe from the change of variable (3.32) that

$$\tilde\lambda^2\, w_{x_0t}\circ\rho = \quad T_{\tilde\lambda,1}f + T_{\tilde\lambda,2}f$$

where $T_{\lambda,i}$ is defined as in (3.23) relative to $L_i, E_i$ for $i=1,2$. Here at $z, L_1 = \frac{\partial}{\partial t}J/\|z\|$, $L_2 = -J\,\frac{\partial\|z\|}{\partial t}/\|z\|^2$, and at $r$, $E_1=\Lambda'$, $E_2 = r\Lambda''$. Let $g^i, i=1,2$, be the square functions defined in (3.24). Then from the above equality and Theorem 3.25 we have

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^n}\Lambda_1^2\,\omega\,\lambda^3 dz d\lambda &\le c\int_0^\infty\int_{\mathbb{R}^n}(w_{x_0t}\circ\rho)^2\,\omega\,\lambda^3 dz d\lambda \\ &\le c\sum_{i=1}^2\|g^i f\|_{2,\omega}^2 \;\le\; c_{J,\omega}(1+\beta)^{2N}\,\|f\|_{2,\omega}^2.\end{aligned} \tag{3.42}$$

As for $\Lambda_2$, observe from (b) of Lemma 2.8 with $\theta=1, \sigma=0=|\phi|$ that $|\frac{\partial}{\partial t}P_{\gamma\lambda}A|\le c(1+\beta)^2\,\lambda^{-1}$ and so $|\Lambda_2|\le c(1+\beta)^2\,\lambda^{-1}|w_{x_0x_0}\circ\rho|$. Using (3.32) and Theorem 3.10 in the same way that we treated $w_t\circ\rho$ earlier, it follows that (3.33) is true with $w_t$ replaced by $w_{x_0x_0}$. Hence the last inequality in (3.42) is valid with $\Lambda_1$ replaced by $\Lambda_2$. Finally from the usual nontangential maximum-Carleson measure argument we get (3.42) for $\Lambda_3$. That is, using (2.7), (3.35), and Lemma 2.8 with $\theta=\sigma=1, |\phi|=0$ we find that the last line in (3.42) continues to hold when $\Lambda_1$ is replaced by $\Lambda_3$. We conclude from (3.42) for $\Lambda_i, 1\le i\le 3$, and (3.41), (3.40), that (3.38) is true with II replaced by $I_2$.

It remains to estimate $I_1$. to do this we note that $\tilde Q_\lambda = \lambda\,\mathbb{D}_n P_\lambda$, where $\tilde Q$ satisfies (0.24), as can be seen for example from taking Fourier transforms and using standard multiplier theorems. Thus arguing as in the proof of (3.30), we obtain

$$\begin{aligned} |I_1| &= \left| \int_0^\infty \int_{\mathbb{R}^n} \tfrac{\partial^2}{\partial\lambda^2} S_\lambda f \ \tilde{Q}_\lambda g \, dz d\lambda \right| \\ &\le c_\omega \left( \int_0^\infty \int_{\mathbb{R}^n} |\tfrac{\partial^2}{\partial\lambda^2} S_\lambda f|^2 \, \omega \, \lambda dz d\lambda \right)^{1/2} \end{aligned} \tag{3.43}$$

But

$$\tfrac{\partial^2}{\partial\lambda^2} S_\lambda f \ = \ (w_{x_0 x_0} \circ \rho)(1 + \tfrac{\partial}{\partial\lambda} P_{\gamma\lambda} A)^2 + (w_{x_0} \circ \rho)(\tfrac{\partial^2}{\partial\lambda^2} P_{\gamma\lambda} A).$$

The contribution of the first term on the righthand side of this equality to (3.43) was handled in the estimate of $\Lambda_2$. The contribution of the second term to (3.43) can be estimated using a nontangential maximum-Carleson measure argument, as we see from (2.7), (3.35), and Lemma 2.8 with $\sigma = 2, \theta = |\phi| = 0$. Thus we can make the desired estimates for $|I_1|$. From this discussion and our earlier estimate for $|I_2|$ we conclude that (3.38) is also true with II replaced by I. In view of (3.29), the proof of (3.28) is now complete. As mentioned earlier, (3.28) implies Theorem 1.10 for $S = S_A$.

We now turn to the second class of operators, $U_{A,B} = U$, considered in Theorem 1.10 (see (1.8)). For specificity we consider

$$Uf(z) \equiv \int_{\mathbb{R}^n} J(z-v) \frac{B(z) - B(v)}{\|z-v\|} \sin\left( \frac{A(z) - A(v)}{\|z-v\|} \right) f(v) dv,$$

where $\|A\|_{\text{comm}} \le \beta$, $\|\beta\|_{\text{comm}} \le \beta_0$, and $J$ satisfies (3.26). We make the same deductions as in the previous case : again to prove (3.27) for $U$ it suffices to prove (3.28) with $S$ replaced by $U$. Moreover we may replace the sine in the definition of $U$ by a certain odd function $\Psi \in C_0^\infty(\mathbb{R})$. Finally, we may assume that $J$ is supported in a parabolic annulus, and $A, B \in C_0^\infty(\mathbb{R}^n)$. We employ the same strategy we used for estimating $\|\mathbb{D}_n S f\|$. We set

$$U_\lambda f(z) \equiv \int_{\mathbb{R}^n} J(z-v) \frac{P_\lambda B(z) - B(v)}{\|z-v\|} \Psi\left( \frac{P_{\gamma\lambda} A(z) + \lambda - A(v)}{\|z-v\|} \right) f(v) dv,$$

and we write for $g \in C_0^\infty(\mathbb{R}^n)$ with $\|g\|_{2,\omega} = 1$, that

$$\|\mathbb{D}_n U f\|_{2,\omega} \;=\; \sup \left| \int_0^\infty \int_{\mathbb{R}^n} \mathbb{D}_n U f \;\, g dz d\lambda \right|,$$

where the supremum is taken over all such $g$. Now as previously,

$$\begin{aligned}
-\int_{\mathbb{R}^n} \mathbb{D}_n U f \, g dz \;&=\; \int_0^\infty \int_{\mathbb{R}^n} \tfrac{\partial}{\partial \lambda} (\mathbb{D}_n U_\lambda f \; P_\lambda g) \, dz d\lambda \\
&= \int_0^\infty \int_{\mathbb{R}^n} \mathbb{D}_n \tfrac{\partial}{\partial \lambda} U_\lambda f \;\; P_\lambda g \, dz d\lambda + \int_0^\infty \int_{\mathbb{R}^n} \mathbb{D}_n U_\lambda f \;\; \tfrac{\partial}{\partial \lambda} P_\lambda g \, dz d\lambda \\
&= I \,+\, II.
\end{aligned} \tag{3.44}$$

Since $\frac{\partial P_\lambda}{\partial \lambda} = \mathbb{D} \tilde{Q}_\lambda$ and $\mathbb{D}_n = i \mathbb{D}^{-1} \frac{\partial}{\partial t}$, we have

$$\begin{aligned}
|II| &= \left| \int_0^\infty \int_{\mathbb{R}^n} \tfrac{\partial}{\partial t} U_\lambda f \, \tilde{Q}_\lambda g \, dz d\lambda . \right| \\
&\le\; c_\omega \left( \int_0^\infty \int_{\mathbb{R}^n} | \tfrac{\partial}{\partial t} U_\lambda f \, |^2 \, \omega \, \lambda dz d\lambda \right)^{1/2} .
\end{aligned} \tag{3.45}$$

Setting

$$w(x_{00}, x_0, z) \equiv \int_{\mathbb{R}^n} J(z-v) \, \frac{x_{00} - B(v)}{\|z - v\|} \Psi \left( \frac{x_0 - A(v)}{\|z - v\|} \right) f(v) dv$$

and letting $\mu(\lambda, z) = (P_\lambda B(z), \lambda + P_{\gamma\lambda} A(z), z)$ we see that $w \circ \mu(\lambda, z) = U_\lambda f(z)$. Hence, at $(\lambda, z) = (\lambda, x, t)$

$$\begin{aligned}
\tfrac{\partial}{\partial t} U_\lambda f &= \tfrac{\partial}{\partial t} (w \circ \mu) \\
&= w_t \circ \mu + (w_{x_0} \circ \mu) \, \tfrac{\partial}{\partial t} P_{\gamma\lambda} A + (w_{x_{00}} \circ \mu) \, \tfrac{\partial P_\lambda B}{\partial t} .
\end{aligned} \tag{3.46}$$

To handle the third term on the righthand side of (3.46), We note that (3.35) remains valid with $w_{x_0}$ replaced by $w_{x_{00}}$, thanks to (3.34). From this note, (2.7), and Lemma 2.8 with $\theta = 1, \sigma = |\phi| = 0$, we see that the usual nontangential maximum-Carleson measure argument can be applied to get that (3.37) is true with $w_{x_0}$ replaced by $w_{x_{00}}$. We treat the second term on the righthand side of (3.46) in a similar fashion. Indeed changing variables in the integral defining $w_{x_0}$ according to (3.32), we find that

$$w_{x_0}\circ\mu(\lambda,z)=(P_\lambda B(z)-B(z))\int_{\mathbb{R}^n}\frac{J(z-v)}{\|z-v\|^2}\,\Psi'\left(\frac{A(z)-A(v)+\tilde{\lambda}}{\|z-v\|}\right)f(v)dv$$

$$+\int_{\mathbb{R}^n}J(z-v)\frac{B(z)-B(v)}{\|z-v\|^2}\,\Psi'\left(\frac{A(z)-A(v)+\tilde{\lambda}}{\|z-v\|}\right)f(v)dv$$

$$=E_1f\,(\tilde{\lambda},z)+E_2f\,(\tilde{\lambda},z).$$

Using (3.34) and $|P_\lambda B-B|\le c\beta_0\lambda$ (see (0.10), it is easily shown as in the proof of (3.35) that $E_if, i=1,2$, are nontangentially bounded by $L^2_\omega(\mathbb{R}^n)$ functions with norms $\le c_\omega\,\beta_0(1+\beta)^N\,\|f\|_2$. From this fact, (2.7), and Lemma 2.8, we get (3.37) in this case. To handle the first term on the right hand side of (3.46) we shall need another lemma.

**Lemma 3.47** *If $\|B\|_{\text{comm}}\le\beta_0$, and $\omega$ is an $A_2$ weight, then the measure*

$$d\nu(\lambda,z)=\ |B(z)-P_\lambda B(z)|^2\,\omega(z)dz\frac{d\lambda}{\lambda^3}$$

*is a weighted Carleson measure on $\mathbb{R}^{n+1}_+$ with*

$$\nu(B_r(z_0)\times(0,r))\ \le\ c_\omega\,\beta_0^2\,\omega(B_r(z_0))$$

*whenever $r>0$ and $z\in\mathbb{R}^n$.*

**Proof:** To give a rigorous proof we integrate in $\lambda$ over $(\epsilon,r)$ and let $\epsilon\to 0$. Integrating by parts with respect to $\lambda$ we obtain

$$\nu(B_r(z_0)\times(\epsilon,r))\ \le\ |\text{ boundary terms }|+\left|\int_\epsilon^r\int_{B_r(z_0)}(B-P_\lambda B)\tfrac{\partial}{\partial\lambda}P_\lambda B\,\omega\,dz\frac{d\lambda}{\lambda^2}\right|.$$

The boundary terms are easy to handle since as noted above, $|P_\lambda B-B|\le c\beta_0\lambda$. To estimate the integral on the righthand side of the above equality we use " Cauchy 's inequality with $\epsilon$ ' s " and then hide the small term on the left, to conclude that it suffices to show

$$\int_{B_r(z_0)}\int_0^r |\tfrac{\partial}{\partial\lambda}P_\lambda B|^2\,dz\frac{d\lambda}{\lambda} \,\le\, c_\omega\,\beta_0^2\,\omega(B_r(z)),$$

in order to prove Lemma 3.47. The above inequality is a consequence of Lemma 2.8 for $B$ with $\sigma = 1, \theta = |\phi| = 0$. This completes the proof of Lemma 3.47. $\square$

We now use Lemma 3.47 to estimate the first term on the right hand side of (3.46). Changing variables in the integral defining $w_t$ according to (3.32), we find that

$$\tilde\lambda\, w_t\circ\mu \,=\, \sum_{i=1}^{2}\tilde R_{\tilde\lambda,i}f + \tilde\lambda^{-1}[P_\lambda B(z)-B(z)]\sum_{i=1}^{2}\Theta_{i,\tilde\lambda}, \tag{3.48}$$

where $\tilde R_{\lambda,i}$, $i=1,2$, is defined relative to $H_i, F_i$ as in (3.17) and at $z$, $H_1 = \|z\|\frac{\partial}{\partial t}(J/\|z\|)$, $H_2 = -J\frac{\partial}{\partial t}\|z\|/\|z\|$, while at $r$, $F_1=\Psi, F_2 = r\Psi'$. Moreover,

$$\Theta_{i,\lambda}(z) = \lambda^2\int_{\mathbb{R}^n}\frac{H_i(z-v)}{\|z-v\|}\,F_i\left(\frac{A(z)-A(v)+\lambda}{\|z-v\|}\right)dz.$$

Let $\tilde G^i, i=1,2$, be the square functions corresponding to $\tilde R_{\lambda,i}$. Then from Theorem 3.19 we deduce

$$\sum_{i=1}^{2}\|\tilde G^i\|_{2,\omega} \,\le\, c_\omega\,\beta_0(1+\beta)^N\,\|f\|_{2,\omega}.$$

Moreover as in the proof of (3.35) we see that $\Theta_i, i=1,2$ are nontangentially bounded by $L^2_\omega(\mathbb{R}^n)$ integrable functions. Using this fact, Lemma 3.47, and the above inequality in (3.48) we find that (3.33) is still true. Putting these estimates in (3.45) we conclude (3.38) for $Uf$.

Next we consider $I$. We shall be brief since the ideas should now be familiar. Arguing as we did for $S$, we get upon integration by parts

$$\begin{aligned}-I \,&=\, \int_0^\infty\int_{\mathbb{R}^n}\mathbb{D}_n\tfrac{\partial^2}{\partial\lambda^2}U_\lambda f\;\,P_\lambda g\,\lambda dz d\lambda\\ &\quad+\int_0^\infty\int_{\mathbb{R}^n}\mathbb{D}_n\tfrac{\partial}{\partial\lambda}U_\lambda f\;\,\tfrac{\partial}{\partial\lambda}P_\lambda g\,\lambda dz d\lambda\\ &= I_1\,+\,I_2.\end{aligned}\tag{3.49}$$

As in (3.40) it follows that

$$|I_2|^2 \leq c_\omega \int_{\mathbb{R}^n} |\tfrac{\partial^2}{\partial t\, \partial\lambda} U_\lambda f|^2 \, \omega \, \lambda^3 \, dz d\lambda \tag{3.50}$$

Again since $U_\lambda(\lambda, z) = w \circ \mu(\lambda, z)$ we can compute $\frac{\partial^2}{\partial t \partial \lambda} U_\lambda f$ in terms of derivatives of $w \circ \mu$. We obtain an equality similar to (3.41) with $\mu, U_\lambda$ replacing $\rho, S_\lambda$ but with three additional terms involving partial derivatives of $w$ in which one of the subscripts is $x_{00}$. All terms can be handled by using either Theorem 3.19, Theorem 3.25, or the usual nontangential maximum-Carleson measure argument involving Lemmas 2.8 and 3.47. We omit the details.

Similarly we can write

$$I_1 = -\int_0^\infty \int_{\mathbb{R}^n} \tfrac{\partial^2}{\partial \lambda^2} U_\lambda f \ \tilde{Q}_\lambda g \, dz d\lambda$$

and this term may be handled like its analogue involving $S_\lambda$, again with straightforward modifications. We omit the details. Thus we have obtained the desired estimates for $I, II$, in this situation. From (3.44) we conclude that (3.28) holds with $S$ replaced by $U$. The alternative proof of Theorem 1.10 is complete. $\square$

**4. Square Function Estimates for Derivatives of the Single Layer Potential.**

In this section we give analogues of the square function estimates of Theorem 3.10 and Theorem 3.25 when our square functions arise by taking derivatives of a single layer potential. These estimates will be very useful in the proof of Theorem 1.16. To this end, recall that $\|A\|_{\text{comm}} \leq \beta < \infty$ (see (0.9)), and $\Omega$ is the graph domain defined relative to $A$ as in (0.1). Throughout this section we suppose that $\gamma \leq \frac{1}{2}$ satisfies (2.2), and $\|\mathbb{D}_n A\|_* \leq \epsilon_0 \leq \gamma^3$. Also, recall for $f \in L^2(\Omega)$, that $u \circ \rho = \mathcal{S} f \circ \rho$ is the single layer potential of $f$ in $\Omega$ (see (1.1)), lifted by $\rho$ in (2.4) to $\mathbb{R}^{n+1}_+$. With this notation we prove

**Theorem 4.1** *Let $u, \rho, A, \gamma, \epsilon_0, f$ be as above. Then for $z = (x,t) \in I\!\!R^n$,*

$$(i) \quad \int_0^\infty \int_{I\!\!R^n} |\, u_{x_i x_j} \circ \rho(\lambda, z)\,|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \|f\|_2^2 \text{ for } 0 \le i, j \le n-1,$$

$$(ii) \quad \int_0^\infty \int_{I\!\!R^n} |\, u_{x_j t} \circ \rho(\lambda, z)\,|^2 \, dz \, \lambda^3 d\lambda \;\le\; c_\beta \|f\|_2^2 \text{ for } 0 \le j \le n-1,$$

$$(iii) \quad \int_0^\infty \int_{I\!\!R^n} |\, D^t_{1/2}(u_{x_j} \circ \rho)(\lambda, z)\,|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \, \|f\|_2^2 \text{ for } 0 \le j \le n-1,$$

$$(iv) \quad \int_0^\infty \int_{I\!\!R^n} |\, D^t_{3/2}(u \circ \rho)(\lambda, z)\,|^2 \, dz \, \lambda^3 d\lambda \;\le\; c_\beta \, \|f\|_2^2 \text{ for } 0 \le j \le n-1.$$

We note that $(i)$ is the key estimate in Theorem 4.1. In fact (i) implies $(ii) - (iv)$, as we shall see in the proof of Theorem 4.1. We know three different proofs of $(i)$. In addition to the following proof which is nearly identical to the arguments in section 3, we shall indicate in a remark at the end of this section, a " pde " inspired proof which as mentioned in section 1, was adapted from a proof of Dahlberg, Kenig, Pipher, and Verchota for the Laplacian. The third proof uses the absolute continuity results for caloric measure of [LM, ch 3] and the square function estimates of [Br3]. Of the three proofs, only the present one is of a " real variable " character, relying essentially upon the geometry of the domain and not on pde or absolute continuity results.

To prove $(i)$ we shall need to slightly alter Theorem 3.10. To this end let $F \in C^1(I\!\!R)$ be as in (3.7) and let $H \in C^1(I\!\!R^n \setminus \{0\})$ satisfy (3.6) as well as

$$|H(z)| \;\le\; \frac{c|z_n|^{3/2}}{\|z\|^{d+4}}.$$

Recall our various notations for points in $I\!\!R^n : z = (x,t) = (z', z_n)$ and put

$$R_\lambda f(z) \;\equiv\; \lambda \int_{I\!\!R^n} H(z-v)\, F\left(\frac{A(z) - A(v) + \lambda}{|z_n - v_n|^{1/2}}\right) f(v)\, dv. \tag{4.2}$$

Finally define $Gf$ relative to $R_\lambda f$ as in (3.9). Then we have the following analogue of Theorem 3.10.

**Theorem 4.3** *Suppose that for $H, F$ as above we have either (a) $F$ is odd and $H$ is odd in $z'$ for each fixed $z_n$ or (b) $F$ is even, $H$ is even in $z'$ for each fixed $z_n$, while one of the integrals : $\int_{\mathbb{R}} F(r)dr$, $\int_S H(\sigma)|\sigma_n|^{1/2}(1+\sigma_n^2)d\sigma$, vanishes. If $\|A\|_{\mathrm{comm}} \leq \beta < \infty$, and $f$ is as above, then there exists a positive integer $N$ depending only on $d$ such that*

$$\|Gf\|_2 \leq c_{F,H}(1+\beta)^N \|f\|_2.$$

**Proof:** In Theorem 4.3, $S$ denotes the unit sphere in $\mathbb{R}^n$ which is the notation used in section 3. We define $Q_\lambda^* f(z)$ as in the display above (3.11) only now we replace $\|z-v\|$ in the argument of $F$ by $|z_n - v_n|^{1/2}$. It is easily checked that (3.12)-(3.16) remain true under this alteration. Thus the proof of Theorem 4.3 reduces, as in the proof of Theorem 3.10, to showing that $Q_\lambda^* 1 \equiv 0$. Introducing parabolic polar coordinates as in the proof of this theorem, we deduce that

$$Q_\lambda^* 1(z) = \lambda \int_S \left( \int_0^\infty F\left( \frac{\langle \vec{a}, \sigma' \rangle + \lambda/\rho}{|\sigma_n|^{1/2}} \right) \frac{d\rho}{\rho^2} \right) H(\sigma)(1+\sigma_n^2)\, d\sigma.$$

As in the discussion after (3.16) we use the change of variables $\rho \to \lambda\rho$, then $r = 1/\rho$, then $r \to r - \langle \vec{a}, \sigma' \rangle$, and finally, $r \to |\sigma_n|^{1/2} r$, to rewrite the above display as

$$Q_\lambda^* 1(z) = \int_S \left( \int_{\frac{\langle \vec{a}, \sigma' \rangle}{|\sigma_n|^{1/2}}}^\infty F(r)\, dr \right) H(\sigma)|\sigma_n|^{1/2}(1+\sigma_n^2) d\sigma.$$

If $F$ is odd, then the function inside the big parentheses is even as a function of $\sigma'$ for fixed $\sigma_n$. Since $H$ is odd in $\sigma'$ for fixed $\sigma_n$, we see that the righthand side of the above equality is identically zero in this case. Otherwise, let $b = \int_{\mathbb{R}} F(r)\, dr$ and note that if $\psi(s) = \int_s^\infty F(r)dr$, then $\psi(s) + \psi(-s) = b$, as we see from the evenness of $F$. Using this fact and evenness of $H$ we deduce that

$$2Q_\lambda^* 1(z) = b \int_S H(\sigma)|\sigma_n|^{1/2}(1+\sigma_n^2) d\sigma = 0,$$

thanks to our hypotheses. $\square$

**Proof of Theorem 4.1**. We now use Theorem 4.3 to prove $(i)$ of Theorem 4.1. Let $\tilde\rho(\lambda, z) = (\lambda + A(z), z)$ for $z = (x, t)$ and observe from the change of variable (3.32) as well as the discussion following this display, that it suffices to prove $(i)$ with $\rho$ replaced by $\tilde\rho$. For the reader's convenience we recall from (1.1) that if $(X, t) = (x_0, x, t) \in \mathbb{R}^{n+1}$ and

$$W(X,t) \;=\; (4\pi t)^{-n/2} \,\exp\left\{\frac{-|X|^2}{4t}\right\}\,\chi_{(0,\infty)}\,(t)$$

is the usual Gaussian, then

$$\mathcal{S}f(X,t) \;\equiv\; \int_{-\infty}^{t}\int_{\partial\Omega_s} W(X-Q,t-s)\,f(Q,s)\,d\sigma_s(Q)\,ds.$$

Next for $0 \le i, j \le n-1$, we compute

$$\begin{aligned} W_{x_ix_j}(X,t) &= c\,\frac{x_ix_j}{t^{2+n/2}}\,\exp\left\{\frac{-|X|^2}{4t}\right\}\chi_{(0,\infty)},\; i\neq j,\\ W_{x_jx_j}(X,t) &= c\left[\frac{-1}{2t^{1+n/2}}+\frac{x_j^2}{4t^{2+n/2}}\right]\exp\left\{\frac{-|X|^2}{4t}\right\}\chi_{(0,\infty)}. \end{aligned} \tag{4.4}$$

Suppose first that $i = 0$ and $1 \le j \le n-1$ in $(i)$ (with $\tilde\rho$ replacing $\rho$). We write $d\sigma_s$ in graph coordinates and replace $(X, t)$ by $\tilde\rho(\lambda, z) = (\lambda + A(z), z)$ in (4.4). Using the resulting equality we get

$$\lambda\, u_{x_0x_j}\circ\tilde\rho(\lambda,z) \equiv\; R_{\lambda,j}\tilde f(z),$$

where $\tilde f(z) = \sqrt{1 + |\nabla_{z'}A(z)|^2}\,(f\circ\tilde\rho)(0, z)$, $z \in \mathbb{R}^n$, and $R_{\lambda,j}\tilde f$ is defined as in (4.2) relative to $H_j, F$ for $1 \le j \le n-1$. Here $F(r) = re^{-r^2/4}$ while

$$H_j(z) \equiv c\frac{z_j}{z_n^{(n+3)/2}}\,\exp\left\{\frac{-|z'|^2}{4z_n}\right\}\chi_{(0,\infty)}(z_n).$$

Since $H$ is odd in $z'$ for each fixed $z_n$ and $F$ is odd, we may apply Theorem 4.3 to deduce $(i)$ of Theorem 4.1 when $i = 0, 1 \leq j \leq n-1$.

Next if $i \neq j$, $1 \leq i, j \leq n-1$, then $\lambda\, u_{x_i x_j} \circ \tilde{\rho}(\lambda, z) = R_\lambda^{i,j} \tilde{f}(z)$, where $R_\lambda^{i,j} \tilde{f}(z)$ is defined as in (4.2) relative to $H_{i,j}, F^*$. In this case $F^*(r) = e^{-r^2/4}$ and

$$H_{i,j}(z) = c \frac{z_i z_j}{z_n^{2+n/2}} \exp\left\{ \frac{-|z'|^2}{4z_n} \right\} \chi_{(0,\infty)}(z_n).$$

Clearly, $F^*$ is even and $H_{i,j}$ is even as a function of $z'$ for fixed $z_n$. Also since $H_{i,j}$ is odd as a function of $z_j$ for fixed $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$, we see that

$$\int_S H_{i,j}(\sigma) |\sigma_n|^{1/2} \, (1 + \sigma_n^2) \, d\sigma = 0.$$

Thus again we can apply Theorem 4.3 to conclude that $(i)$ of Theorem 4.1 holds when $1 \leq i, j \leq n-1$ and $i \neq j$.

To finish the proof of estimate $(i)$ of Theorem 4.1, we consider separately the cases $u_{x_0 x_0}$ and $u_{x_j x_j}, 1 \leq j \leq n-1$. In the latter case we again have $\lambda u_{x_j x_j} \circ \tilde{\rho}(\lambda, z) = R_\lambda^j \tilde{f}(z)$, where $R_\lambda^j \tilde{f}$ is defined relative to $H^j, F^*$ with $F^*$ as above and

$$H^j(z) = c \left[ \frac{-1}{2z_n^{1+n/2}} + \frac{z_j^2}{4z_n^{2+n/2}} \right] \exp\left\{ \frac{-|z'|^2}{4z_n} \right\} \chi_{(0,\infty)}.$$

Since $F^*$ is even and $H^j$ is even as a function of $z'$ for fixed $z_n$ we need to show that

$$0 = \int_S H^j(\sigma) |\sigma|^{1/2} \, (1 + \sigma_n^2) d\sigma = (1/\log 2) \int_{\{1 \leq \|z\| \leq 2\}} |z_n|^{1/2} \, H^j(z) dz = I / \log 2, \tag{4.5}$$

where the second equality follows from the homogeneity of $H^j$. Next we note that if $K^j(z) = |z_n|^{1/2} \, H^j(z)$, for $1 \leq j \leq n-1$, then by direct calculation and our choice of $H^j$, we have

$$K^j(z) = \frac{\partial^2}{\partial z_j^2} \left( \frac{1}{z_n^{(n-1)/2}} \exp\left\{ \frac{-|z'|^2}{4z_n} \right\} \chi_{(0,\infty)} \right).$$

This note implies that $I = 0$, as can be seen from either the divergence theorem and homogeneity of $K^j$ or the following argument due to Fabes and Rivière [FR2]. Observe from the above equality that

$$\int_{\mathbb{R}^{n-1}} K^j(z', z_n)dz' = 0 \tag{4.6}$$

for each $z_n > 0$. Moreover, from the definition of $I$ in (4.5) we find for $1 \le j \le n-1$ that

$$I = \int_{\{1 \le z_n \le 4\}} K^j(z)dz + \int_{\left\{\substack{0 \le z_n \le 1 \\ \|z\| \ge 1}\right\}} K^j(z)dz$$

$$- \int_{\left\{\substack{0 \le z_n \le 4 \\ \|z\| \ge 2}\right\}} K^j(z)dz.$$

The difference of the last two terms is zero as we see from homogeneity of $K^j$ ( i.e $K^j(\lambda^\alpha z) = \lambda^{-d} K^j(z)$). Furthermore, the first term equals

$$\int_1^4 \int_{\mathbb{R}^{n-1}} K^j(z', z_n)dz'dz_n = 0,$$

by (4.6). Hence $I = 0$ and so by (4.5) we may apply Theorem 4.3 to conclude $(i)$ when $i = j, 1 \le j \le n-1$.

Finally in the case of $u_{x_0x_0}$, we have $\lambda u_{x_0x_0} \circ \tilde{\rho}(\lambda, z) = \tilde{R}_\lambda \tilde{f}(z)$, where $\tilde{R}_\lambda \tilde{f}$ is defined relative to $\tilde{H}, \tilde{F}$ with

$$\tilde{F}(r) = (r^2/4 - 1/2)e^{-r^2/4} = \tfrac{d^2}{dr^2}\, e^{-r^2/4}$$

and

$$\tilde{H}(z) = c\frac{1}{z_n^{1+n/2}} \exp\left\{\frac{-|z'|^2}{4z_n}\right\} \chi_{(0,\infty)}(z_n).$$

Clearly $\tilde{H}$ is even in $\sigma'$ for fixed $\sigma_n$ and $\tilde{F}$ is even. Also $\int_{\mathbb{R}} \tilde{F}(r)dr = 0$, by the Fundamental Theorem of Calculus, so Theorem (4.3) again implies $(i)$ in this case. This concludes our treatment of $(i)$.

To prove $(ii)$ of Theorem 4.1 we could first prove a slight alteration of Theorem 3.25 and then proceed as in $(i)$. The reader is invited to fill in the details for this argument. Another proof of $(ii)$ is to use the fact that $u_t$ is a solution to the heat equation in $\Omega$ and standard estimates for the heat equation in caloric balls, to deduce as in (2.19) that at$(\lambda, x, t)$

$$|\, u_{x_j\,t} \circ \rho\,|^2 \,\leq\, c_\beta\, \lambda^{-n-4} \int_{B_{\lambda/2}(x,t)} \int_{\lambda/2}^{3\lambda/2} \sum_{i,j=0}^{n-1} (u_{x_i x_j} \circ \rho)^2\, (\tilde{\lambda}, y, s) d\tilde{\lambda} dy ds.$$

Using $(i)$ and integrating the above inequality over $\mathbb{R}^{n+1}_+$ we get $(ii)$ for $0 \leq j \leq n-1$.

In the proof of $(iii)$ and $(iv)$ of Theorem 4.1 we assume that $A, f \in C_0^\infty(\mathbb{R}^n)$, in order to justify integration by parts. To see this assumption is no restriction we note that $f, A$ are the pointwise limit of such functions and $u$, as well as all its derivatives, are pointwise limits of the corresponding potentials and their derivatives. Using this observation and the Fatou Lemma one gets $(iii), (iv)$, for a general $A, f$ from the smooth case. To begin the proof of $(iii)$, we integrate by parts in $\lambda$ to get

$$\int_0^\infty \int_{\mathbb{R}^n} |\, D^t_{1/2}(u_{x_j} \circ \rho)\,|^2\, dz\, \lambda d\lambda \,=\, -\int_0^\infty \int_{\mathbb{R}^n} \, D^t_{1/2}(u_{x_j} \circ \rho)_\lambda\, D^t_{1/2}(u_{x_j} \circ \rho)\, dz\, \lambda^2 d\lambda \,=\, J. \tag{4.7}$$

To justify this integration by parts, we note from our smoothness assumption for $\lambda \geq 1$ and $0 \leq j \leq n-1$ that

$$|u_{x_j}(\lambda, z)| \leq \frac{c_{A,f}\|f\|_1}{(\lambda + |z|)^d}$$

$$|u_{x_j\,t}(\lambda, z)| \leq \frac{c_{A,f}\|f\|_1}{(\lambda + |z|)^{d+2}}.$$

Now for $\lambda \geq 1$ and $0 \leq j \leq n-1$, we can estimate $D^t_{1/2}(u_{x_j} \circ \rho)$ in terms of the above derivatives, as in (2.15), (2.16), and (2.22). Doing this and using Lemma2.8$(b)$ we get for $\lambda \geq 1, 0 \leq j \leq n-1$, that

$$\lambda |D^t_{1/2}(u_{x_j} \circ \rho)(\lambda, z)| \,\leq\, \frac{c_{f,A}\|f\|_1}{(\lambda + \|z\|)^d}.$$

Integrating the integrand on the lefthand side of (4.7) by parts over $[0,R]\times \mathbb{R}^n$ and letting $R\to\infty$, we obtain from the above estimate that (4.7) is true. To estimate $J$ in (4.7), we use Schwarz's inequality, self-adjointness of $D^t_{1/2}$, and the fact that $D^t_{1/2} * D^t_{1/2} \,=\, c\, H\,\frac{\partial}{\partial t}$, where $H$ is the one dimensional Hilbert transform, to obtain

$$J \,\leq\, c\,\left(\int_{\mathbb{R}^n}\int_0^\infty |(u_{x_j}\circ\rho)_\lambda|^2\,dz\lambda d\lambda\right)^{1/2}\left(\int_{\mathbb{R}^n}\int_0^\infty |(u_{x_j}\circ\rho)_t|^2\,dz\lambda^3\,d\lambda\right)^{1/2} \,=\, c\,J_1\,J_2.$$

Now

$$(u_{x_j}\circ\rho)_\lambda \equiv (u_{x_jx_0}\circ\rho)\,(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A),$$

so $J_1 \,\leq\, c_\beta\|f\|_2$ thanks to $(i)$ of Theorem 4.1 and (2.2). Also,

$$(u_{x_j}\circ\rho)_t \,\equiv\, u_{x_j\,t}\circ\rho \,+\, (u_{x_jx_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)$$

so $J_2 \,\leq\, c_\beta(1+\gamma^{-1}\|\mathbb{D}_nA\|_*\,)\,\|f\|_2$ thanks to $(i),(ii)$ of Theorem 4.1 and Lemma 2.8 (b) with $\theta=1, \sigma=|\phi|=0$. Using the above estimates for $J_1, J_2$ in the displays containing $J$ and the fact that $\|\mathbb{D}_nA\|_*\leq\gamma$, we get $(iii)$.

To conclude the proof of Theorem 4.1, it remains to prove estimate $(iv)$. To do so we integrate by parts in $\lambda$ to obtain

$$\int_0^\infty\int_{\mathbb{R}^n}|\,D^t_{3/2}(u\circ\rho)|^2dz\lambda^3\,d\lambda \,=\, -\tfrac{1}{2}\int_0^\infty\int_{\mathbb{R}^n} D^t_{3/2}(u\circ\rho)_\lambda\,D^t_{3/2}(u\circ\rho)dz\lambda^4\,d\lambda$$

$$=\, -\tfrac{1}{2}\int_0^\infty\int_{\mathbb{R}^n} D^t_1(u\circ\rho)_\lambda\,D^t_2(u\circ\rho)\,dz\lambda^4\,d\lambda \,=\, L.$$

This integration by parts can be justified using our smoothness assumption on $A, f$ as in the proof of $(iii)$. From the above equality, Schwarz's inequality, and boundedness of the Hilbert transform on $L^2(\mathbb{R})$, we deduce

$$L \,\leq\, c\left(\int_{\mathbb{R}^n}\int_0^\infty |\tfrac{\partial}{\partial t}(u\circ\rho)_\lambda|^2\,dz\lambda^3 d\lambda\right)^{1/2}\left(\int_{\mathbb{R}^n}\int_0^\infty |\tfrac{\partial^2}{\partial t^2}(u\circ\rho)|^2\,dz\lambda^5\,d\lambda\right)^{1/2} \,=\, c\,L_1\,L_2.$$

Now

$$\frac{\partial}{\partial t}(u\circ\rho)_\lambda \equiv \frac{\partial}{\partial t}\Big[(u_{x_0}\circ\rho)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)\Big]$$

$$= (u_{x_0\,t}\circ\rho)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\,) + (u_{x_0x_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)$$

$$+\ (u_{x_0}\circ\rho)(\tfrac{\partial^2}{\partial t\partial\lambda}\,P_{\gamma\lambda}A).$$

The contribution of the first term on the righthand side of the last equality to $L_1$ can be handled immediately by $(ii)$ and (2.2). The second term yields the desired estimate as we see from $(i)$, (2.2), and Theorem 2.8 $(b)$ with $\theta=1, \sigma=|\phi|=0$. Finally the contribution of the third term is handled using (2.7), Lemma 2.8 $(a)$ with $\theta=1=\sigma, |\phi|=0$, and $(ii)$ of Lemma 2.14. Thus $L_1 \leq c_\beta\,(1+\gamma^{-1}\|I\!\!D_nA\|_*\,)\,\|f\|_2$.

To estimate $L_2$ we note that

$$\frac{\partial^2}{\partial t^2}(u\circ\rho) \equiv \frac{\partial}{\partial t}\Big[(u_t\circ\rho)+(u_{x_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)\Big]$$

$$= u_{tt}\circ\rho + 2(u_{t\,x_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)$$

$$+\ (u_{x_0x_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)^2 \ +\ (u_{x_0}\circ\rho)(\tfrac{\partial^2}{\partial t^2}P_{\gamma\lambda}A).$$

The contribution of the first term on the righthand side of the above equality to $L_2$ can be handled using $(ii)$ and local interior estimates for solutions to the heat equation as in the proof of $(ii)$. The second term is estimated using $(ii)$ and Lemma 2.8$(b)$, while the third term is treated using $(i)$ and Lemma 2.8$(b)$. The fourth term is handled using (2.7), Lemma 2.8$(a)$ with $\theta=2, \sigma=|\phi|=0$ and $(ii)$ of Lemma 2.14. Altogether we get, $L_2 \leq c_\beta(1+\gamma^{-3}\|I\!\!D_nA\|_*\,)\,\|f\|_2$. Using these estimates for $L_1, L_2$ in the displays for $L$ and the fact that $\|I\!\!D_nA\|_* \leq \gamma^3$, we get $(iv)$. The proof of Theorem 4.1 is now complete . $\square$

Next we show that Theorem 4.1 implies the following corollary.

**Corollary 4.8** *Let $u' = \mathcal{D}f$ and suppose that $\rho, A, \gamma, \epsilon_0, f$ are as in Theorem 4.1. Then for*

$z = (x,t) \in I\!\!R^n$,

$$(i) \quad \int_0^\infty \int_{I\!\!R^n} |\, u'_{x_j} \circ \rho(\lambda, z)\,|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \|f\|_2^2 \text{ for } 0 \;\le j\; \le n-1,$$

$$(ii) \quad \int_0^\infty \int_{I\!\!R^n} |\, u'_t \circ \rho(\lambda, z)\,|^2 \, dz \, \lambda^3 d\lambda \;\le\; c_\beta \|f\|_2^2,$$

$$(iii) \quad \int_0^\infty \int_{I\!\!R^n} |\, D^t_{1/2}(u' \circ \rho)(\lambda, z)\,|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \, \|f\|_2^2.$$

**Proof** : To prove Corollary 4.8, we note from the definition of $\mathcal{D}$ that

$$u' \;=\; \sum_{i=0}^{n-1} \tfrac{\partial}{\partial x_i} \mathcal{S}\, k_i,$$

where

$$k_i(y,s) = \; -\tfrac{A_{y_i}}{\sqrt{1+|\nabla_y A|^2}} f(y,s), \; 1 \le i \le n-1, \text{ and } k_0(y,s) = \tfrac{1}{\sqrt{1+|\nabla_y A|^2}} f(y,s).$$

Using this note we get Corollary 4.8 by applying Theorem 4.1 $(i)-(iii)$ to each term of the above sum defining $u'$. □

**Remark.** Here we indicate another proof of $(i)$ of Theorem 4.1 due to Dahlberg, Kenig, Pipher, and Verchota. Again we may assume that $A, f \in C_0^\infty(I\!\!R^n)$, in order to justify our integrations by parts. For fixed $j, 0 \le j \le n-1$, the argument of the above authors gives

$$-\tfrac{1}{2} \int_{I\!\!R^n} (u_{x_j} \circ \rho)^2(0,z) \, dz \;+\; \int_0^\infty \int_{I\!\!R^n} \frac{(1 + \frac{\partial}{\partial \lambda} P_{\gamma\lambda} A)^2}{1 + |\nabla_x P_\lambda A|^2} \, |\nabla u_{x_j} \circ \rho|^2 \, dz \lambda d\lambda \;=\; G \tag{4.9}$$

where $G$ means a good term which can be estimated using Cauchy's inequality with $\epsilon$'s (to hide the small term on the lefthand side of (4.9)) and (2.7), Lemma 2.8, Lemma 2.14 (i.e. a nontangential maximum-Carleson measure argument). Clearly (4.9) and Lemma 2.14 imply $(i)$ of Theorem 4.1.

To prove (4.9) we write

$$
\begin{aligned}
-\tfrac{1}{2}\int_{\mathbb{R}^n} (u_{x_j}\circ\rho)^2\,dz \;=\; & \int_0^\infty\!\!\int_{\mathbb{R}^n} (u_{x_j}\circ\rho)\,(u_{x_j}\circ\rho)_\lambda\,dz\,d\lambda \\
= -\int_0^\infty\!\!\int_{\mathbb{R}^n} [(u_{x_j}\circ\rho)_\lambda]^2\,dz\,\lambda d\lambda \;-\; & \int_0^\infty\!\!\int_{\mathbb{R}^n} (u_{x_j}\circ\rho)\,(u_{x_j}\circ\rho)_{\lambda\lambda}\;dz\lambda d\lambda \\
= \;-\int_0^\infty\!\!\int_{\mathbb{R}^n} (u_{x_jx_0}\circ\rho)^2(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^2\,dz\lambda d\lambda - & \int_0^\infty\!\!\int_{\mathbb{R}^n} (u_{x_j}\circ\rho)(u_{x_jx_0^2}\circ\rho)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^2\,dz\lambda d\lambda \\
-\;\int_0^\infty\!\!\int_{\mathbb{R}^n} (u_{x_j}\circ\rho)(u_{x_jx_0}\circ\rho)(\tfrac{\partial^2}{\partial\lambda^2}P_{\gamma\lambda}A)\,dz\,\lambda d\lambda = T_1+T_2+G_1, &
\end{aligned}
\tag{4.10}
$$

where $G_1$ is a good term as described above. To estimate $T_2$ we note that since $u_{x_j}$ is a solution to the heat equation in $\Omega$ we have

$$
\begin{aligned}
(1+|\nabla_x P_\lambda A|^2)\,(u_{x_jx_0^2}\circ\rho) \;=\; & \sum_{i=1}^{n-1}[-(u_{x_jx_i}\circ\rho)_{x_i} \;+\; (\tfrac{\partial}{\partial x_i}P_\lambda A\,)\,(u_{x_jx_0}\circ\rho)_{x_i}] \\
& + (1+|\nabla_x P_\lambda A|^2)(u_{x_j\,t}\circ\rho)\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
T_2 = \int_0^\infty\!\!\int_{\mathbb{R}^n} \frac{(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^2}{1+|\nabla_x P_\lambda A|^2}(u_{x_j}\circ\rho) & \sum_{i=1}^{n-1}[(u_{x_jx_i}\circ\rho)_{x_i} \;-\; (\tfrac{\partial}{\partial x_i}P_\lambda A)\,(u_{x_jx_0}\circ\rho)_{x_i}\,]\,dz\,\lambda d\lambda \\
-\int_0^\infty\!\!\int_{\mathbb{R}^n} \Big(2\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A+(\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^2\Big)\; & (u_{x_j}\circ\rho)\,(u_{x_j\,t}\circ\rho)dz\,\lambda d\lambda \\
-\int_0^\infty\!\!\int_{\mathbb{R}^n} (u_{x_j}\circ\rho)[(u_{x_j}\circ\rho)_t-(u_{x_jx_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)]dz\,\lambda d\lambda & \\
= T_3 \;+\; G_2 \;+\; G_3. &
\end{aligned}
\tag{4.11}
$$

Goodness of $G_2$ follows from Cauchy's inequality with $\epsilon's$, local estimates for solutions to the heat equation as in the proof of $(ii)$, and the usual nontangential maximum-Carleson measure argument. Goodness of $G_3$ is obtained by noting that the first term in brackets in the integrand of $G_3$ integrates to zero. As for $T_3$ we integrate by parts once again to get

$$
T_3 = \int_0^\infty\!\!\int_{\mathbb{R}^n} \frac{(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^2}{1+|\nabla_x P_\lambda A|^2}\sum_{i=1}^{n-1}[-(u_{x_jx_i}\circ\rho)^2 \;+\; (\tfrac{\partial}{\partial x_i}P_\lambda A)^2\,(u_{x_jx_0}\circ\rho)^2\,]dz\lambda d\lambda + G_4
$$

where again $G_4$ is a good term. We first put this expression for $T_3$ into (4.11). Second we put the resulting expression for $T_2$ into (4.10). Adding $T_1$ and $T_2$ we get (4.9) after some juggling. $\square$

**5. The Dirichlet, Neumann, and Regularity Problems.** In this section we begin the proof of Theorem 1.16 for small $\|\mathbb{D}_n A\|_*$ . We shall reduce the proof for existence of a solution in the various regularity problems to a main lemma which we then prove in section 6. In section 7 we prove uniqueness in these regularity problems. We begin with the Neumann problem (Theorem 1.14). Given $f \in L^2(\partial\Omega)$ we again let $u = \mathcal{S}f$ and set $\rho(\lambda, x, t) = (\lambda + P_{\gamma\lambda}A(x,t), x, t)$. From Lemma (2.29)($\beta$) we see that

$$\lim_{(\lambda,y,s)\to(0,x,t)} \langle \nabla u \circ \rho(\lambda, y, s),\, n_t \circ \rho(0, x, t) \rangle \,=\, (\tfrac{1}{2}f + K^* f) \circ \rho(0, x, t),$$

for a.e $(x,t) \in \mathbb{R}^n$, where $n_t$ is the outer unit normal to $\partial\Omega_t$ considered as a subset of $\mathbb{R}^n$ and

$$K^* f(P,t) \equiv \text{ pv } \int_{-\infty}^{t} \int_{\partial\Omega_s} \frac{\partial}{\partial n_t} W(P-Q, t-s) f(Q,s) d\sigma_s(Q)\, ds\,.$$

Here $W$ is the Gaussian (see (1.1) or (4.4)). Thus to prove existence of a layer potential solution to the Neumann problem (Theorem 1.14) it suffices to show that the mapping $f \to \frac{1}{2}f + K^* f$ is 1-1 and onto $L^2(\partial\Omega)$. From Lemma 2.14 $(ii)$ and Lemma 2.29 $(\beta)$ we see that this mapping is into $L^2(\partial\Omega)$ with constants depending only on $\beta$. To prove 1-1 and onto, we claim that it is enough to prove

$$\|f\|_2 \,\le\, c_\beta \min\{\, \|\tfrac{1}{2}f + K^* f\|_2\,,\, \|\tfrac{1}{2}f - K^* f\|_2 \,\} \tag{5.1}$$

whenever $f \circ \rho(0,x,t), (x,t) \in \mathbb{R}^n$, is in $C_0^\infty(\mathbb{R}^n)$. Indeed since such functions $f$ are dense in $L^2(\partial\Omega)$, and (5.1) implies the above mapping is invertible, it suffices to show that $f \to \frac{1}{2}f + K^* f$ is onto $L^2(\partial\Omega)$. To see that this mapping is onto we use a method of continuity argument which in this context was first seen by the authors in [K, p 150] . Let $L_A$ be the linear operator defined for $g \in L^2(\mathbb{R}^n)$ by $L_A\, g(x,t) = (\frac{1}{2}I + K_A^*)(g \circ \rho^{-1})(A(x,t),x,t)$, $(x,t) \in \mathbb{R}^n$, where $K_A^* \equiv K^*$ as in (1.3) for $\partial\Omega$ given by the graph of $A$. Consider the operators $(L_{A\lambda})$ for $\lambda \in [0,1]$ acting on $L^2(\mathbb{R}^n)$. We note that $L_0 = \frac{1}{2}I$ maps $L^2(\mathbb{R}^n)$ onto itself and also

$$\| \tfrac{d}{d\lambda} L_{A\lambda}\, g \,\|_2 \,\le\, c_\beta \, \|g\|_2,$$

as follows from Theorem 1.9 and a Fourier transform type argument (see [H2, (12)]). Using the above note, (5.1), and a continuity argument we get that $L_A$ is onto. Hence we need only prove (5.1) in order to prove existence in Theorem 1.14 when $\|A\|_{\mathrm{comm}} \le \beta < \infty$ and $\|\mathbb{D}_n A\|_* \le \epsilon_0$, where $0 < \epsilon_0 = \epsilon_0(\beta)$. A corresponding theorem also holds for $\mathbb{R}^n \setminus \bar{\Omega}$, as we see from Lemma 2.38 and (5.1). Finally, for use in the Dirichlet problem, we note that Theorem 1.14 implies a similar theorem for the adjoint heat equation, since $u(x,-t)$ is a solution to the adjoint heat equation in $\{(X,t) : x_0 > A(x,-t)\}$ and $A(x,-t)$ has the same properties as $A(x,t)$. In particular if $\tilde{K}$ is defined as in Lemma 2.29 $(\beta)$ with $K^*, \mathcal{S}$, replaced by $\tilde{K}$ and the adjoint heat kernel, then (5.1) remains valid with $K^*$ replaced by $\tilde{K}$.

Next we consider the Dirichlet problem (Theorem 1.13). From the above reasoning, Lemmas 2.14 $(i)$, and 2.29 $(\alpha)$, we see that in order to prove existence in Theorem 1.13, we need only show that (5.1) holds with $K^*$ replaced by $K$. In fact (5.1) for $K^*$ implies (5.1) for $K$ since $K, \tilde{K}$, are adjoints of each other as operators on $L^2(\partial\Omega)$.

Finally to prove existence in Theorem 1.15 we have to show that the mapping $f \to S_b f$ from $L_2(\partial\Omega)$ to $L_{1,1/2}(\partial\Omega)$ is 1-1 and onto. From Lemma 2.14 $(ii), (iii)$ and Lemma 2.29 $(\gamma), (\delta)$, we deduce that this mapping is into. Moreover, using the continuity argument above, but this time invoking Theorem 1.10, we see that it suffices to show

$$\|f\|_2 \,\le\, c_\beta \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} \tag{5.2}$$

in order to establish existence in Theorem 1.15 when $\|A\|_{\text{comm}} \leq \beta$ and $\|\mathbb{D}_n A\|_* \leq \epsilon_0$. Thus to complete the proof of existence in Theorem 1.16 we need to prove (5.1) and (5.2). To prove (5.1) and (5.2) we adopt the strategy of Brown [Br1], Shen [Sh], and Verchota [V] mentioned in section 1. To recall this strategy let $\Omega^- = \mathbb{R}^n \setminus \bar{\Omega}, u^+ = u|_\Omega, u^- = u|_{\Omega^-}$, as in section 1. Let $u_n^\pm(P,t) = \lim_{(Y,s)\to(P,t)} \langle \nabla u^\pm \, , \, n_t\rangle(Y,s)$ where the limit is taken nontangentially in $\Omega$, $\Omega^-$, respectively. Existence a.e with respect to the measure in (0.12) is guaranteed by Lemma 2.38 and $(\beta)$ of Lemma 2.29. From these lemmas and the triangle inequality we find

$$\|f\|_2 \leq \|(\tfrac{1}{2}I + K^*)f\|_2 + \|(\tfrac{1}{2}I - K^*)f\|_2 = \|u_n^+\|_2 + \|u_n^-\|_2 \, . \tag{5.3}$$

We shall prove for $\epsilon_0$ sufficiently small in Theorem 1.16 that

$$\max\{\|u_n^+\|_2 \, , \|u_n^-\|_2 \} \leq \tfrac{1}{4}\|f\|_2 + c_\beta^* \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} \, , \tag{5.4}$$

$$\|S_b f\|_{L_{1,1/2}(\partial\Omega)} \leq \theta\|f\|_2 + c_{\beta,\theta} \min\{\|u_n^+\|_2 \, , \|u_n^-\|_2 \}, \tag{5.5}$$

where $\theta$ can be chosen small with $\epsilon_0$. Using (5.4) in (5.3) we see that (5.2) is true. To get (5.1) we first use (5.5) in (5.4) to deduce

$$\begin{aligned} \max\{\|(\tfrac{1}{2}I + K^*)f\|_2 \, , \|(\tfrac{1}{2}I - K^*)f\|_2 \} &= \max\{\|u_n^+\|_2, \|u_n^-\|_2 \} \\ &\leq (\tfrac{1}{4} + c_\beta^*\theta)\|f\|_2 + c_{\beta,\theta} \min\{\|u_n^+\|_2 \, , \|u_n^-\|_2 \} \\ &= (\tfrac{1}{4} + c_\beta^*\theta)\|f\|_2 + c_{\beta,\theta} \min\{\|(\tfrac{1}{2}I + K^*)f\|_2 \, , \|(\tfrac{1}{2}I - K^*)f\|_2 \}, \end{aligned}$$

where $c_\beta^*$ is as in (5.4). Putting the above inequality in (5.3) we see from the resulting inequality that (5.1) is true provided $\epsilon_0$ is chosen so that $c_\beta^*\theta \leq 1/8$.

To prove (5.4) and (5.5), let $e_0$ be a unit vector directed along the positive $x_0$ axis. We note that at $(X,t) \in \Omega$,

$$(1/2)\nabla \cdot (e_0 \, |\nabla u^+|^2) = \nabla \cdot (u_{x_0}^+ \, \nabla u^+) - u_{x_0}^+ \, u_t^+$$

where as usual $\nabla\cdot$ denotes the divergence operator and we have used $u_t^+ = \Delta u^+$. A similar formula holds in $\Omega^-$ with $u^+$ replaced by $u^-$. For fixed $t > 0$, we use the above equality and apply the divergence theorem in $\{(X,t) : x_0 > \epsilon + P_\epsilon A(x,t)$ and $|X| < 1/\epsilon\}$. Letting $\epsilon \to 0$ we see from $(ii)$ of Lemma (2.14) and simple estimates, using $f \circ \rho(0,x,t) \in C_0^\infty(\mathbb{R}^n)$, that for a.e $t \in \mathbb{R}$,

$$\tfrac{1}{2}\int_{\partial\Omega_t} \langle n_t, e_0\rangle\, |\nabla u^+|^2\, d\sigma_t \;=\; \int_{\partial\Omega_t} u^+_{x_0}\, u^+_{n_t}\, d\sigma_t \;-\; \int_{\Omega_t} u^+_{x_0}\, u^+_t\, dX\,, \tag{5.6}$$

where the last integral on the righthand side of (5.6) is interpreted as a principal value. Similarly,

$$\tfrac{1}{2}\int_{\partial\Omega_t} \langle n_t, e_0\rangle\, |\nabla u^-|^2\, d\sigma_t \;=\; \int_{\partial\Omega_t} u^-_{x_0}\, u^-_{n_t}\, d\sigma_t \;+\; \int_{\Omega_t^-} u^-_{x_0}\, u^-_t\, dX. \tag{5.7}$$

where again the solid integral is interpreted as a principal value. Our goal is to show that if $\gamma \le \frac{1}{2}$ satisfies (2.2), and $\|\mathbb{D}_n A\|_* \le \gamma^{8+d}$, then

$$\left|\int_{\Omega^\pm} u^\pm_{x_0}\, u^\pm_t\, dX dt\right| \le c_\beta \gamma^{1/2}\, \|f\|_2^2 \,+\, c_\beta\, \|f\|_2\, (\, \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)}\, \|u^\pm_n\|_2\,)^{1/2}\,, \tag{5.8}$$

where we have written $\Omega^+$ for $\Omega$. We claim that (5.8) implies (5.4). To prove this claim, we write at a point $(P,t) \in \partial\Omega_t$,

$$\nabla u^+ = \langle \nabla u^+, n_t\rangle\, n_t + \langle \nabla u^+, T\rangle\, T \;= u^+_n\, n_t \,+\, u^+_T\, T,$$

where $T$ is a unit vector in the tangent space to $\partial\Omega_t$ at $(P,t)$. We note that

$$u^+_{x_0} = \langle e_0, n_t\rangle\, u_n \,+ \langle e_0, T\rangle u_T,$$

$$1 \le \frac{c_\beta}{\sqrt{1+|\nabla_x A|^2}} \;= -c_\beta\, \langle e_0, n_t\rangle.$$

Using this note in (5.6), and integrating with respect to t we obtain after some juggling

$$\|u^+_n\|_2^2 \le c_\beta \left( \|u^+_T\|_2^2 + \left|\int_\Omega u^+_t\, u^+_{x_0}\, dX dt\right|\right). \tag{5.9}$$

In the sequel we shall write $u \circ \rho$ for $u \circ \rho|_{\partial\mathbb{R}^{n+1}_+}$ when there is no chance of confusion. From Lemma 2.29 $(\gamma)$ and the fact that (see (0.17))

$$\|u^+\|_{L_{1,1/2}(\partial\Omega)} \approx \|\nabla_x(u\circ\rho)\|_2 \,+\, \|D^t_{1/2}(u\circ\rho)\|_2,$$

we deduce, $\|u_T\|_2 \le c\|S_b f\|_{L_{1,1/2}(\partial\Omega)}$. Also from (5.8) and Cauchy's inequality with $\epsilon$ ' s we see for $\gamma$ sufficiently small that

$$\left|\int_\Omega u^+_{x_0}\, u^+_t dXdt\right| \le \tfrac{1}{8}\|f\|_2 + \tfrac{1}{2}\|u^+_n\|_2 + c_\beta\|S_b f\|_{L^2_{1,1/2}(\partial\Omega)}\,.$$

Using these inequalities in (5.9) we find that (5.4) is true for $u^+_n$. A similar argument using (5.7) can be given for $u^-_n$. Thus (5.8) implies (5.4).

To prove (5.5) we shall need in addition to (5.8),

$$\|D^t_{1/2}\,(u\circ\rho)\|_2^2 \le c_\beta\,\gamma^{1/2}\,\|f\|_2^2 + c_{\beta,\gamma}\,\min\{\|u^+_n\|_2\,,\,\|u^-_n\|_2\}\,\|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} \tag{5.10}$$

provided $\epsilon_0$ is small enough. To see that (5.10) and (5.8) imply (5.5) we first deduce from (5.6), (5.7), as in the proof of (5.4) that

$$\|u_T\|_2^2 \le c_\beta\left(\|u^\pm_n\|_2^2 + \left|\int_{\Omega^\pm} u^\pm_{x_0}\, u^\pm_t\, dXdt\right|\right),$$

where we have written $\Omega^+$ for $\Omega$, and $u_T$ for $u^\pm_T$ since both functions have the same tangential derivatives on $\partial\Omega$. Adding (5.10) to both sides of the above inequality, using (5.8) and Cauchy's inequality with $\epsilon$'s, we get (5.5) provided $\gamma$ and thus also $\epsilon_0 \le \gamma^{8+d}$ are chosen small enough (depending on $\beta$ ). Thus we need only prove (5.8) and (5.10). To prove these " main estimates " is really the heart of the matter. It is in proving them that we shall be forced to deal with obstacles which do not arise in the cylinder case. In this section we shall reduce the proof of (5.8) and (5.10) to the following lemma (Lemma 5.11). We then prove Lemma 5.11 in section 6 to complete the proof of existence in Theorem 1.16.

To state this lemma let $\mathbb{R}^{n+1}_- = \{(\lambda, z) : \lambda < 0, z \in \mathbb{R}^n\}$ and as in Lemma 2.38, let $\rho_-(\lambda, z) = \rho(-\lambda, z)$ when $\lambda < 0$. Finally set $\rho_+ = \rho$.

**Lemma 5.11** *With the above notation and $u = \mathcal{S}f$, we have*

$$(i)\ \int_{\mathbb{R}^{n+1}_{\pm}} \left| D^t_{1/4}(u^{\pm}_{x_j} \circ \rho_{\pm})(\lambda, z)\right|^2 dz d\lambda \ \le\ c_{\beta}\, \|f\|_2^2 \ \textit{for}\ 0 \le j \le n-1,$$

$$(ii)\ \int_{\mathbb{R}^{n+1}_{\pm}} \left| D^t_{3/4}(u^{\pm} \circ \rho_{\pm})(\lambda, z)\right|^2 dz d\lambda \le c_{\beta} \|u^{\pm}_n\|_2 \, \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} \ +\ c_{\beta}\, \gamma \|f\|_2^2$$

*whenever* $\|\mathbb{D}_n A\|_* \le \epsilon_0 \le \gamma^{8+d}$.

Next we reduce the proof of (5.10) to Lemma 5.11. We shall prove this inequality only for $u^+$ as the proof for $u^-$ is identical. We shall write $u$ for $u^+$. To begin we pass to graph coordinates and write

$$\begin{aligned}
c_{\beta}^{-1}\, \|D^t_{1/2} u\,\|_2^2 \ &=\ \int_{\mathbb{R}^n} |D^t_{1/2}(u \circ \rho)(0, z)|^2 dz \\
&= -\int_0^{\infty}\int_{\mathbb{R}^n} \tfrac{\partial}{\partial \lambda}[D^t_{1/2}(u \circ \rho)(\lambda, z)]^2 dz d\lambda = -2\int_0^{\infty}\int_{\mathbb{R}^n} D^t_{1/2}(u \circ \rho)\, D^t_{1/2}(u \circ \rho)_{\lambda}\, dz d\lambda \\
&= -2\int_0^{\infty}\int_{\mathbb{R}^n} D^t_{1/2}(u \circ \rho)\, D^t_{1/2}(u_{x_0} \circ \rho)\, dz d\lambda \qquad (5.12)\\
&\qquad - 2\int_0^{\infty}\int_{\mathbb{R}^n} D^t_{1/2}(u \circ \rho) D^t_{1/2}(\, [u_{x_0} \circ \rho] \tfrac{\partial P_{\lambda} A}{\partial \lambda}\,) dz d\lambda \\
&\qquad\qquad = I_1 \ +\ I_2,
\end{aligned}$$

where again we have used $(u \circ \rho)_{\lambda} = (u_{x_0} \circ \rho)(1 + \frac{\partial}{\partial \lambda} P_{\gamma\lambda} A)$. Also $z = (x, t)$ in the integrands and all integrations are easily justified using Lemma 2.29($\delta$), Lemma 2.14, as well as the fact that $f \circ \rho(0, x, t)$ is smooth with compact support. Using self adjointness of fractional derivative operators, Schwarz's inequality, and $D^t_{1/2} * D^t_{1/2} = c\, H \frac{\partial}{\partial t}$ where $H =$ Hilbert transform, we find

$$\begin{aligned}
|I_2| \ &=\ c \left| \int_0^{\infty}\int_{\mathbb{R}^n} H(u \circ \rho)_t\, (u_{x_0} \circ \rho\,)\, \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A\, dz d\lambda \right| \\
&\le c\, \left( \int_0^{\infty}\int_{\mathbb{R}^n} [(u \circ \rho)_t]^2\, dz \lambda d\lambda \right)^{1/2} \left( \int_0^{\infty}\int_{\mathbb{R}^n} (u_{x_0} \circ \rho\,)^2\, (\tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A)^2\, dz \frac{d\lambda}{\lambda} \right)^{1/2} \qquad (5.13)\\
&= c\, M_1\, M_2.
\end{aligned}$$

Now $(u \circ \rho)_t = u_t \circ \rho + (u_{x_o} \circ \rho) \frac{\partial}{\partial t} P_{\gamma\lambda} A$ and $u_t = \Delta u$, so we can use Theorem 4.1 (i) on the contribution of the first term to $M_1$ while we can use the usual nontangential maximum-Carleson measure argument ((2.7) and Lemma 2.8) to estimate the contribution of the second term to $M_1$. We get $M_1 \leq c_\beta (1 + \gamma^{-1} \| I\!\!D_n A \|_* ) \, \|f\|_2$. Also, $M_2 \leq c_\beta \, \gamma \|f\|_2$, as we see again from (2.7) and Lemma 2.8. Putting these estimates for $M_1, M_2$ in (5.13) and using $\| I\!\!D_n A\|_* \leq \gamma$, we conclude

$$|I_2| \, \leq \, c_\beta \, \gamma \, \|f\|_2^2 \, . \tag{5.14}$$

As for $I_1$, we again use self adjointness and Schwarz's inequality to obtain

$$\begin{aligned}
|I_1| \, &= \, 2 \left| \int_0^\infty \int_{I\!\!R^n} (D^t_{3/4} (u \circ \rho) D^t_{1/4} (u_{x_0} \circ \rho) \, dz d\lambda \right| \\
&\leq \left( \int_0^\infty \int_{I\!\!R^n} \left[ D^t_{3/4} (u \circ \rho) \right]^2 dz d\lambda \right)^{\frac{1}{2}} \left( \int_0^\infty \int_{I\!\!R^n} \left[ D^t_{1/4} (u_{x_0} \circ \rho) \right]^2 dz d\lambda \right)^{\frac{1}{2}} .
\end{aligned}$$

Using Lemma 5.11 and Cauchy's inequality with $\epsilon$ 's, we conclude that

$$\begin{aligned}
|I_1| \leq \, & c_\beta \|f\|_2 \left[ \|u_n\|_2 \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} \, + \, \gamma \|f\|_2^2 \right]^{1/2} \\
\leq \, & c_\beta \, \gamma^{1/2} \, \|f\|_2^2 \, + c_\beta \, \gamma^{-1/2} \, \|u_n\|_2 \, \|S_b f\|_{L^2_{1,1/2}(\partial\Omega)} .
\end{aligned}$$

From this estimate for $I_1$ and (5.14) for $I_2$ we see in view of (5.12) that (5.10) is true for $\epsilon_0 \leq \gamma^{8+d}$, and $\gamma$ sufficiently small. Finally in this section we consider (5.8). Using the above differentiation formulas for $(u \circ \rho)_t$ and $(u \circ \rho)_\lambda$, we have

$$\begin{aligned}
\int_\Omega u_{x_0} \, u_t \, dX dt \, &= \int_0^\infty \int_{I\!\!R^n} \, u_{x_0} \circ \rho \; u_t \circ \rho \; (1 + \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A) \, dz dt \\
&= \int_0^\infty \int_{I\!\!R^n} \, u_{x_0} \circ \rho \; (u \circ \rho)_t \; dz d\lambda - \int_0^\infty \int_{I\!\!R^n} \, (u_{x_0} \circ \rho)^2 \, \tfrac{\partial}{\partial t} P_{\gamma\lambda} A \, dz d\lambda \\
&\qquad + \int_0^\infty \int_{I\!\!R^n} \, u_{x_0} \circ \rho \; u_t \circ \rho \; \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A \, dz d\lambda \\
&= I + E_1 + E_2 .
\end{aligned} \tag{5.15}$$

We estimate $E_2$ almost exactly as we did $I_2$ in (5.13) to get

$$\begin{aligned}
|E_2| \, &\leq \, c \left( \int_0^\infty \int_{I\!\!R^n} [u_t \circ \rho]^2 \, dz \lambda d\lambda \right)^{1/2} \left( \int_0^\infty \int_{I\!\!R^n} (u_{x_0} \circ \rho)^2 \, (\tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A)^2 \, dz \frac{d\lambda}{\lambda} \right)^{1/2} \\
&\leq c_\beta \, \gamma (1 + \gamma^{-1} \, \| I\!\!D_n A \|_* ) \, \|f\|_2^2 \, \leq \, c_\beta \, \gamma \, \|f\|_2^2 .
\end{aligned} \tag{5.16}$$

Also with the aid of Lemma 5.11 we treat $I$ similar to $I_1$. We obtain

$$\begin{aligned}
|I| \;&=\; c\left|\int_0^\infty\!\!\int_{\mathbb{R}^n} H D^t_{3/4}(u\circ\rho) D^t_{1/4}(u_{x_0}\circ\rho)\,dz d\lambda\right| \\
&\le c\left(\int_0^\infty\!\!\int_{\mathbb{R}^n}\left[D^t_{3/4}(u\circ\rho)\right]^2 dz d\lambda\right)^{\frac12}\left(\int_0^\infty\!\!\int_{\mathbb{R}^n}\left[D^t_{1/4}(u_{x_0}\circ\rho)\right]^2 dz d\lambda\right)^{\frac12} \qquad (5.17)\\
&\le\; c_\beta\gamma^{1/2}\,\|f\|_2^2 + c_\beta\|f\|_2\;\left[\|u_n\|_2\|\|S_b f\|_{L^2_{1,1/2}(\partial\Omega)}\right]^{1/2}.
\end{aligned}$$

To handle $E_1$ we integrate by parts in $\lambda$ to find that

$$\begin{aligned}
E_1 = 2&\int_0^\infty\!\!\int_{\mathbb{R}^n}(u_{x_0}\circ\rho)_\lambda\,(u_{x_0}\circ\rho)\,\tfrac{\partial}{\partial t}P_{\gamma\lambda}A\,dz\lambda d\lambda \\
&+\int_0^\infty\!\!\int_{\mathbb{R}^n}(u_{x_0}\circ\rho)^2\tfrac{\partial^2}{\partial t\partial\lambda}P_{\gamma\lambda}A\,dz\,\lambda d\lambda \;=\; E_{11}+E_{12}\,.
\end{aligned} \qquad (5.18)$$

By (2.2) and the usual differentiation rules we have $|(u_{x_0}\circ\rho)_\lambda|\;\le\;2|u_{x_0x_0}|$ . Thus by Schwarz's inequality ,

$$\begin{aligned}
|E_{11}|\;&\le 4\int_0^\infty\!\!\int_{\mathbb{R}^n}|u_{x_0x_0}\circ\rho|\,|u_{x_0}\circ\rho|\,|\tfrac{\partial}{\partial t}P_{\gamma\lambda}A|\,dz\lambda d\lambda \\
&\le\left(\int_0^\infty\!\!\int_{\mathbb{R}^n}(u_{x_0x_0}\circ\rho)^2 dz\lambda d\lambda\right)^{\frac12}\left(\int_0^\infty\!\!\int_{\mathbb{R}^n}(u_{x_0}\circ\rho)^2\,(\tfrac{\partial}{\partial t}P_{\gamma\lambda})^2\,dz\lambda d\lambda\right)^{\frac12}
\end{aligned}$$

The first term on the right hand side of the last inequality can be estimated using Theorem 4.1 (i) while the second term can be handled using (2.7) and Lemma 2.8 $(a)$ with $\theta=1, |\phi|=\sigma=0$. Thus

$$|E_{11}|\;\le\;c_\beta\gamma^{-1}\,\|\mathbb{D}_nA\|_*\,\|f\|_2^2\;\le\;c_\beta\,\gamma\|f\|_2^2, \qquad (5.19)$$

since $\|\mathbb{D}_nA\|_*\le\gamma^2$. Finally in this section we consider $E_{12}$. To handle this term we integrate by parts in the $t$ variable to find

$$E_{12} = -2\int_0^\infty\!\!\int_{\mathbb{R}^n}\;u_{x_0}\circ\rho\;(u_{x_0}\circ\rho)_t\;\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}\,A\,dz\lambda d\lambda.$$

But $(u_{x_0}\circ\rho)_t\;=\;u_{x_0\,t}\circ\rho\,+\,(u_{x_0x_0}\circ\rho)\,\frac{\partial}{\partial t}P_{\gamma\lambda}A,$ so $|E_{12}|$ can be estimated using Theorem4.1$(i),(ii)$ and Lemma 2.8 $(a)$ and $(b)$. We get

$$|E_{12}|\;\le\;c_\beta\,\gamma\,\|f\|_2^2\,.$$

Using this estimate for $E_{12}$ and the estimate for $E_{11}$ in (5.19) we obtain

$$|E_1| \leq c_\beta \, \gamma \|f\|_2^2 \, .$$

In view of this inequality, (5.16), (5.17) and (5.15) we conclude that (5.8) is valid once we have proved Lemma 5.11. Thus it remains to prove Lemma 5.11 in order to complete the proof of existence in Theorem 1.16 for small $\|\mathbb{D}_n A\|_*$ .

**6. Proof of Theorem 1.16.** In this section we prove our main lemma, Lemma 5.11, thereby completing the proof of existence in Theorem 1.16 for small $\|\mathbb{D}_n A\|_*$. We shall prove Lemma 5.11 only for $u = u^+$, since the proof for $u^-$ is identical to the proof for $u^+$. We begin by assuming that $\gamma \leq \frac{1}{2}$, satisfies (2.2) and $\|\mathbb{D}_n A\|_* \leq \epsilon_0 \leq \gamma^3$, which will be sufficient for our present needs. Later in this section we shall require that $\|\mathbb{D}_n A\|_* \leq \epsilon_0 \leq \gamma^{8+d}$.

**Proof of Lemma 5.11 :** We prove the easier estimate $(i)$ first. To do this we integrate the integral in $(i)$ by parts with respect to $\lambda$ and use self adjointness of $D^t_{1/4}$ to get

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} \left[ D^t_{1/4}(u_{x_j} \circ \rho) \right]^2 \, dz d\lambda &= -2 \int_0^\infty \int_{\mathbb{R}^n} (u_{x_j} \circ \rho)_\lambda \; D^t_{1/2}(u_{x_j} \circ \rho) \, dz \, \lambda d\lambda \\
&\leq 2 \left( \int_0^\infty \int_{\mathbb{R}^n} \left[ (u_{x_j} \circ \rho)_\lambda \right]^2 \, dz \, \lambda d\lambda \right)^{\frac{1}{2}} \left( \int_0^\infty \int_{\mathbb{R}^n} \left[ D^t_{1/2}(u_{x_j} \circ \rho) \right]^2 dz \, \lambda d\lambda \right)^{\frac{1}{2}}
\end{aligned}$$

for $0 \leq j \leq n-1$. The integrals in the last inequality are estimated using $(i), (iii)$ of Theorem 4.1 and once again the observation that $|(u_{x_j} \circ \rho)_\lambda| \leq 2 |u_{x_j x_0}|$, thanks to (2.2). Thus $(i)$ of Lemma 5.11 is valid.

Next we treat the more difficult $(ii)$ of Lemma 5.11. To ease our writing we put $\tilde{\omega} = H D^t_{1/2} \, (u \circ \rho)$ on $\mathbb{R}^{n+1}_+$. Our game plan is to follow the strategy of Shen [Sh] and implicity that of Brown [Br 1], by first using self adjointness of $D^t_{3/4}$ to rewrite the integral in $(ii)$, and second integrating by parts in the space variable, to get some cancellation. First using self adjointness we have,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} [D^t_{3/4}(u \circ \rho)]^2 \, dz d\lambda &= \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega}\, (u \circ \rho)_t \; dz d\lambda \\
= \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega}\, (u_t \circ \rho)\, (1 + \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A) \, dz d\lambda &- \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega}\, (u_t \circ \rho) \; \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A \, dz d\lambda \\
&+ \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega}\, (u_{x_0} \circ \rho) (\tfrac{\partial}{\partial t} P_{\gamma\lambda} A) \, dz d\lambda \\
&= II + E_3 + E_4.
\end{aligned} \tag{6.1}$$

We handle $E_3$ using Schwarz's inequality, the fact that $u_t = \Delta u$, Theorem 4.1 $(i)$, (2.7), and Lemmas 2.8, 2.14. We find

$$|E_3| \;\le\; c_\beta \, \gamma \, \|f\|_2^2 \, . \tag{6.2}$$

To treat $E_4$ we integrate by parts in $\lambda$ to deduce that

$$\begin{aligned}
-E_4 = \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega}_\lambda \, (u_{x_0} \circ \rho) (\tfrac{\partial}{\partial t} P_{\gamma\lambda} A) \, dz \lambda d\lambda &+ \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega}\, (u_{x_0} \circ \rho)_\lambda (\tfrac{\partial}{\partial t} P_{\gamma\lambda} A) \, dz \lambda d\lambda \\
+ \int_0^\infty \int_{\mathbb{R}^n} \tilde{\omega} (u_{x_0} \circ \rho) (\tfrac{\partial^2}{\partial t \, \partial \lambda} P_{\gamma\lambda} A) \, dz \; \lambda d\lambda & \\
= E_{41} + E_{42} + E_{43}. &
\end{aligned} \tag{6.3}$$

Using Schwarz's inequality, Theorem 4.1 $(i)$, Lemma 2.14 $(iii)$, (2.7) and Lemma 2.8 $(a)$ with $\theta = 1, |\phi| = \sigma = 0$, we obtain

$$|E_{42}| \;\le\; c_\beta \, \gamma^{-1} \|\mathbb{D}_n A\|_* \, \|f\|_2^2 \;\le c_\beta \, \gamma \|f\|_2^2, \tag{6.4}$$

since $\|\mathbb{D}_n A\|_* \le \gamma^3$. To handle $E_{41}$ we argue as previously to get

$$|E_{41}| \;\le\; c_\beta \, \gamma^{-1} \|\mathbb{D}_n A\|_* \, \|f\|_2 \left( \int_0^\infty \int_{\mathbb{R}^n} \left[ D^t_{1/2} (u \circ \rho)_\lambda \right]^2 \; dz \, \lambda d\lambda \right)^{\frac{1}{2}} = c_\beta \, \gamma^{-1} \|\mathbb{D}_n A\|_* \, \|f\|_2 \, E_{44}. \tag{6.5}$$

We square $E_{44}$ and integrate by parts with respect to $\lambda$ to obtain

$$
\begin{aligned}
(E_{44})^2 &= -\int_0^\infty \int_{\mathbb{R}^n} D^t_{1/2}(u\circ\rho)_{\lambda\lambda}\ D^t_{1/2}(u\circ\rho)_\lambda\ dz\,\lambda^2 d\lambda \\
&= -c\int_0^\infty \int_{\mathbb{R}^n} (u\circ\rho)_{\lambda\lambda}\ H(u\circ\rho)_{\lambda t}\, dz\,\lambda^2\, d\lambda \\
&\le c\left(\int_0^\infty \int_{\mathbb{R}^n} [(u\circ\rho)_{\lambda\lambda}]^2\, dz\lambda\, d\lambda\right)^{\frac{1}{2}} \left(\int_0^\infty \int_{\mathbb{R}^n} [(u\circ\rho)_{\lambda t}]^2\, dz\lambda^3\, d\lambda\right)^{\frac{1}{2}} \\
&= c\, E_{45}\, E_{46}
\end{aligned} \tag{6.6}
$$

Now

$$
(u\circ\rho)_{\lambda\lambda} = (u_{x_0x_0})(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^2 + (u_{x_0}\circ\rho)\,\tfrac{\partial^2}{\partial\lambda\partial\lambda}P_{\gamma\lambda}A,
$$

so we can use Theorem 4.1 ($i$), (2.2), (2.7), and Lemmas 2.8, 2.14, to find that $|E_{45}| \le c_\beta\|f\|_2$. Similarly to estimate $|E_{46}|$ we note as in the proof of Theorem 4.1 ($ii$) that

$$
(u\circ\rho)_{\lambda t} = \ (u_{x_0\,t}\circ\rho)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\,) + (u_{x_0x_0}\circ\rho)(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)\,\tfrac{\partial}{\partial t}P_{\gamma\lambda}A\ +\ (u_{x_0}\circ\rho)\,\tfrac{\partial^2}{\partial\lambda\partial t}P_{\gamma\lambda}A\,.
$$

The contribution of the first term on the righthand side of this equality to $E_{46}$ is estimated using Theorem 4.1 ($ii$), (2.2), and the fact that $u_{x_0\,t} = \Delta u_{x_0}$. The contribution of the second term is treated using Lemma 4.1 ($i$) and Lemma 2.8 ($b$) with $\theta = 1, |\phi| = \sigma = 0$. The contribution of the third term is estimated by Lemma 2.14 ($ii$) and Lemma 2.8 ($a$) with $\theta = 1 = \sigma, |\phi| = 0$. Since $\|\mathbb{D}_n A\|_* \le \lambda^3$, we find that

$$
|E_{46}|\ \le\ c_\beta\,\|f\|_2\,.
$$

Putting the above estimates for $E_{45}, E_{46}$ into (6.6) and taking square roots, we see that

$$
|E_{44}|\ \le\ c_\beta\,\|f\|_2\,.
$$

Next using this estimate for $|E_{44}|$ and $\|\mathbb{D}_n A\| \le \gamma^3$ in (6.5), we obtain

$$
|E_{41}|\ \le\ c_\beta\gamma\,\|f\|_2^2\,. \tag{6.7}
$$

To finish our treatment of $E_4$, it remains to consider $E_{43}$. To do so we integrate by parts in $t$ to get

$$\begin{aligned} -E_{43} \;=\; & c \int_0^\infty \int_{\mathbb{R}^n} D^t_{3/2}(u\circ\rho)\,(u_{x_0}\circ\rho)\,\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\,dz\lambda d\lambda \\ & + \int_0^\infty \int_{\mathbb{R}^n} HD^t_{1/2}(u\circ\rho)\,(u_{x_0}\circ\rho)_t\,\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\,dz\lambda d\lambda. \end{aligned}$$

The first term on the righthand side of this equality is estimated using Schwarz's inequality, Theorem 4.1 $(iv)$, Lemma 2.8, and the usual nontangential maximum-Carleson measure argument. The second term can be handled just like the term $E_{12}$ following (5.19). Indeed this term is identical to $-\frac{1}{2}E_{12}$ , except that $HD^t_{1/2}(u\circ\rho)$ replaces $u_{x_0}\circ\rho$ in it. But we also have control of the nontangential maximal function of $HD^t_{1/2}(u\circ\rho)$ thanks to Lemma 2.14 $(iii)$. Hence

$$|E_{43}| \;\le\; c_\beta\gamma\|f\|_2^2\,.$$

From this estimate for $E_{43}$, (6.7), (6.4), and (6.3) we conclude that

$$|E_4| \;\le\; c_\beta\,\gamma\,\|f\|_2^2. \tag{6.8}$$

From (6.8), (6.2), and (6.1) we see that in order to complete the proof of Lemma 5.11 we need to estimate $II$. For this purpose we put $\omega = \tilde\omega\circ\rho^{-1}$ and note from the divergence theorem that

$$\int_{\Omega_t} \omega\,\tfrac{\partial u}{\partial t}\;dX = \int_{\partial\Omega_t} \omega\,u_n\,d\sigma_t \;-\; \int_{\Omega_t} \langle\nabla u,\nabla\omega\rangle\;dX,$$

where we have also used $u_t = \Delta u$. We integrate this inequality with respect to $t$ over $\mathbb{R}$ and use $\rho^{-1}$ to change variables in $II$. Since the Jacobian of this transformation is equal to $(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A)^{-1}$ we have

$$\begin{aligned} II = \textstyle\int_\Omega \omega\,\frac{\partial u}{\partial t}\,dX dt = \int_{\partial\Omega} \omega\,u_n\,d\sigma_t\,dt \;-\; \int_\Omega \langle\nabla\omega,\nabla u\rangle\,dX dt \\ =\; B_1 + III. \end{aligned} \tag{6.9}$$

For the boundary term $B_1$ we see from Schwarz's inequality and $\|A\|_{\text{comm}} \le \beta$, that

$$|B_1| \;\le\; c_\beta\,\|u_n\|_2\,\|D^t_{1/2}u\|_2\,. \tag{6.10}$$

We have therefore reduced matters to showing that $III$ is small. This will require some work. To estimate $III$, we first consider the term $\omega_{x_0}\, u_{x_0}$ in the integrand of the integral defining $III$. We transfer the corresponding integral back to $\mathbb{R}^{n+1}_+$ using $\rho$ as a change of variable. Since the Jacobian of this transformation is $1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A$, and

$$\omega_{x_0}\circ\rho\,(1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\,)=\tilde{\omega}_\lambda = HD^t_{1/2}\left(u_{x_0}\circ\rho\,[1+\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A]\right), \tag{6.11}$$

we obtain

$$\begin{aligned}
-\textstyle\int_\Omega \omega_{x_0}\, u_{x_0}\, dX\, dt \;&=\; -\int_0^\infty\int_{\mathbb{R}^n}\tilde{\omega}_\lambda\; u_{x_0}\circ\rho\, dz d\lambda \\
&= -\int_0^\infty\int_{\mathbb{R}^n} HD^t_{1/2}(u_{x_0}\circ\rho)\, u_{x_0}\circ\rho\, dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n} HD^t_{1/2}\left(u_{x_0}\circ\rho\;\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\; u_{x_0}\circ\rho\, dz d\lambda \\
&= 0\,+\,\int_0^\infty\int_{\mathbb{R}^n}(u_{x_0}\circ\rho\;\tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A)\;HD^t_{1/2}(u_{x_0}\circ\rho)\, dz\, d\lambda,
\end{aligned}$$

where we have used the antisymmetry of $HD^t_{1/2}$. The last term in this inequality is estimated using Schwarz's inequality, Theorem 4.1, (2.7), and Lemmas 2.8, 2.14. We find that

$$\left|\int_\Omega \omega_{x_0}\, u_{x_0}\, dX\, dt\right| \;\leq\; c_\beta\,\gamma\|f\|_2^2\,. \tag{6.12}$$

We now consider the rest of the integrand in the integral defining $III$. Again we use $\rho$ to change variables in the above integral and note that the Jacobian of this transformation is $1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A$. In addition to (6.11) we observe that at $(\lambda, z)$,

$$\omega_{x_j}\circ\rho = \tilde{\omega}_{x_j}\,-\,\tilde{\omega}_\lambda\;\frac{\frac{\partial}{\partial x_j}P_{\gamma\lambda}A}{(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A)}$$

and

$$\tilde{\omega}_{x_j}\;=\;HD^t_{1/2}(u_{x_j}\circ\rho)+HD^t_{1/2}\left([u_{x_0}\circ\rho]\,\tfrac{\partial}{\partial x_j}P_{\gamma\lambda}A\right)\,.$$

Using these observations we get

$$
\begin{aligned}
-\sum_{j=1}^{n-1} \int_{\Omega} \omega_{x_j} \, u_{x_j} \, dX dt \;=\; & -\sum_{j=1}^{n-1} \int_0^\infty \int_{\mathbb{R}^n} HD^t_{1/2}(u_{x_j} \circ \rho) \, u_{x_j} \circ \rho \; dz d\lambda \\
& - \sum_{j=1}^{n-1} \int_0^\infty \int_{\mathbb{R}^n} HD^t_{1/2}(u_{x_j} \circ \rho) \, u_{x_j} \circ \rho \; \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A \, dz \, d\lambda \\
& - \sum_{j=1}^{n-1} \int_0^\infty \int_{\mathbb{R}^n} \left( \left[ HD^t_{1/2} \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] \, u_{x_0} \circ \rho \right) \; u_{x_j} \circ \rho \, dz d\lambda \\
& - \sum_{j=1}^{n-1} \int_0^\infty \int_{\mathbb{R}^n} HD^t_{1/2} \left\{ (u_{x_0} \circ \rho) \; \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right\} \, u_{x_j} \circ \rho \; \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A \, dz d\lambda \\
& + \sum_{j=1}^{n-1} \int_0^\infty \int_{\mathbb{R}^n} HD^t_{1/2} \left\{ (u_{x_0} \circ \rho) \; \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A \right\} \, u_{x_j} \circ \rho \; \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \, dz d\lambda \\
& = N_1 + N_2 + N_3 + N_4 + N_5,
\end{aligned}
\tag{6.13}
$$

where we are using the standard notation

$$
[T, B] \, g \;= T(Bg) - B(Tg)
$$

for the commutator of $T$ and $B$ acting on $g$. Now $N_1 = 0$, by the antisymmetry of $HD^t_{1/2}$. Next using Schwarz's inequality, (2.7), Lemmas 2.8, 2.14, and Theorem 4.1, we deduce that $|N_2| \;\le\; c_\beta \, \gamma \|f\|_2^2$. To treat $N_4, N_5$, it is enough by antisymmetry of $HD^t_{1/2}$ to consider terms of the form

$$
\int_0^\infty \int_{\mathbb{R}^n} HD^t_{1/2} \left\{ (u_{x_i} \circ \rho) \; \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right\} \, u_{x_l} \circ \rho \; \tfrac{\partial}{\partial \lambda} P_{\gamma\lambda} A \, dz d\lambda \;=\; N_6,
$$

where $0 \le i, l \le n-1$. Using Schwarz's inequality, (2.7), Lemmas 2.8 and 2.14, we deduce

$$
|N_6| \;\le\; c_\beta \, \gamma \|f\|_2 \left( \int_0^\infty \int_{\mathbb{R}^n} \left| HD^t_{1/2} \{ (u_{x_i} \circ \rho) \, \tfrac{\partial}{\partial x_j} \, P_{\gamma\lambda} A \} \, \right|^2 \; \lambda dz dt \right)^{1/2} \;=\; c_\beta \, \gamma \|f\|_2 \, N_7 \, . \tag{6.14}
$$

Moreover, using Theorem 4.1 $(iii)$ and Lemma 2.8 $(b)$ with $|\phi| = 1$, $\theta = \sigma = 0$, we find that

$$
\begin{aligned}
c^{-1}\,|N_7|^2 \;\le\; & \int_0^\infty\!\!\int_{\mathbb{R}^n} \left|\tfrac{\partial}{\partial x_j}\, P_{\gamma\lambda}A \; HD^t_{1/2}(u_{x_i}\circ\rho)\;\right|^2 dz\,\lambda d\lambda \\
& + \int_0^\infty\!\!\int_{\mathbb{R}^n} \left|\left[HD^t_{1/2}\,,\; \tfrac{\partial}{\partial x_j} P_{\gamma\lambda}A\right]\; u_{x_i}\circ\rho\right|^2 dz\,\lambda d\lambda \\
& \le\; c_\beta \|f\|_2^2 \;+\; \int_0^\infty\!\!\int_{\mathbb{R}^n} \left|\left[HD^t_{1/2}\,,\; \tfrac{\partial}{\partial x_j} P_{\gamma\lambda}A\right] u_{x_i}\circ\rho\right|^2\; dz\,\lambda d\lambda \\
& =\; c_\beta\,\|f\|_2^2 \;+\; N_8\,.
\end{aligned}
\tag{6.15}
$$

To bound $N_8$ we prove the following lemma.

**Lemma 6.16** *Given* $R \ge 2$, *and* $i \in [0, n-1]$, *we have*

$$
N_8 \;\le\; c_\beta\,(\,R^{-2} + R^{2+d}\,\gamma^{-4}\,\|\mathbb{D}_n A\|_*^2\,)\;\|f\|_2^2.
$$

Before proving this lemma we note several of its consequences when $R = \gamma^{-1}$ and $\|\mathbb{D}_n A\|_* \le \epsilon_0 \le \gamma^{8+d}$. First from Lemma 6.16, (6.14), and (6.15) we see that

$$
\sum_{i\neq 3} |N_i| \;\le\; c_\beta\,\gamma\|f\|_2^2.
\tag{6.17}
$$

Second we show that $N_3$ can also be estimated in terms of $N_8$. To do this we integrate by parts in $\lambda$, so that

$$
\begin{aligned}
N_3 \;=\; & \sum_{j=1}^{n-1} \int_0^\infty\!\!\int_{\mathbb{R}^n} \left\{\left[HD^t_{1/2}\,, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda}A\right]\; u_{x_0}\circ\rho\right\}\;(u_{x_j}\circ\rho)_\lambda\, dz\,\lambda d\lambda \\
& + \sum_{j=1}^{n-1} \int_0^\infty\!\!\int_{\mathbb{R}^n} \left\{\left[HD^t_{1/2}\,, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda}A\right]\;(u_{x_0}\circ\rho)_\lambda\right\}\; u_{x_j}\circ\rho\, dz\,\lambda d\lambda \\
& + \sum_{j=1}^{n-1} \int_0^\infty\!\!\int_{\mathbb{R}^n} \left\{\left[HD^t_{1/2}\,, \tfrac{\partial^2}{\partial\lambda\partial x_j} P_{\gamma\lambda}A\right]\; u_{x_0}\circ\rho\right\}\; u_{x_j}\circ\rho\, dz\,\lambda d\lambda \\
& = N_{31} + N_{32} + N_{33}\,.
\end{aligned}
\tag{6.18}
$$

We can handle $N_{31}$ immediately using Schwarz's inequality, Lemma 6.16 with $R, \epsilon_0$ as above, Theorem 4.1 $(i)$, and (2.2). Furthermore the self adjointness of the commutator $\left[HD^t_{1/2}\,, \frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right]$ permits us to treat $N_{32}$ in exactly the same way as $N_{31}$. We obtain $|N_{31}| + |N_{32}| \le c_\beta\gamma\,\|f\|_2^2$. As for $N_{33}$, we write

$$\begin{aligned} N_{33} \;&=\; -\sum_{j=1}^{n-1}\int_0^\infty\int_{\mathbb{R}^n} \tfrac{\partial}{\partial\lambda\partial x_j}P_{\gamma\lambda}A\ HD^t_{1/2}(u_{x_0}\circ\rho)\ u_{x_j}\circ\rho\,dz\,\lambda d\lambda \\ &\quad+\sum_{j=1}^{n-1}\int_0^\infty\int_{\mathbb{R}^n} HD^t_{1/2}\left\{(u_{x_0}\circ\rho)\,\tfrac{\partial}{\partial\lambda\partial x_j}P_{\gamma\lambda}A\right\}\ u_{x_j}\circ\rho\,dz\,\lambda d\lambda. \end{aligned} \tag{6.19}$$

By the anti-symmetry of $HD^t_{1/2}$, these terms are of essentially the same form. Mere boundedness of these terms is easy, but we need them to be small, which requires some work. After integrating by parts in $x_j$, it is enough to consider terms of the form

$$N_{34} = \int_0^\infty\int_{\mathbb{R}^n} \tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\ HD^t_{1/2}(u_{x_i}\circ\rho)\ (u_{x_l}\circ\rho)_{x_j}\,dz\,\lambda d\lambda$$

and

$$N_{35} = \int_0^\infty\int_{\mathbb{R}^n} \tfrac{\partial}{\partial\lambda}P_{\gamma\lambda}A\ HD^t_{1/2}\left\{(u_{x_i}\circ\rho)_{x_j}\right\}\ u_{x_l}\circ\rho\,dz\,\lambda d\lambda,$$

where $0 \le i, l \le n-1$. We note that

$$|(u_{x_l}\circ\rho)_{x_j}| \le |u_{x_l x_j}\circ\rho| + |(u_{x_l x_0}\circ\rho)\,\tfrac{\partial}{\partial x_j}P_{\gamma\lambda}A\,| \;\le\; (1+\beta)\sum_{k,m=0}^{n-1}|u_{x_k x_m}\circ\rho|, \tag{6.20}$$

where we have used Lemma 2.8 $(b)$ with $\sigma = \theta = 0, |\phi| = 1$. From this note, (2.2), and Theorem 4.1 $(i), (iii)$, we find $|N_{34}| \;\le\; c\gamma\,\|f\|_2^2$. Also by Schwarz's inequality, (2.7), and Lemmas 2.8, 2.14, we have

$$\begin{aligned} |N_{35}| \;&\le\; c_\beta\,\gamma\|f\|_2\left(\int_0^\infty\int_{\mathbb{R}^n}\left[HD^t_{1/2}\left\{(u_{x_i}\circ\rho)_{x_j}\right\}\right]^2 dz\,\lambda^3 d\lambda\right)^{1/2} \\ &= c_\beta\,\gamma\|f\|_2\,N_{36}\,. \end{aligned} \tag{6.21}$$

From the self adjointness of $D^t_{1/2}$ and Schwarz's inequality we deduce

$$N_{36}^2 \le \left(\int_0^\infty \int_{\mathbb{R}^n} |(u_{x_i}\circ\rho)_{x_j}|^2\,dz\,\lambda d\lambda\right)^{1/2} \left(\int_0^\infty \int_{\mathbb{R}^n} |\,(u_{x_i}\circ\rho)_{x_j\,t}\,|^2\,dz\,\lambda^5 d\lambda.\right)^{1/2} \tag{6.22}$$
$$= N_{37}\,N_{38}.$$

From (6.20) and Theorem 4.1($i$), we have $N_{37}\le c_\beta\|f\|_2$. As to $N_{38}$, observe that

$$(u_{x_i}\circ\rho)_{x_j\,t} \;=\; [u_{x_ix_j}\circ\rho \,+\, (u_{x_ix_0}\circ\rho)\,\tfrac{\partial}{\partial x_j}P_{\gamma\lambda}A\,]_t$$
$$= u_{x_ix_jt}\circ\rho \,+\, (u_{x_ix_jx_0}\circ\rho)(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A\,) + (u_{x_ix_0t}\circ\rho)(\tfrac{\partial}{\partial x_j}P_{\gamma\lambda}A)$$
$$+\,(u_{x_ix_0x_0}\circ\rho)\,(\tfrac{\partial}{\partial t}P_{\gamma\lambda}A)\,(\tfrac{\partial}{\partial x_j}P_{\gamma\lambda}A)\,+\,(u_{x_ix_0}\circ\rho)\,\tfrac{\partial^2}{\partial t\partial x_j}P_{\gamma\lambda}A.$$

Using Lemma 2.8 ($b$), Theorem 4.1 ($i$), and local interior estimates for solutions to the heat equation (see the proof of Theorem 4.1 ($ii$)), we can handle the contribution of the second, fourth and fifth terms on the righthand side of the above equality, to the integral defining $N_{38}$. The contribution of the first and third terms to this integral are treated using Theorem 4.2 ($ii$), Lemma 2.8 ($b$) and local estimates as above. Altogether we get

$$N_{38}\le c_\beta(1+\tfrac{\|D_nA\|_*}{\gamma^2})\|f\|_2 \;\le\; c_\beta\,\|f\|_2.$$

From these estimates for $N_{37}, N_{38}$, and (6.22) we deduce first that $|N_{36}| \;\le\; c_\beta\,\|f\|_2,$ and second from (6.21) that $|N_{35}| \;\le\; c_\beta\,\gamma\|f\|_2^2$. Next from this estimate for $|N_{35}|$, our earlier estimate for $|N_{34}|$, and (6.19), we have $|N_{33}| \le\; c_\beta\,\gamma\|f\|_2^2$. In view of (6.18) and our previous estimates for $N_{31}, N_{32}$, we conclude that

$$|N_3|\le\; c_\beta\,\gamma\|f\|_2^2\,. \tag{6.23}$$

Thus Lemma 6.16 implies the estimate in (6.23) for $N_3$. Finally we note that (6.23), (6.17), and (6.13) yield

$$\left|\sum_{j=1}^{n-1}\int_\Omega \omega_{x_j}\,u_{x_j}\,dX dt\right| \le c_\beta\gamma\|f\|_2^2.$$

This inequality together with (6.12) gives, $|III| \leq c_\beta \, \gamma \|f\|_2^2$. Putting this estimate for $III$ in (6.9) and using (6.10), we get an estimate for $II$. Combining our estimate for $II$, together with our earlier estimates for $E_3, E_4$, in (6.2), (6.8), we see in view of (6.1) that $(ii)$ of Lemma 5.11 is true. Hence to complete the proof of Lemma 5.11 and so also the proof of existence in Theorem 1.16, we need only prove Lemma 6.16.

**Proof of Lemma 6.16** To begin the proof of Lemma 6.16, we integrate by parts in $\lambda$ and use Cauchy's inequality with $\epsilon$'s to obtain

$$
\begin{aligned}
&\int_0^\infty \int_{I\!R^n} \left| \left[ HD_{1/2}^t \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] u_{x_i} \circ \rho \right|^2 \, dz \, \lambda d\lambda \\
&\quad = - \int_0^\infty \int_{I\!R^n} \left[ HD_{1/2}^t \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] (u_{x_i} \circ \rho) \; \left[ HD_{1/2}^t \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] (u_{x_i} \circ \rho)_\lambda \, dz \, \lambda^2 d\lambda \\
&\quad - \int_0^\infty \int_{I\!R^n} \left[ HD_{1/2}^t \, , \, \tfrac{\partial^2}{\partial\lambda \partial x_j} P_{\gamma\lambda} A \right] (u_{x_i} \circ \rho) \; \left[ HD_{1/2}^t \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] (u_{x_i} \circ \rho) \, dz \, \lambda^2 d\lambda \\
&\quad \leq c_\beta \int_0^\infty \int_{I\!R^n} \left| \left[ HD_{1/2}^t \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] (u_{x_i} \circ \rho)_\lambda \right|^2 \, dz \, \lambda^3 d\lambda \\
&\quad + c_\beta \int_0^\infty \int_{I\!R^n} \left| \left[ HD_{1/2}^t \, , \, \tfrac{\partial^2}{\partial\lambda \partial x_j} P_{\gamma\lambda} A \right] u_{x_i} \circ \rho \right|^2 \, dz \, \lambda^3 d\lambda \\
&\qquad\qquad = c_\beta ( T_1 \, + \, T_2 ) \, .
\end{aligned} \tag{6.24}
$$

We treat $T_1$ first. We have

$$
\begin{aligned}
\left[ HD_{1/2}^t \, , \, \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A \right] g(x,t) \; &\equiv \; \int_{\{s: |s-t| \leq R^2 \lambda^2\}} k(t-s) \, \left( \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A(x,s) - \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A(x,t) \right) \, g(x,s) \, ds \\
&+ \int_{\{s: |s-t| > R^2 \lambda^2\}} k(t-s) \, \left( \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A(x,s) - \tfrac{\partial}{\partial x_j} P_{\gamma\lambda} A(x,t) \right) \, g(x,s) \, ds \\
&= F_\lambda(x,t) \, + \, G_\lambda(x,t),
\end{aligned} \tag{6.25}
$$

where $k(t-s) \equiv c \text{ sgn } (t-s) \, |t-s|^{-3/2}$, whenever $s, t \in I\!R$, $s \neq t$. Using Lemma 2.8 $(b)$ with $|\phi| = 1, \theta = \sigma = 0$, we deduce as in [S1, Thm 1, p 62]

$$
\lambda \, G_\lambda(x,t) \; \leq \; c_\beta \, R^{-1} \, M_n(g(x,\cdot)(t), \tag{6.26}
$$

where $M_n$ denotes the one dimensional Hardy Littlewood maximal function acting in the $t$ variable while the other variables are fixed. Moreover, using Lemma 2.8($b$) with $|\phi| = \theta = 1, \sigma = 0$, and again arguing as in [S1, Thm 1, p 62] we find that

$$\begin{aligned}\lambda|F_\lambda(x,t)| &= \lambda \left| \int_0^1 \int_{\{s:|s-t|\leq R^2\lambda^2\}} k(t-s)\,(t-s)\, \tfrac{\partial^2}{\partial z_n \partial x_j} P_{\gamma\lambda} A(x, s+\theta(t-s))\, g(x,s)\, ds\, d\theta \right| \\ &\leq c_\beta\, \gamma^{-2}\, \|I\!\!D_n A\|_*\, \lambda^{-1} \int_{\{s:|s-t|\leq R^2\lambda^2\}} |t-s|^{-1/2}\, g(x,s) ds \\ &\leq\; c_\beta R\, \gamma^{-2} \|I\!\!D_n A\|_*\, M_n g(x,t)\,.\end{aligned} \tag{6.27}$$

Here $\frac{\partial}{\partial z_n}$ denotes the partial with respect to the time variable. Putting $g = (u_{x_i} \circ \rho)_\lambda$, into (6.26), (6.27), and using (6.25), (6.24), (6.20), as well as Theorem 4.1 ($i$), we obtain

$$|T_1| \;\leq\; c_\beta\, (\, R^{-2} + R^2 \gamma^{-4}\, \|I\!\!D_n A\|_*^2\, ) \|f\|_2^2\,. \tag{6.28}$$

Next we turn to $T_2$ in (6.24). We make a similar decomposition,

$$\begin{aligned}&\left[ H D^t_{1/2}\, ,\, \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A \right] g(x,t) \\ &\qquad \equiv \int_{\{s:|s-t|\leq R^2\lambda^2\}} k(t-s)\, \left( \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A(x,s) - \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A(x,t) \right)\, g(x,s)\, ds \\ &\qquad + \int_{\{s:|s-t|> R^2\lambda^2\}} k(t-s)\, \left( \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A(x,s) - \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A(x,t) \right)\, g(x,s)\, ds \\ &\qquad = \tilde{F}_\lambda(x,t) \,+\, \tilde{G}_\lambda(x,t).\end{aligned}$$

As in (6.26), we get at $(x,t)$

$$\lambda|\tilde{G}_\lambda| \;\leq\; c_\beta R^{-1}\, M_n \left( g\, \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A \right) \,+\, c_\beta R^{-1}\, |\phi_{R\lambda} * g(x,\cdot)\, \tfrac{\partial^2}{\partial \lambda \partial x_j} P_{\gamma\lambda} A\,|,$$

where

$$\phi_{R\lambda}\,(t) \;\equiv\; R\lambda\, k(t)\, \chi_{\{s:|s|>R^2\lambda^2\}}(t).$$

As in [S1, Thm 1, p 62] ). we again see that $\phi_{R\lambda} * g$ is nontangentially bounded by $cM_n N_* g$. Using the above inequality with $g = u_{x_i} \circ \rho$, the Hardy Littlewood maximal theorem, (2.7), Lemma 2.8 ($a$)

with $\sigma = |\phi| = 1, \theta = 0$, and Lemma 2.14 $(ii)$, we find that the contribution of $\tilde{G}_\lambda$ to the integral defining $T_2$ is $\leq c_\beta R^{-2} \|f\|_2^2$.

Finally to handle the contribution of $\tilde{F}$ to the integral defining $T_2$, we first note that if $|s-t| \leq R^2\lambda^2$, then

$$\left| \tfrac{\partial^2}{\partial\lambda\partial x_j} P_{\gamma\lambda} A(x,s) - \tfrac{\partial^2}{\partial\lambda\partial x_j} P_{\gamma\lambda} A(x,t) \right| \leq c_\beta |s-t| \, M_n^{R\lambda} \left( \tfrac{\partial^3}{\partial\lambda\partial x_j \partial z_n} P_{\gamma\lambda} A(x,\cdot) \right)(t), \qquad (+)$$

where $M_n^{R\lambda}$ denotes the truncated maximal function defined by

$$M_n^{R\lambda} h(x,t) = \sup_{0<a\leq(R\lambda)^2} \left[ (2a)^{-1} \int_{-a}^{a} |h(x, z_n+t)| dz_n \right]$$

whenever $h(x,\cdot)$ is locally integrable. Second we note that if $z = (x,t)$, then

$$d\nu(\lambda, x, t) = \left[ M_n^{R\lambda} \left( \tfrac{\partial^3}{\partial\lambda\partial x_j\partial z_n} P_{\gamma\lambda} A \right) \right]^2 (z) \, dz \, \lambda^5 d\lambda$$

is a Carleson measure on $\mathbb{R}_+^{n+1}$ with

$$\nu\left((0,r) \times B_r(z_0)\right) \leq c_\beta \, \gamma^{-4} \|\mathbb{D}_n A\|_*^2 \, R^d \, r^d \tag{6.29}$$

whenever $r > 0$. Indeed if $\chi$ denotes the characteristic function of $(0, crR) \times B_{crR}(z_0)$, then for $c$ large enough we see from the definition of $\|z\|, M_n^{R\lambda}$, that if $h(\lambda, x, t) = \frac{\partial^3}{\partial\lambda\partial x_j\partial z_n} P_{\gamma\lambda} A(x,t)$, then $M_n^{R\lambda} h(\lambda, x, t) \leq M_n(h\chi)(\lambda, x, t)$, whenever $(\lambda, x, t) \in (0,r) \times B_r(z_0)$. Using this observation, the Hardy Littlewood maximal theorem, and Lemma 2.8 $(a)$ with $\sigma = \theta = |\phi| = 1$, we get

$$\nu\left((0,r) \times B_r(z_0)\right) \leq c_\beta \int_0^\infty \int_{\mathbb{R}^n} M_n(\chi h)^2(\lambda, z) \, dz \, \lambda^5 d\lambda$$

$$\leq c_\beta \int_0^{crR} \int_{B_{crR}(z_0)} h^2 \, dz \, \lambda^5 d\lambda \leq c_\beta \, \gamma^{-4} \|\mathbb{D}_n A\|_*^2 \, R^d \, r^d \, .$$

Thus (6.29) is true.

Using (+) and arguing as in (6.27) we get for $h$ defined as above,

$$\lambda |\tilde{F}_\lambda(x,t)| \leq c_\beta \lambda M_n^{R\lambda} h(x,t) \int_{\{s : |s-t| \leq (R\lambda)^2\}} |s-t|^{-1/2} g(x,s) ds$$

$$\leq c_\beta R \, M_n^{R\lambda} h(x,t) \; \psi_{R\lambda} * g(x,\cdot)(t),$$

where

$$\psi_{R\lambda}(t) \equiv (R\lambda)^{-1}\,|t|^{-1/2}\,\chi_{\{s:|s|\le (R\lambda)^2\}}\,(t)\,.$$

If $g = u_{x_i}\circ\rho$, then the contribution of $\tilde{F}_\lambda$ to $T_2$ is therefore dominated by

$$c_\beta R^2 \int_0^\infty \int_{\mathbb{R}^n} [\psi_{R\lambda} * |g|(z)]^2 \left(M_n^{R\lambda}h(z)\right)^2 dz\,\lambda^5 d\lambda$$

$$\le c_\beta R^2 \int_0^\infty \int_{\mathbb{R}^n} |g|^2\,\psi_{R\lambda} * (M_n^{R\lambda}h)^2\,dz\,\lambda^5\,d\lambda,$$

where we have used Schwarz to write

$$(\psi_{R\lambda} * |g|)^2 \;\le\; \psi_{R\lambda} * |g|^2,$$

since $\|\psi_{R\lambda}\|_1 \;\equiv\; c$, and then used self adjointness of $\psi_{R\lambda}$. Next we simply replace the Carleson measure $d\nu$ defined above by

$$d\hat{\nu}(\lambda, z) = \psi_{R\lambda} * (M_n^{R\lambda}h)^2\,dz\,\lambda^5 d\lambda$$

and observe that the Carleson norm of this measure is bounded by a constant multiple of the Carleson norm of $\nu$ (by a similar argument to the one we used for $\nu$). Using this observation we finally get that the contribution of $\tilde{F}_\lambda$ to the integral defining $T_2$ is at most, $c_\beta\,R^{d+2}\,\gamma^{-4}\,\|\mathbb{D}_n A\|_*^2\|f\|_2^2$. Thus,

$$T_2 \le c_\beta(R^{-2} \,+\, R^{d+2}\,\gamma^{-4}\,\|\mathbb{D}_n A\|_*^2\,)\,\|f\|_2^2\,.$$

Putting this estimate and the estimate for $T_1$ in (6.28) into (6.24), we conclude that Lemma 6.16 is true. □

In view of the remarks preceding the proof of this lemma, we have also now proved Lemma 5.11 and existence in Theorem 1.16 for small $\|\mathbb{D}_n A\|_*$. □

**7. Uniqueness in Theorem 1.16.** In this section we prove uniqueness in Theorem 1.16. We begin with the Dirichlet problem (Theorem 1.13). Assume for some $f \in L^2(\partial\Omega)$ that there exist two solutions $u, v$, to the heat equation in $\Omega$ with nontangential limits equal to $f$ a.e. with respect to the surface measure defined in (0.12), where the limit is taken relative to parabolic cones defined as in (1.11) for some fixed $a > 0$. If $\tilde{N}_* u$, $\tilde{N}_* v$, are the nontangential maximal functions of $u, v$, defined relative to this $a$ as in (1.12) and if both functions are in $L^2(\partial\Omega)$, then we shall show that $u \equiv v$. To this end put $h = u - v$ and note that $h$ has nontangential limits zero a.e. while $\tilde{N}_* h \in L^2(\partial\Omega)$. Given $\epsilon > 0$, set $\Omega(\epsilon) = \{(x_0, x, t) : x_0 > \epsilon + P_{\gamma\epsilon}A(x,t) \text{ and } (x,t) \in \mathbb{R}^n\}$. We note that $\|\mathbb{D}_n P_{\gamma\epsilon} A\|_* \le c\|\mathbb{D}_n A\|_*$ and $\|P_{\gamma\epsilon}A\|_{\mathrm{comm}} \le c\|A\|_{\mathrm{comm}}$. Thus we can apply Theorem 1.16 in $\Omega(\epsilon)$, for sufficiently small $\epsilon_0$. Let $h_\epsilon = \mathcal{D} f_\epsilon$, be the solution to the $L^2(\partial\Omega(\epsilon))$ Dirichlet problem for the heat equation in $\Omega(\epsilon)$ with boundary values : $h_\epsilon = h$ a.e on $\partial\Omega(\epsilon)$ in the nontangential limiting sense. Existence is guaranteed by our existence proof in Theorem 1.16. We claim that in fact

$$h_\epsilon \equiv h \tag{7.1}$$

in $\Omega(\epsilon)$. Once this claim is proved we easily obtain that $h \equiv 0$. Indeed, let $\tilde{N}_{*,\epsilon} h$ denote the nontangential maximal function of $h$ in $\Omega(\epsilon)$ defined in parabolic cones relative to the above $a$. We note from (2.2) that $\Omega(\epsilon_1) \supset \Omega(\epsilon_2)$ whenever $0 < \epsilon_1 < \epsilon_2$. Using this note and simple geometry it is easily seen that $\tilde{N}_{*,\epsilon}\, h \le \tilde{N}_*\, h$. Also from (7.1) and Theorem 1.16 applied to $\Omega(\epsilon)$, we have

$$\|N_{*,\epsilon} h\|_{L^2(\partial\Omega(\epsilon))} \le c\,\|h\|_{L^2(\partial\Omega(\epsilon))}.$$

We first write this inequality in graph coordinates, and then use dominated convergence as $\epsilon \to 0$. Using $\|P_{\gamma\epsilon}A\|_{\mathrm{comm}} \le c\beta$, it follows that $\tilde{N}_{*,\epsilon}\, h \equiv 0$ for each $\epsilon > 0$. Thus claim (7.1) implies uniqueness in Theorem 1.13.

To prove (7.1) we begin by observing that the kernel of the double layer potential(see (1.3)) is continuous on $\partial\Omega(\epsilon)$ (in fact $C^\infty$), as we see from the smoothness of $P_{\gamma\epsilon}A$. Using this fact, smoothness of $h|_{\partial\Omega(\epsilon)}$, and Lemma 2.29 ($\alpha$), we find first that $f_\epsilon$ is a continuous function on $\partial\Omega(\epsilon)$ and second that $h_\epsilon$ extends continuously to the closure of $\Omega(\epsilon)$, (also denoted $h_\epsilon$). Then for fixed $\epsilon > 0$, we see that $g = h_\epsilon - h$ is a solution to the heat equation in $\Omega(\epsilon)$, which is continuous in the closure of $\Omega(\epsilon)$, and $g \equiv 0$ on $\partial\Omega(\epsilon)$. Also, $\tilde{N}_{*,\epsilon}\, g \in L^2(\partial\Omega(\epsilon))$. We extend $g$ to a continuous function on $\mathbb{R}^{n+1}$, by defining $g \equiv 0$ in $\mathbb{R}^{n+1} \setminus \bar{\Omega}(\epsilon)$. We show that these properties of $g$ and the maximum principle for the heat equation imply $g \equiv 0$. To do this given $(X,t) \in \partial\Omega(\epsilon)$ with $X = (x_0, x)$, let

$$Q_\rho(X,t) = \{(y_0, y, s) : |y_i - x_i| \le \rho, 0 \le i \le n-1, \text{ and } |s-t| \le \rho^2\}.$$

Let $G_\rho$ be the solution to the continuous Dirichlet problem for the heat equation in $Q_\rho(X,t)$ with $G_\rho = |g|$ on $\partial Q_\rho(X,t)$. Then from the maximum principle for the heat equation we have $|g| \le G_\rho$ in $Q_\rho(X,t)$. Now $G_\rho$ can be calculated explicitly in terms of a Poisson integral of its boundary values. More simply, we can compare $G_\rho$ to the Poisson integral of certain halfplane solutions. Doing this, and using the maximum principle for the heat equation, we deduce that if $(Y,s) \in Q_{\rho/2}(X,t)$, then

$$g(Y,s)^2 \le G_\rho(Y,s)^2 \le c\rho^{-n} \int_{F_1} g^2\, dS + c\,\rho^{-n-1} \int_{F_2} g^2 dS, \tag{7.2}$$

where $dS$ denotes surface area on $\partial Q_\rho(X,t)$,

$$F_1 = \{(Z,\tau) \in \partial Q_\rho(X,t) : |\tau - t| = \rho^2\},$$

and $F_2 = \partial Q_\rho(X,t) \setminus F_1$. Given $R > 0$ we integrate the righthand side of (7.2) over $\rho \in (R, 2R)$. From the resulting integral and the lefthand side of (7.2), we deduce that for $(Y,s) \in Q_{R/2}(X,t)$,

$$g(Y,s)^2 \le c\, R^{-(n+1)}\, \|\tilde{N}_* g\|_2^2.$$

Letting $R \to \infty$, we obtain that $g \equiv 0$. Thus claim (7.1) is true and the proof of uniqueness in Theorem 1.13 is now complete.

Next we consider uniqueness in Theorem 1.15. Let $f \in L^2_{1,1/2}(\partial\Omega)$ and suppose as above that there exist two solutions $u, v$, to the heat equation in $\Omega$ with nontangential limits a.e. equal to $f$ in $L^2_{1,1/2}(\partial\Omega)$. We extend $u, v$ to almost every point in $\partial\Omega$, by defining each function to be equal to its nontangential limit whenever this limit exists. We also denote these extensions by $u, v$. If $h = u - v$, then on $\partial\Omega$ we have $h \equiv 0$ in $L^2_{1,1/2}(\partial\Omega)$ and from the definition of this space we see that $h = c$, a.e. on $\partial\Omega$. We assume, as we may, that $c = 0$, so $h = 0$, a.e on $\partial\Omega$. Using this fact we show that

$$h = \mathcal{S} h_n \tag{7.3}$$

in $\Omega$. Once (7.3) is proved we can apply (5.8), (5.9) with $f = h_n$ to conclude for $\gamma$ small enough that

$$\|h_n\|_2 \le c_\beta \, \|h\|_{L^2_{1,1/2}(\partial\Omega)} = 0.$$

Thus $h_n = 0$ a.e. on $\partial\Omega$ and so $h \equiv 0$ in $\Omega$, which proves uniqueness in Theorem 1.15.

To prove (7.3) fix $(X', t') \in \partial\Omega$ and $(X, t) \in \Omega$. Choose $\rho > 0$ so large and $\epsilon > 0$ so small that $(X, t) \in \Omega(\epsilon) \cap Q_\rho(X', t')$, where $\Omega(\epsilon)$, $Q_\rho(X', t')$ are defined as above. Let $k = h\psi$ in $\Omega$ where $\psi \in C_0^\infty(Q_{2\rho}(X', t'))$ with $\psi \equiv 1$ on $\bar{Q}_{3\rho/2}(X', t')$ and

$$\|\psi_t\|_\infty + \sum_{i,j=0}^{n-1} \|\psi_{x_i x_j}\|_\infty \le c\, \rho^{-2}. \tag{7.4}$$

If $0 < \delta < 1$, then for $s \in (-\infty, t - \delta)$, we can apply Green's second identity in $\Omega(\epsilon) \cap (\mathbb{R}^n \times \{s\})$ with one of our functions $k$ and the other

$$(Y, s) \to W(X - Y, t - s) = (4\pi(t - s))^{-n/2} \exp\left\{ -\frac{|X - Y|^2}{4(t - s)} \right\} \chi_{(0,\infty)}(t - s).$$

We note from (1.1) that $W(X-Y,t-s)$ is the kernel of the single layer potential in $\Omega$. Integrating the resulting equality over $s\in(-\infty,t-\delta)$ we obtain

$$\begin{aligned} I = \int_{-\infty}^{t-\delta}\int_{\Omega_s(\epsilon)} [k(Y,s)\,W(X-Y,t-s)]_s \, dY\, ds \\ = \int_{-\infty}^{t-\delta}\int_{\partial\Omega(\epsilon)_s} k_n(Y,s)W(X-Y,t-s)\,d\sigma_{\epsilon,s}\,ds \\ - \int_{-\infty}^{t-\delta}\int_{\partial\Omega(\epsilon)_s} k(Y,s)W_n(X-Y,t-s)\,d\sigma_{\epsilon,s}\,ds \\ - \int_{-\infty}^{t-\delta}\int_{\Omega(\epsilon)_s} (\,2\langle\nabla h,\nabla\psi\rangle\,+\,h\,\Delta\psi\,-h\,\psi_s\,)W(X-Y,t-s)\,dY ds \\ = H_1+H_2+H_3, \end{aligned} \tag{7.5}$$

where $\sigma_\epsilon=\sigma_{\epsilon,s}, n=n_{\epsilon,s}(Y,s)$ are defined as in (0.12), (0.14). We note from $\|A\|_{\text{comm}}\le\beta$ that if $h(A(y,s),y,s)=0$, then

$$|h|(\lambda+P_{\gamma\epsilon}A(y,s),y,s)\le\,c_\beta\,\lambda\tilde N_*\,(|\nabla h|)(A(y,s),y,s) \tag{7.6}$$

when $\lambda\ge\epsilon$. Using (7.6) and letting $\epsilon\to0$, we find from dominated convergence, that

$$\lim_{\epsilon,\delta\to0} H_2=0. \tag{7.7}$$

Also, from dominated convergence we have

$$\lim_{\epsilon,\delta\to0} H_1=\int_{\mathbb{R}}\int_{\partial\Omega_s} k_n(Y,s)W(X-Y,t-s)\,d\sigma_s\,ds. \tag{7.8}$$

Next we see from (7.6) and our choice of $\psi$ that if

$$Q=[Q_{2\rho}(X',t')\setminus Q_{3\rho/2}(X',t')]\cap\Omega(\epsilon),$$

and $X'=(x_0',x')$, then for $c$ sufficiently large

$$\begin{aligned} |H_3|\,\le c\rho^{-n}\int_Q(\rho^{-2}|h|\,+\,\rho^{-1}|\nabla h|)\,dY ds \\ \le c\rho^{-n}\int_{B_{c\rho}(x',t')}\tilde N_*(|\nabla h|)\,dy\,ds\,\le c\rho^{(1-n)/2}\,\|\tilde N_*(\nabla h\|)\|_2\,. \end{aligned}$$

Hence

$$\lim_{\rho \to \infty} H_3 = 0 \tag{7.9 }$$

independently of $\epsilon, \delta \in (0,1)$. To treat $I$ let

$$\theta_\epsilon(\lambda, x, t) = (\lambda + \epsilon + P_{\gamma\epsilon} A(x,t), x, t),$$

when $(\lambda, x, t) \in \mathbb{R}^{n+1}_+$ and put $w(Y,s) = k(Y,s) W(X - Y, t - s)$. Since the Jacobian of the above transformation is $\equiv 1$, we have

$$\begin{aligned} I &= \int_{-\infty}^{t-\delta} \int_{\Omega_s(\epsilon)} w_s \, dY \, ds = \int_{-\infty}^{t-\delta} \int_{\mathbb{R}^{n-1}} \int_0^\infty w_s \circ \theta \, d\lambda \, dy \, ds \\ &= \int_{-\infty}^{t-\delta} \int_{\mathbb{R}^{n-1}} \int_0^\infty [\, (w \circ \theta)_s - (w \circ \theta)_\lambda \, \tfrac{\partial}{\partial s} P_{\gamma\epsilon} A \,] \, d\lambda \, dy \, ds \\ &= \int_{\mathbb{R}^{n-1}} \int_0^\infty w \circ \theta(\lambda, y, t - \delta) \, d\lambda \, dy \; + \int_{-\infty}^{t-\delta} \int_{\mathbb{R}^{n-1}} w \circ \theta(0, y, s) \, \tfrac{\partial}{\partial s} P_{\gamma\epsilon} A \, dy \, ds \\ &= I_1 + I_2 \end{aligned} \tag{7.10}$$

Letting $\delta \to 0$ we deduce from the usual " approximate identity type " limiting argument that $I_1 \to h(X,t)$, independently of $\epsilon, \rho$. Also from (7.6), Lemma 2.8 ($c$) with $\theta = 1, \sigma = |\phi| = 0$, and dominated convergence, we deduce that

$$|I_2| \le c \, \epsilon \int_{\partial\Omega(\epsilon)} \tilde{N}_*(|\nabla h|) \, |\psi| \, |\tfrac{\partial}{\partial s} P_{\gamma\epsilon} A \,| \, d\sigma_s \, ds \to 0$$

as $\epsilon, \delta \to 0$, We let $\delta \to 0$, then $\epsilon \to 0$, and finally $\rho \to \infty$. Using the above inequalities for $I_1, I_2$, in (7.10) we obtain $I \to h(X,t)$. Using this fact, (7.5), (7.7), (7.8) and (7.9), we get that (7.3) holds in a certain limiting sense. On the other hand from Sobolev type estimates using $h_n \in L^2(\partial\Omega)$, it is easily shown that the integral in (7.3) converges absolutely for all $(X,t) \in \Omega$. Thus (7.3) is true and the proof of uniqueness in Theorem 1.15 is now complete.

Finally we consider uniqueness in Theorem 1.14. Again assume for some $f \in L^2(\partial\Omega)$ that there exist two solutions $u, v$, to the heat equation in $\Omega$ with normal derivatives equal to $f$ a.e. on $\partial\Omega$ in the nontangential sense of Theorem 1.14. If for some fixed $a > 0$, we have $\tilde{N}_*(|\nabla u|),\ \tilde{N}_*|\nabla v| \in L^2(\partial\Omega)$, we shall show $u \equiv v$. For this purpose, set $h = u - v$. We first prove that

$$\|h\|_{L^2_{1,1/2}(\partial\Omega(\epsilon))} < \infty. \tag{7.11}$$

Once (7.11) is proved we can use uniqueness in Theorem 1.15 to conclude that $h = Sq_\epsilon + c$ in $\Omega(\epsilon)$ for some $q_\epsilon \in L^2(\partial\Omega(\epsilon))$. From (5.1) and dominated convergence, it follows that

$$\|q_\epsilon\|_2 \le c_\beta \|h_{n_\epsilon}\|_2 \to 0, \quad \text{as} \quad \epsilon \to 0.$$

From this inequality and basic Sobolev estimates we see that $h = c$ in $\Omega$ which proves uniqueness in Theorem 1.14.

To prove (7.11) we shall need several lemmas. In order to state these lemmas define $\rho_\epsilon : \bar{\mathbb{R}}^{n+1}_+ \to \bar{\Omega}(\epsilon)$, by

$$\rho_\epsilon(\lambda, x, t) = (\lambda + \epsilon + P_{\gamma(\lambda+\epsilon)}A(x,t), x, t) = (x_0, x, t).$$

For fixed $\epsilon, \tilde{a} > 0$, put $g = h \circ \rho_\epsilon$ and let $N_*g(x,t)$ be the nontangential maximal function of $g$ defined relative to $\Gamma_{\tilde{a}}(0,x,t)$, where $\tilde{a}$ is chosen so small that $N_*g \le \tilde{N}_{*,\epsilon}\, h$, at points corresponding under the transformation $(0,x,t) \to \rho_\epsilon(0,x,t)$. Also, for given $R > 0$, set

$$\begin{aligned}
Q' &= \{ (\lambda, x, t) \in \mathbb{R}^{n+1}_+ : \lambda < 2R,\ (x,t) \in B_{2R}(0,0) \},\\
Q'' &= \{ (\lambda, x, t) \in \mathbb{R}^{n+1}_+ : 2R < \lambda < 4R,\ (x,t) \in B_{2R}(0,0) \},\\
Q''' &= \{ (\lambda, x, t) \in \mathbb{R}^{n+1}_+ : \lambda < 6R,\ (x,t) \in B_{6R}(0,0) \},\\
m_{Q'}\, g &= |Q'|^{-1} \int_{Q'} g(X,t)\, dX dt\,.
\end{aligned}$$

With this notation we prove

**Lemma 7.12** *We have*

$$R^{-3} \int_{Q'} |\, g - m_{Q'}g\,|^2\, dX\, dt \le c_\beta \|N_*(|\nabla g|)\|_2^2\,.$$

**Proof** We first note that

$$T = R^{-3}\int_{Q'} |g - m_{Q'}g|^2\, dX\, dt \;\le\; c\, R^{-n-5}\int_{Q'}\int_{Q'} |\, g(Y,s) - g(X,t)\,|^2\, dY ds\, dX dt.$$

Second we let $\chi$ be the characteristic function of $Q''$ and suppose that $(X',t') = (x_0', x', t'), (Y', s') = (y_0', y', s')$ are points in $Q'$. We put $(X,t) = (x_0' + 2R, x', t'), (Y,s) = (y_0' + 2R, y', s')$ and observe that $(X,t), (Y,s) \in Q''$. Using the triangle inequality and the mean value theorem of differential calculus we see that

$$|g(X',t') - g(Y',s')| \;\le\; cR\,[\, N_*(|\nabla g|)(0,x',t') + N_*(|\nabla g|)(0,y',s)\,] \;+\; |g(X,t) - g(Y,s)|.$$

Third, we let $\chi$ denote the characteristic function of $Q''$ and note that

$$|g(X,t) - g(Y,s)| \;\le\; cR^2\, M_n(g_t\,\chi)(X,t) \;+\; cR\sum_{i=0}^{n-1} M_i(g_{x_i}\chi)(X,t),$$

where $M_i, 0 \le i \le n-1$, denotes the one dimensional maximal function in $x_i$, while the other variables are held constant, and $M_n$ is the maximal function in the time variable. Using these notes, the Hardy-Littlewood Maximal Theorem, and integrating over $Q' \times Q'$, we find that

$$\begin{aligned}
T \;&\le\; c\,\|N_*(|\nabla g|)\|_2^2 + c\,R\int_{Q''} M_n(g_t\,\chi)^2\, dX dt \\
&\qquad + cR^{-1}\left(\sum_{i=0}^{n-1}\int_{Q''} M_i(|\nabla g|\chi)^2\, dX dt\right) \\
&\le c\|N_*(|\nabla g|)\|_2^2 + cR\int_{Q''} |g_t|^2\, dX dt \\
&\qquad + c\,R^{-1}\int_{Q''} |\nabla g|^2\, dX dt \\
&\le c\|N_*(|\nabla g|)\|_2^2 + cR\int_{Q''} |g_t|^2\, dX dt\,.
\end{aligned}$$

To handle the integral involving $g_t$ we observe that if $X = (\lambda, x)$, then from Lemma 2.8 $(b)$, and the fact that $h_t = \Delta h$ in $\Omega$, we have at $(X, t)$

$$|g_t| \le |h_t \circ \rho_\epsilon| \,+\, |h_{x_0} \circ \rho_\epsilon|\, |\tfrac{\partial}{\partial t} P_{\gamma(\epsilon+\lambda)} A|$$

$$\le c \sum_{i=0}^{n-1} |h_{x_i x_i} \circ \rho_\epsilon| + c_\beta R^{-1} \, |h_{x_0} \circ \rho_\epsilon| \, .$$

Using this observation, local interior estimates for second derivatives of solutions to the heat equations (in terms of the first derivatives), and Lemma 2.8 $(b)$ we see that

$$R \int_{Q''} |g_t|^2 \, dX dt \;\le\; c_\beta \, R^{-1} \int_{Q'''} |\nabla g|^2 \, dX dt$$

$$\le c_\beta \, \|N_*(|\nabla g|)\|_2^2.$$

In view of this inequality and the above inequality for $T$, we conclude that Lemma 7.12 is true. □

Next let $\phi \in C_0^\infty((-2R, 2R) \times B_{2R}(0,0))$ with $\phi \equiv 1$ in $(-R, R) \times B_R(0,0)$ and

$$\|\phi_t\|_\infty \,+\, \|\nabla^2 \phi\|_\infty \;\le\; c\, R^{-2},$$

where $|\, \nabla^2 \phi \,|^2$ is defined at $X = (\lambda, x)$ by

$$|\, \nabla^2 \phi \,|^2 \;=\; \sum_{i,j=1}^{n-1} \phi^2_{x_i x_j} \,+\, \sum_{i=1}^{n-1} \phi^2_{x_i \lambda} \,+\, \phi^2_{\lambda\lambda} \, . \tag{7.13}$$

For $k = (g - m_{Q'}\, g)\, \phi$, we shall prove

**Lemma 7.14** *If $z = (x, t)$ and $|\, \nabla^2 \, \cdot |^2$ is as in (7.13), then*

*(i)* $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} |\, \nabla^2 k \,|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \|\tilde{N}_*(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))}$

*(ii)* $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} |k_t|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \|\tilde{N}_*(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))}$

*(iii)* $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} |\, \nabla D^t_{1/2} k \,|^2 \, dz \, \lambda d\lambda \;\le\; c_\beta \, \|\tilde{N}_*(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))} \, .$

**Proof:** To begin the proof of Lemma 7.14 we claim that

$$\begin{aligned} &(i) \quad \int_0^\infty \int_{\mathbb{R}^n} |\, h_{x_i x_j} \circ \rho_\epsilon(\lambda, z)\,|^2\, dz\, \lambda d\lambda \,\leq\, c_\beta \|\tilde{N}_*(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))} \text{ for } 0 \leq i, j \leq n-1, \\ &(ii) \quad \int_0^\infty \int_{\mathbb{R}^n} |\, h_{x_j\, t} \circ \rho_\epsilon(\lambda, z)\,|^2\, dz\, \lambda^3 d\lambda \,\leq\, c_\beta \|\tilde{N}_*(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))} \text{ for } 0 \leq j \leq n-1\,. \end{aligned} \tag{7.15}$$

To prove this claim we note that since $\tilde{N}_{*,\epsilon}(|\nabla h|) \in L^2(\partial\Omega(\epsilon))$ and each component of $\nabla h$ is a solution to the heat equation in $\Omega(\epsilon)$, we can apply Theorem 1.13 to $h_{x_j}$, $0 \leq j \leq n-1$ in $\Omega(\epsilon)$. Doing this we see that each component of $\nabla h$ can be written as the double layer potential of an $L^2(\partial\Omega(\epsilon))$ function with norm $\leq c\|\tilde{N}_{*,\epsilon}(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))}$. Using this fact, the fact that $\tilde{N}_{*,\epsilon} \leq \tilde{N}_*$, and Corollary 4.8, we get claim (7.15). Next we note from Lemma 2.8 $(b)$, as in many previous differentations, that at $(\lambda, x, t)$

$$\begin{aligned} |\, \nabla^2 k\,|^2 \,\leq\, c_\beta\, |\, \nabla^2 h\,|^2 \circ \rho_\epsilon \,+\, (h_{x_0} \circ \rho_\epsilon)^2\, |\, \nabla^2 P_{\gamma(\lambda+\epsilon)} A\,|^2 \\ +\, c|\nabla g|^2\, |\nabla \phi|^2 \,+\, c|\, g - m_{Q'}\, g\,|^2\, |\, \nabla^2 \phi\,|^2. \end{aligned}$$

The contribution of the first term on the righthand side of this inequality to the integral in Lemma 7.14 $(i)$ is handled using (7.15)$(i)$. The contribution of the second term is estimated from (2.7) and Lemma 2.8 $(b)$ . The third term is treated easily using Lemma 2.8 $(b)$ and the fact that $\phi \equiv 0 \in \mathbb{R}^{n+1}_+ \setminus Q'$, while the estimate for the fourth term can be obtained from Lemma 7.12. Thus $(i)$ of Lemma 7.14 is true.

$(ii)$ of Lemma 7.14 is proved similarly, using $h_t = \Delta h$ in $\Omega(\epsilon)$. We omit the details. To prove $(iii)$ of Lemma 7.14, we integrate by parts in $\lambda$ and use Schwarz's inequality, as well as self adjointness of $D^t_{1/2}$, to get

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |\, \nabla D^t_{1/2} k\,|^2\, dz\, \lambda d\lambda \,=\, -\int_0^\infty \int_{\mathbb{R}^n} \langle \nabla D^t_{1/2} k_\lambda\,,\, \nabla D^t_{1/2} k\,\rangle\, dz\, \lambda^2 d\lambda \\ \leq\, c\, \left( \int_0^\infty \int_{\mathbb{R}^n} |\, \nabla k_\lambda\,|^2\, dz\, \lambda d\lambda \right)^{1/2} \left( \int_0^\infty \int_{\mathbb{R}^n} |\, \nabla k_t\,|^2\, dz\, \lambda^3 d\lambda \right)^{1/2} \\ \leq c_\beta \|\tilde{N}_*(|\nabla h|)\|_{L^2(\partial\Omega(\epsilon))} \,\cdot\, J, \end{aligned} \tag{7.16}$$

where

$$J^2 = \int_0^\infty \int_{I\!R^n} |\,\nabla k_t\,|^2 \, dz\, \lambda^3 d\lambda,$$

and we have used Lemma (7.14) $(i)$ to get the last inequality in (7.16). Differentiating and using Lemma 2.8 $(b)$ we see that at $(\lambda, x, t) = (\lambda, z)$, we have

$$|\nabla k_t|^2 \,\le\, c_\beta \,|\nabla h_t \circ \rho_\epsilon|^2 \,+\, c_\beta \,|\nabla h_{x_0} \circ \rho_\epsilon|^2 \,|\tfrac{\partial}{\partial t} P_{\epsilon+\lambda}\, A|^2$$

$$+ \,|h_{x_0} \circ \rho_\epsilon|^2 \,|\nabla \tfrac{\partial}{\partial t} P_{\epsilon+\lambda} A|^2 \,+\, |\nabla g|^2 \,|\phi_t|^2$$

$$+ \,|g_t|^2 \,|\nabla \phi|^2 \,+\, |g - m_{Q'} g|^2 \,|\nabla \phi_t|^2.$$

The contribution of the first term on the righthand side of the integral defining $J$ can be handled using (7.15)$(ii)$. The second term is treated using Lemma 2.8 $(b)$ and (7.15)$(i)$ while the third is estimated using (2.7) and Lemma 2.8 $(a)$. The fourth and sixth terms have essentially already been considered in the estimate of $|\,\nabla^2\, k|^2$. Finally the fifth term can be handled in a way similar to the integrand in Lemma 7.14 $(ii)$. Doing this we get

$$J \,\le\, c_\beta \,\|\tilde{N} h\|_{L^2(\partial\Omega(\epsilon))}\,.$$

Putting this estimate in (7.16) we see that Lemma 7.14 $(iii)$ is true. The proof of Lemma 7.14 is now complete. □

Armed with Lemma 7.14 we are now ready to prove (7.11). To do this we note from (0.17) that it suffices to show

$$\|D^t_{1/2} g|_{I\!R^n}\|_2 < \infty \tag{7.17}$$

since $\tilde{N}_{*,\epsilon}(|\nabla h|) \in L^2(\partial\Omega(\epsilon))$. Here we have identified $I\!R^n$ with $\partial I\!R^{n+1}_+$. Moreover to avoid decay problems we shall first prove (7.11) with $g$ replaced by $k$, where $k$ is as above. Indeed, as in (5.12) we deduce

$$\begin{aligned}\|D^t_{1/2}k|_{\mathbb{R}^n}\|_2^2 \;&=\; -2\int_0^\infty\int_{\mathbb{R}^n}(D^t_{1/2}k)(D^t_{1/2}k_\lambda)\,dz\,d\lambda\\ &\le\; 2\left(\int_0^\infty\int_{\mathbb{R}^n}(D^t_{1/4}k_\lambda)^2\,dz\,d\lambda\right)^{1/2}\left(\int_0^\infty\int_{\mathbb{R}^n}(D^t_{3/4}k)^2\,dz\,d\lambda\right)^{1/2}\\ &= 2M_1\,M_2.\end{aligned} \tag{7.18}$$

Integrating by parts with respect to $\lambda$ we find

$$\begin{aligned}M_1^2 \;&=\; -2\int_0^\infty\int_{\mathbb{R}^n}D^t_{1/4}k_{\lambda\lambda}\;D^t_{1/4}k_\lambda\,dz\,\lambda d\lambda\\ &= -2\int_0^\infty\int_{\mathbb{R}^n}k_{\lambda\lambda}\;D^t_{1/2}k_\lambda\,dz\,\lambda d\lambda\\ &\le 2\left(\int_0^\infty\int_{\mathbb{R}^n}k^2_{\lambda\lambda}\;dz\,\lambda d\lambda\right)^{1/2}\left(\int_0^\infty\int_{\mathbb{R}^n}(D^t_{1/2}k_\lambda)^2 dz\,\lambda d\lambda\right)^{1/2}\\ &\le c\,\|\tilde N(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))}\end{aligned} \tag{7.19}$$

where we have used Lemma 7.14 to get the last line. Likewise

$$\begin{aligned}M_2^2 \;&=\; -2\int_0^\infty\int_{\mathbb{R}^n}D^t_{3/4}k_\lambda\;D^t_{3/4}k dz\,\lambda d\lambda\\ &= -2\int_0^\infty\int_{\mathbb{R}^n}D^t_{1/2}k_\lambda\;D^t_{1}k dz\,\lambda d\lambda\\ &\le c\left(\int_0^\infty\int_{\mathbb{R}^n}(D_{1/2}k_\lambda)^2\;dz\,\lambda d\lambda\right)^{1/2}\left(\int_0^\infty\int_{\mathbb{R}^n}(k_t)^2\,dz\,\lambda d\lambda\right)^{1/2}\\ &\le c\,\|\tilde N(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))},\end{aligned} \tag{7.20}$$

thanks again to Lemma 7.14. Using (7.19), (7.20) in (7.18) we conclude that (7.17) is true for $k$.

To prove (7.17) for $g$ we note that the norm squared in (7.17) is $\approx$

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{(g(x,s)-g(x,t))^2}{(s-t)^2}\,dx\,ds\,dt\,.$$

Now from (7.17) for $k$ and the fact that $k\equiv g-c$ on $(-R,R)\times B_R(0,0)$ we see that

$$\begin{aligned}\int_{-R}^R\int_{-R}^R\int_{\mathbb{R}^{n-1}}\frac{(g(x,s)-g(x,t))^2}{(s-t)^2}\,dx\,ds\,dt \;&\le\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{(k(x,s)-k(x,t))^2}{(s-t)^2}\,dx\,ds\,dt\\ &\le c\|D^t_{1/2}k|_{\mathbb{R}^n}\|_2^2 \;\le\; c\|\tilde N_*(|\nabla h|)\|^2_{L^2(\partial\Omega(\epsilon))}.\end{aligned}$$

Letting $R \to \infty$ we deduce from this inequality and the above note that (7.17) is true. Thus (7.11) is true and from our earlier remarks we conclude uniqueness in Theorem 1.14. The proof of uniqueness in Theorem 1.16 for small $\|I\!D_n A\|_*$ is now complete. $\square$

**8. Equivalence of Two Conditions.** In this section we shall prove (0.19) and (0.20) which in view of our previous work will establish Theorem 1.16 when $\|D^t_{1/2} A\|_*$ is small. To this end suppose that

$$|A(x,t) - A(y,t)| \;\le\; a_1\,|x-y| \quad \text{for } x, y \in I\!R^{n-1},\, t \in I\!R, \tag{8.1}$$

Our smoothness conditions in time are

$$\|I\!D_n A\|_* \;\le\; a_2 < \infty\,, \tag{8.2}$$

$$\|D^t_{1/2} A\|_* \;\le\; a_3 < \infty\,. \tag{8.3}$$

We prove

**Theorem 8.4** *Let $A : I\!R^n \to I\!R$ satisfy (8.1). Then conditions (8.2)-(8.3) are equivalent in the large in the sense that one condition implies the other for some choice of $a_i$, $i = 2, 3$. These conditions are also equivalent in the small in the sense that for fixed $a$ and given $\epsilon > 0$, there exists $\delta_i, i = 2, 3$, such that if $a_i \le \delta_i$ in condition $(i)$, then condition $(j)$ holds with $a_j \le \epsilon$, whenever $2 \le j \le 3$.*

**Proof:** To avoid decay considerations on the Fourier transform side we assume as we may that $A \in C_0^\infty(I\!R^n)$ and $A(0) = 0$. Indeed, (8.2)-(8.3) are unchanged if we replace $A$ by $A - A(0)$ in these inequalities. Convoluting $A - A(0)$ with an approximate identity and then multiplying the resulting convolution by suitable cutoff functions, we obtain a sequence of $C_0^\infty(I\!\!R^n)$ functions which converge to $A - A(0)$ uniformly on compact subsets of $I\!\!R^n$. Applying the estimates which follow to this sequence and then taking a limit, we get Theorem 8.4 for $A$. We continue under the above assumption. By definition we have

$$(D_n A)\hat{}\,(z,\tau) = \hat{A}\,\left(m(z,\tau)\,|\tau|^{1/2}\right),\ (z,\tau) \in I\!\!R^n,$$

where $\hat{}$, as in section 1, denotes the Fourier transform on $I\!\!R^n$ and

$$m(z,\tau) = \frac{\tau}{|\tau|^{1/2}\,||(z,\tau)||}\,.$$

We note that $m$ is not smooth enough to apply standard multiplier theorems (see [S1, Thm 3, p 96]). To overcome this difficulty let $\phi \in C_0^\infty(I\!\!R)$ be an even function with $\phi = 1$ on $(2/K, 3/2)$, and with support in $(1/K, 2), (-2, -1/K)$, where $K \geq 2$ is a large constant to be chosen later. We also choose $\phi$ so that $||\frac{\partial^l}{\partial x}\phi||_\infty \leq\ 100K^l$, $0 \leq l \leq n+4$. We write

$$(D_n A)\hat{}\,(z,\tau) = \hat{A}\left(m^+(z,\tau)\,|\tau|^{1/2} + \tfrac{|z|^2}{||(z,\tau)||}\,m^{++}(z,\tau)\right),$$

where

$$m^+(z,\tau) = m(z,\tau)\phi\left(\tfrac{\tau}{||(z,\tau)||^2}\right),$$

and

$$m^{++}(z,\tau) = \tfrac{|\tau|^{1/2}\,m(z,\tau)\,||(z,\tau)||}{|z|^2}\,(1-\phi)\left(\tfrac{\tau}{||(z,\tau)||^2}\right).$$

Put $m_j^{++}(z,\tau) = \frac{z_j}{||(z,\tau)||}\,m^{++}(z,\tau)$ for $1 \leq j \leq n-1$. We shall show after (8.10) that there exist kernels $L_j^{++}$ corresponding to $m_j^{++}$ and $L^+$ coresponding to $m^+$ so that

$$D_n\,A = cL^+ * D_{1/2}^t A\ +\ c\sum_{j=1}^{n-1} L_j^{++} * A_{z_j} \tag{8.5}$$

where the convolution is interpreted in the principal value sense.

To go the other way we note that

$$(D^t_{1/2}A)\hat{} = c\hat{A}\left(\tfrac{\tau}{||(z,\tau)||}\, m_1(z,\tau) + \tfrac{|z|^2}{||(z,\tau)||} m_2(z,\tau)\right)$$

where

$$m_1(z,\tau) = \operatorname{sgn}\tau \,\tfrac{||(z,\tau)||}{\tau\,|\tau|^{1/2}}\, \phi\left(\tfrac{\tau}{||(z,\tau)||^2}\right),$$

and

$$m_2(z,\tau) = \tfrac{|\tau|^{1/2}\,||(z,\tau)||}{|z|^2}\,(1-\phi)\left(\tfrac{\tau}{||(z,\tau)||^2}\right).$$

Put $m_{2,j} = \frac{z_j}{||(z,\tau)||}\, m_2(z,\tau)$ for $1 \le j \le n-1$. Again we shall show after (8.10) the existence of kernels $L_1, L_{2,j}$, corresponding to these multipliers such that

$$D^t_{1/2}\, A = cL_1 * D_n A \, + \, c\sum_{j=1}^{n-1} L_{2,j} * A_{z_j} \tag{8.6}$$

where all integrals are principal values. We note that $m_1$ and $m^+$ are infinitely differentiable away from the origin and parabolically homogeneous of degree 0. From this note we see that we can repeat the argument in [S2, ch 4, Prop 2] with $|\cdot|$ replaced by $\|\cdot\|$ to conclude that $L_1, L^+$ exist, have average zero on spheres about the origin, are parabolically homogeneous of degree $n+1$, and satisfy (2.10) with $R_j$ replaced by $L_1, L^+$. Using these facts it follows from a well known argument, often called " Peetre's lemma, " that each operator also maps BMO boundedly into BMO. In fact one can track down the constants in the proposition mentioned above. A crude estimate shows these constants depend only on the $L^\infty$ norm of the first $n+4$ partials of the multiplier. From this observation it is easily seen that either of the convolution operators corresponding to $m^+, m_1$, map BMO into BMO with norm $\le cK^{n+4}$. Hence

$$\begin{aligned} \|L_1 * D_n A\|_* &\le cK^{n+4}\,\|D_n A\|_*, \\ \|L^+ * D^t_{1/2}A\|_* &\le cK^{n+4}\,\|D^t_{1/2}A\|_*. \end{aligned} \tag{8.7}$$

We could do better but this is all we need. Let $H$ be any one of the alleged kernels, $L_j^{++}, L_{2,j}$, for $1 \leq j \leq n-1$. We shall prove that if $f \in L^\infty(I\!\!R^n)$, then $H$ exists and

$$\|Hf\|_* \leq cK^{-7/16} \|f\|_\infty. \tag{8.8}$$

From (8.5) - (8.8) with $K = 4$ we see that

$$\begin{aligned}
&(-) \quad ||D_n A||_* \leq c\,||D^t_{1/2} A||_* + c\,||\nabla_x A||_\infty, \\
&(--) \quad ||D^t_{1/2} A||_* \leq c\,||D_n A||_* + c\,||\nabla_x A||_\infty.
\end{aligned} \tag{8.9}$$

(8.8), (8.9) clearly imply conditions (8.2) and (8.3) are equivalent in the large in the sense of Theorem 8.4. To show (8.2) and (8.3) are equivalent in the small for given $a, \epsilon$, we first define $K$ by $(a+1)K^{-7/16} = \epsilon/c_1$ where $c_1 \geq 2$. Second we put $\delta_i = \epsilon\, K^{-(n+5)}$, for $i = 2, \dots, n-1$. If $c_1$ is chosen large enough (how large depends only on n), then from (8.5) - (8.8) we deduce that conditions (8.2), (8.3) are equivalent in the small.

To prove (8.8) let $\tilde{m}$ denote the multiplier corresponding to $H$. We note that if $\alpha$ is a nonegative integer and $\beta = (\beta_1, \dots, \beta_{n-1})$ is a multi index, then

$$\begin{aligned}
&(a) \quad |\tfrac{\partial}{\partial \tau^\alpha} \tfrac{\partial}{\partial z^\beta}\, \tilde{m}(z,\tau)| \leq c|\tau|^{\frac{1}{2}-\alpha} \,\|(z,\tau)\|^{-(|\beta|+1)} \text{ for } 0 \leq \alpha + |\beta| \leq n+4, \\
&(b) \quad |\tfrac{\partial}{\partial \tau^\alpha} \tfrac{\partial}{\partial z^\beta}\, (z_i\tilde{m})(z,\tau) \leq c|\tau|^{\frac{1}{2}-\alpha}\|(z,\tau)\|^{-|\beta|} \text{ for } 0 \leq \alpha + |\beta| \leq n+4, 1 \leq i \leq n-1, \\
&(c) \quad |\tfrac{\partial}{\partial \tau^\alpha} \tfrac{\partial}{\partial z^\beta}\, (\tau\tilde{m})(z,\tau)| \leq c|\tau|^{\frac{3}{2}-\alpha}\|(z,\tau)\|^{-(|\beta|+1)} \text{ for } 0 \leq \alpha + |\beta| \leq n+4\,.
\end{aligned} \tag{8.10}$$

Also we observe that the support of $\tilde{m}$ is contained in

$$\left\{(z,\tau) : 0 \leq |\tau| \leq \frac{2\|(z,\tau)\|^2}{K}\right\} \tag{ 8.11 }$$

Using (8.10), (8.11), we shall show that if $(z,\tau) \in I\!\!R^n$, then $H$ exists and

$$|H(z,\tau)| \leq c \min\left\{K^{-3/2}|z|^{-(n+1)}, K^{-7/16}|\tau|^{-\frac{17}{16}}|z|^{-n+\frac{9}{8}}\right\}. \tag{ 8.12 }$$

$$|\nabla_z H(z,\tau)| \le c \min\left\{K^{-3/2}|z|^{-(n+2)}, K^{-7/16}|\tau|^{-\frac{17}{16}}|z|^{-n+\frac{1}{8}}\right\}. \tag{8.13}$$

$$|\tfrac{\partial}{\partial\tau} H(z,\tau)| \le c \min\left\{K^{-5/2}|z|^{-(n+3)}, K^{-23/16}|\tau|^{-\frac{17}{16}}|z|^{-(n+\frac{7}{8})}\right\}. \tag{8.14}$$

We claim that (8.12)-(8.14) along with $\|\hat{H}\|_\infty \le cK^{-1/2}$, are enough to imply that the principal value convolution operator corresponding to $H$ exists, and maps $L^\infty$ into BMO with norm $\le cK^{-7/16}$. Existence follows as in [S1, ch 2, sec 3] with $|\cdot|$ replaced by $\|\cdot\|$. To show boundedness suppose $x$ is a point in $\mathbb{R}^n$ and $dx$ Lebesgue measure on $\mathbb{R}^n$. Then from the above note and " Peetre's argument, " we see that it suffices to show

$$\int_{\{\|x\|\ge 1\}} |H(x-y) - H(x)|dx \le cK^{-7/16} \tag{8.15}$$

whenever $\|y\| \le 1/2$ in order to obtain the above norm estimate on $H$ as an operator from $L^\infty$ to BMO. Now,

$$\int_{\{\|x\|\ge 1\}} |H(x-y) - H(x)|dx \le |\int_{E_1} \dots| + |\int_{E_2} \dots| + |\int_{E_3} \dots|$$

where

$$E_1 = \{(z,\tau) : |z| \ge 1, |\tau| \le 1\}$$

$$E_2 = \{(z,\tau) : |z| \le 1, |\tau| \ge 1\}$$

$$E_3 = \{(z,\tau) : |z| \ge 1, |\tau| \ge 1\}.$$

Using (8.12) it is easily seen that

$$\int_{E_1+E_2} |H(x-y) - H(x)|dx \le cK^{-7/16}.$$

Moreover, using (8.13)-(8.14) and splitting the range of integration into $\{(z,\tau) : \tau \le |z|^2\}$ and $\{(z,\tau) : \tau > |z|^2\}$ it follows that

$$\int_{E_3} |H(x-y) - H(x)|dx \le cK^{-7/16}.$$

Hence (8.15) is a consequence of (8.12) - (8.14). Note that we would have rather proved inequalities like

$$|H(z,\tau)| \le K^{-\gamma} \|(z,\tau)\|^{-d}$$

for some $\gamma > 0$ but the multiplier is not smooth enough in $\tau$ to do this .

To prove (8.12)-(8.14) we write $\tilde{m} = \sum \tilde{m}_i$, where $\tilde{m}_i(z,\tau) = \tilde{m}(z,\tau)\, g_i(\|(z,\tau)\|)$ and $\{g_i\}$ is a partition of unity for $(0,\infty)$, with $g_i \equiv 1$ on $(2^{-i}, 2^{1-i})$ while supp $g_i \subset (2^{-i-1}, 2^{2-i})$ for $i = 0, \pm 1, \pm 2, \ldots$ . Let $\{H_i\}$ be the kernels corresponding to $\{\tilde{m}_i\}$. From (8.11) we deduce that the support of $\tilde{m}_i$ is contained in

$$\left\{(z,\tau) : 0 \le \frac{\tau}{\|(z,\tau)\|^2} \le 2/K, \text{ and } 2^{-i-1} \le \|(z,\tau)\| \le 2^{2-i}\right\}. \tag{8.16}$$

Using (8.16), (8.10), and the observation that the Fourier - inverse Fourier transforms turn derivatives into multiplication by powers we shall show that

$$|H_i(z,\tau)| \le c2^{-(n+1)i} K^{-3/2} \min\{1,\, 2^{i(n+2)}|z|^{-(n+2)}, 2^{i(n+3/2)}\, K^{5/4}|z|^{1-n}\, |\tau|^{-5/4}\}, \tag{8.17}$$

$$|\nabla_z H_i(z,\tau)| \le c2^{-(n+2)i} K^{-3/2} \min\{1,\, 2^{i(n+3)}|z|^{-(n+3)}, 2^{i(n+5/2)}\, K^{5/4}|z|^{-n}\, |\tau|^{-5/4)}\}, \tag{8.18}$$

$$|\tfrac{\partial}{\partial\tau} H_i(z,\tau)| \le c2^{-(n+3))i} K^{-5/2} \min\{1,\, 2^{i(n+4)}|z|^{-(n+4)}, 2^{i(n+7/2)}\, K^{5/4}|z|^{-(n+1)}|\tau|^{-5/4}\}. \tag{8.19}$$

We prove only (8.17) as the proofs of (8.18), (8.19) are essentially the same. From (8.10)$(a)$ with $\alpha = \beta = 0$ and (8.16) we see that

$$\| H_i \|_\infty \le c\|\hat{H}_i\|_1 \le c2^{-(n+1)i}\, K^{-3/2}.$$

Likewise from (8.13)$(a)$ with $\alpha = 0$, $|\beta| = n+2$, and (8.16) we deduce

$$|z|^{(n+2)}\,|H_i(z,\tau)| \;\le\; c\,\|\sum_{|\beta|=n+2} |\tfrac{\partial}{\partial z^\beta}\,\hat{H}_i\,|\,\|_1 \le c\,2^i\,K^{-3/2}.$$

To obtain the last estimate in (8.17) we note that

$$|t|^{5/4}|z|^{n-1}|H_i(z,\tau)| \;\le\; c\,\|\sum_{|\beta|=n-1} |D^t_{1/4}[\,\tfrac{\partial}{\partial\tau}\,\tfrac{\partial}{\partial z^\beta}\,\hat{H}_i\,]\,|\,\|_1 \tag{8.20}$$

For fixed $i,\beta$ let $\sigma = \frac{\partial}{\partial\tau}\,\frac{\partial}{\partial z^\beta}\,\hat{H}_i$. From (8.13) $(a)$, with $\alpha = 1,2$, $|\beta| = n-1$, and simple estimates for one quarter derivatives, we find that

$$|D^t_{1/4}\,\sigma|(z,\tau) \le c\,2^{ni}\,|\tau|^{-3/4} \text{ for } 0 \;\le |\tau| \;\le\; 8K^{-1}\,\|(z,\tau)\|^2 \;\le\; K^{-1}\,2^{8-2i}\,,$$

$$|D^t_{1/4}\sigma|(z,\tau) \le c\,2^{(n-1)i}K^{-1/2}\,|\tau|^{-5/4} \text{ for } |\tau| >\; 4K^{-1}\,\|(z,\tau)\|^2\,,$$

$$|D^t_{1/4}\sigma|(z,\tau) \equiv 0 \text{ if } |z| \;\ge \hat{c}\,2^{-i} \text{ and } \hat{c} \text{ is large enough.}$$

Using the above inequalies in (8.20) we obtain

$$|t|^{5/4}|z|^{(n-1)}\,|H_i(z,\tau)| \;\le\; cK^{-1/4}\,2^{i/2}.$$

Thus (8.17) is true. Let $H = \sum H_i$ whenever the sum converges absolutely.

To prove (8.12) we first assume $|\tau| \le K|z|^2$ and sum $|H_i|$. Using (8.17) we get

$$\begin{aligned}|H(z,\tau)| &\le \sum |H_i(z,\tau)|\\ &\le \sum_{\{i:|z|\le 2^i\}} c2^{-(n+1)i}\,K^{-3/2}\\ &+ \sum_{\{i:|z|>2^i\}} cK^{-3/2}|z|^{-(n+2)}\,2^i\\ &\le cK^{-3/2}|z|^{-(n+1)}.\end{aligned} \tag{8.21}$$

From (8.21) we see that (8.12) is valid when $|\tau| \le K|z|^2$. If $K|z|^2 < |\tau|$ and $\lambda > 0$, we again use

(8.17) to get

$$
\begin{aligned}
|H(z,\tau)| &\le \sum |H_i(z,\tau)| \\
&\le \sum_{\{i:2^{-i}\le\lambda\}} c2^{-(n+1)i}K^{-3/2} \\
&+ cK^{-1/4}|\tau|^{-5/4}\,|z|^{1-n} \sum_{\{i:2^{-i}>\lambda\}} 2^{\frac{i}{2}} \\
&\le cK^{-3/2}\,\lambda^{(n+1)} + cK^{-1/4}|\tau|^{-5/4}\,|z|^{1-n}\,\lambda^{-1/2}\,.
\end{aligned}
\tag{8.22}
$$

Clearly (8.21), (8.22), imply that $H$ exists for every $(z,\tau)\neq(0,0)$. The righthand side of (8.22) is minimized when $\lambda$ is a constant multiple of

$$\left(|\tau|^{-5/4}\,|z|^{1-n}K^{5/4}\right)^{\frac{1}{n+3/2}}\ .$$

Putting this value of $\lambda$ in (8.22) we conclude that

$$|H(z,\tau)| \le cK^{-3/2}\left(|\tau|^{-5/4}\,|z|^{1-n}K^{5/4}\right)^{\frac{n+1}{n+3/2}}\ . \tag{8.23}$$

Now since $K|z|^2 < |\tau|$, and $\frac{5(n+1)}{4(n+3/2)} > 17/16$ we have

$$K^{-3/2}\left(|\tau|^{-5/4}\,|z|^{1-n}K^{5/4}\right)^{\frac{n+1}{n+3/2}} \le K^{-7/16}\,|\tau|^{-17/16}\,|z|^{-n+9/8}.$$

Using this inequality in (8.23) we conclude that (8.12) is also valid when $K|z|^2<|\tau|$. Hence (8.12) holds. (8.13) and (8.14) are proved similarly using (8.18) and (8.19). Thus (8.8) is true. From our earlier remarks we now get Theorem 8.4. $\square$

Note that in the proof of the equivalences, it is rather important that $|\nabla_x A|\in L^\infty$ rather than BMO. Indeed we have shown that all our operators map $L^\infty\to BMO$, but it does not appear obvious that $L_j^{++}$, $L_{2,j}, 1\le j\le n-1$, map $BMO{\to}BMO$ (even though each of these operators maps 1 to 0), essentially because of the lack of smoothness of each multiplier in $\tau$.

**9. Examples.** In this section we prove Theorem 1.17 and Corollaries 1.18, 1.19. As mentioned in section 1, these results show that the work of [LM, ch 3] and Theorem 1.16 of the present paper are sharp. We shall prove our results by an iterative procedure in the spirit of Tom Wolff [W]. To do so we shall need a " rate " theorem of Fabes, Garofalo, and Salsa [FGS, Theorem 3]. In order to state this lemma let $z_0 = (x_0, t_0) \in \mathbb{R}^n$ when $n \geq 2$, and $z = t_0$ when $n = 1$. We ask the reader to please excuse our small change in notation ($x_0$ now denotes a fixed point in $\mathbb{R}^{n-1}$ rather than the first coordinate of a point in $\mathbb{R}^n$). Next suppose $f : \bar{B}_\rho(z_0) \to \mathbb{R}$ satisfies (0.10) with $c\beta = b$. That is,

$$|f(z) - f(v)| \leq b\,\|z - v\| \tag{9.1}$$

whenever $z, v \in \bar{B}_\rho(z_0)$. Set

$$\Omega(B_\rho(z_0)) = \{(\lambda, z) \in \mathbb{R}^{n+1} : z \in B_\rho(z_0) \text{ and } f(z) < \lambda < f(z_0) + 4b\rho\}.$$

With this notation we state the following lemma of Fabes, Garofalo, and Salsa.

**Lemma 9.2** *Let $f$ satisfy (9.1) and suppose $u, v$ are positive solutions to the adjoint heat equation in $\Omega(B_\rho(z_0))$ that are continuous on the closure of this domain with*

$$u(f(z), z) = v(f(z), z) = 0, \quad \textit{whenever } z \in \bar{B}_\rho(z_0).$$

*If $\lambda_0 = f(z_0) + 2b\rho$, $z_1 = (x_0, t_0 - [1/4 + \gamma]\rho^2)$, and $z_2 = (x_0, t_0 + [1/4 + \gamma]\rho^2)$, then*

$$\sup_{(\lambda,z)\in\Omega(B_{\rho/2}(z_0))} \frac{u(\lambda, z)}{v(\lambda, z)} \leq c(b, \gamma)\,\frac{u(\lambda_0, z_1)}{v(\lambda_0, z_2)}\ .$$

We note that FGS proved the above lemma for the heat rather than adjoint heat equation and with $z_1, z_2$ interchanged. However, the above lemma follows from their work using the transformation $t \to -t$.

**Proof of Theorem 1.17 :** To begin we first study a canonical example. Put $A_0(t) = -m\,|t|$ when $t \in I\!R$ and set

$$D_0 = \{(x,t) \in I\!R^2 : A_0(t) < x < 2,\ -4 < t < 4\}$$

for $m \geq 1000$. Let $G_0$ be the Green's function for the adjoint heat equation in $D_0$ with pole at (1, 1). We shall need the following lemma.

**Lemma 9.3** *Given $p$, $1 < p < \infty$, there exists $c_1 = c_1(p)$, such that*

$$c_1 \int_{1/m}^{10/m} \left| \frac{\partial}{\partial x} G_0 \right|^p (A_0(s), s)\, ds \,\geq\, m^{p-1}\,.$$

**Proof:** Let $\omega$ be the bounded caloric function in $D_0$ with boundary values 1 on $\{(-mt, t) : 1/m \leq t \leq 10/m\}$ and 0 on the rest of the parabolic boundary of $D_0$ in the sense of Perron, Wiener, and Brelot. As in [LS] it can be shown that

$$\omega(1,1) = \int_{1/m}^{10/m} \frac{\partial}{\partial x} G_0(A_0(s), s)\, ds. \tag{9.4}$$

Let $\hat\omega$ be the bounded parabolic function in the rectangle, $\Lambda = \{(x,t) : -2 < x < 2,\ 1/m < t < 4\}$, with boundary values, $\hat\omega = 1$ on $(-2,-1) \times \{1/m\}$ and $\hat\omega = 0$ on the rest of the parabolic boundary of $\Lambda$. Using the maximum principle for the heat equation in $\Lambda$ and comparing boundary values of $\omega, \hat\omega$, in $\Lambda \cap D_0$ we see that

$$\omega(1,1) \,\geq\, \hat\omega(1,1) \geq c^{-1}$$

where $c \geq 2$ is an absolute constant independent of $m$ . Putting this inequality in (9.4) and using Hölder's inequality, we deduce that

$$c^{-p} \leq \omega(1,1)^p = \left( \int_{1/m}^{10/m} \frac{\partial}{\partial x} G_0(A_0(s), s)\, ds \right)^p \leq (10/m)^{p-1} \int_{1/m}^{10/m} \left| \frac{\partial}{\partial x} G_0 \right|^p (A_0(s), s)\, ds \,.$$

Clearly the above inequality implies Lemma 9.3. □

In the proof of Theorem 1.17 we shall construct $(D_k)_0^\infty, (A_k)_0^\infty$, so that for each nonnegative integer $k$, we have $D_k \subset D_{k+1}$, $A_{k+1} \leq A_k$, $D = \bigcup_{k=0}^\infty D_k$, and $A(t) = \lim_{k \to \infty} A_k(t)$, $t \in \mathbb{R}$. For this purpose we divide the interval $[4/m^2, 1/2]$ into closed intervals with disjoint interiors and of equal length, $2r_0 = \frac{1}{2m^2}$. Put $\mathcal{F}_0 = \{[\frac{4}{m^2}, \frac{1}{2}]\}$ and let $\mathcal{F}_0^*$ be the collection of all intervals in the above subdivision, together with $[\frac{14}{4m^2}, \frac{4}{m^2}]$. If $I = I_{r_0}(t_0) = \{t : |t - t_0| \leq r_0\} \in \mathcal{F}_0^*$, we define the piecewise linear function $A_1$ on $I$ by

$$A_1(t) = A_0(t_0) - (4/\sqrt{r_0})\, m\, |t - t_0| \text{ for } |t - t_0| \leq r_0/4,$$

while $A_1$ is linear on $[t_0 + r_0/4,\, t_0 + r_0/2]$, $[t_0 - r_0/2,\, t_0 - r_0/4]$, with $A_1(t_0 \pm r_0/2) = A_0(t_0 \pm r_0/2)$. Set $A_1 = A_0$ for $r_0/2 \leq |t - t_0| \leq r_0$. We make this definition for each interval in $\mathcal{F}_0^*$. Put $A_1 = A_0$ on $\mathbb{R} \setminus \left( \bigcup_{I \in \mathcal{F}_0^*} I \right)$. Clearly, $A_1 \leq A_0$. With $A_1$ now defined set

$$D_1 = \{(x,t) : A_1(t) < x < 2,\ -4 < t < 4\}$$

and let $G_1$ be the Green's functions for the adjoint heat equation with pole at ( 1, 1 ) in $D_1$ . Next define $\mathcal{F}_1$ by

$$\mathcal{F}_1 = \{\hat{I} = [t_0 + \tfrac{r_0}{8m^2}, t_0 + r_0/16] : I = I_{r_0}(t_0) \in \mathcal{F}_0^*\}.$$

We claim for some $c_2 = c_2(p)$, that

$$c_2 \int_{\hat{I}} \left| \frac{\partial}{\partial x} G_1 \right|^p (A_1(s), s)\, ds \geq r_0^{1-\frac{p}{2}}\, m^{p-1}\, G_0(x_1(I), t_1(I))^p \tag{9.5}$$

where $x_1(I) = A_0(t_0) + \frac{\sqrt{r_0}}{\sqrt{2}}$, $t_1(I) = t_0 + \frac{r_0}{2}$ and $I, \hat{I}$, are as in the definition of $\mathcal{F}_1$. To prove claim (9.5) define $\Omega(I)$ as in (9.1) by

$$\Omega(I) = \{(x,t) : A_1(t) < x < A_1(t_0) + r_0^{1/2}/2, \text{ and } |t - t_0| < r_0/4\}.$$

We can translate $\Omega(I)$ by $(-A_0(t_0), -t_0)$ and then scale by $4r_0^{-1/2}$ in the $x$ direction, $16/r_0$ in the $t$ direction to get $D_0$ in Lemma 9.3.

Using dilation invariance of the heat equation, it follows that if $\tilde{G}$ denotes the Green's function for the adjoint heat equation in $\Omega(I)$ with pole at $(x_2, t_2) = (A(t_0) + r_0^{1/2}/4, t_0 + r_0/16)$, then

$$\tilde{G}(\tfrac{\sqrt{r_0}}{4}x + A_0(t_0),\, t_0 + \tfrac{r_0\, t}{16}) = (16/r_0)^{1/2}\, G_0(x,t),\ (x,t) \in D_0. \tag{9.6}$$

We claim for some $c_3 > 0$ that

$$c_3\, G_1(x,t) \ \geq\ r_0^{1/2}\, G_1(x_1,t_1)\, \tilde{G}(x,t) \tag{9.7}$$

in $\Omega(I) \setminus Q$, where

$$Q = \{(x,t) : |x - x_2| < \sqrt{r_0}/25,\ |t - t_2| < r_0/625\}.$$

To prove this claim observe from Harnack's inequality for the adjoint heat equation that for some $c \geq 2$, we have $c\,G_1 \geq G_1(x_1,t_1)$ on $\partial Q$ while $G_1 \geq 0 = \tilde{G}$ on the parabolic boundary of $\Omega(I)$. Since $\tilde{G} \leq c r_0^{-1/2}$ on $\partial Q$ we conclude from the maximum principle for the adjoint heat equation applied to $r_0^{1/2}\, G_1(x_1,t_1)\, \tilde{G} - cG_1$ in $\Omega(I) \setminus Q$ that claim (9.7) is true. Using (9.7), the Hopf boundary maximum principle, (9.6), and Lemma 9.3 we deduce

$$\begin{aligned}
\int_{\hat{I}} \left| \tfrac{\partial}{\partial x} G_1 \right|^p (A_1(s),s)\, ds &\geq c^{-1}\, r_0^{p/2}\, G_1(x_1,t_1)^p \int_{\hat{I}} \left| \tfrac{\partial}{\partial x}\tilde{G} \right|^p (A_1(s),s)\, ds \\
&\geq c^{-1}\, r_0^{1-p/2}\, G_1(x_1,t_1)^p \int_{1/m}^{10/m} \left| \tfrac{\partial}{\partial x} G_0 \right|^p (A_0(s),s)\, ds \\
&\geq c^{-1} r_0^{1-p/2}\, m^{p-1}\, G_1(x_1,t_1)^p \\
&\geq c^{-1} r_0^{1-p/2}\, m^{p-1}\, G_0(x_1,t_1)^p.
\end{aligned} \tag{9.8}$$

In the last line of (9.8) we have used the fact that $G_0 \le G_1$ in $D_0$. Thus claim (9.5) is true.

Next we use Lemma 9.2 to show that

$$c \sum_{I \in \mathcal{F}_0^*} r_0^{1-p/2}\, G_0(x_1,t_1)^p \;\ge\; \sum_{I \in \mathcal{F}_0} \int_I \left| \tfrac{\partial}{\partial x} G_0 \right|^p (A_0(s),s)\, ds\,. \tag{9.9}$$

We note that $\mathcal{F}_0$ consists only of $[4/m^2, 1/2]$ so we would not have needed to write the sum on the righthand side of (9.9). However we have purposely written (9.9) this way since we plan to replace $\mathcal{F}_0$, $\mathcal{F}_0^*$ by $\mathcal{F}_k$, $\mathcal{F}_k^*$, in future iterations. To prove (9.9) we note that $A_0$ is Hölder 1/2 with norm $\le 1$ on a scale of $1/m^2$. That is,

$$|\, A_0(t) - A_0(s) \,| \;\le\; m\,|s-t| \;\le\; |s-t|^{1/2}$$

when $|s-t| \le 1/m^2$. Let $I = I_{r_0}(t_0) \in \mathcal{F}_0^*$ with $I \subset [4/m^2, 1/2]$. Set $I' = I_{r_0}(t_0 - 2r_0)$ and define $x_1(I'), t_1(I')$ as above relative to $I'$. We note that $I' \in \mathcal{F}_0^*$. Let $I^* = I_{4r_0}(t_0)$ and put $\Omega^* = \{(x,t) : A_0(t) < x < A_0(t_0) + 16\sqrt{r_0}, t \in I^* \}$. Next put $u$ equal to the restriction of $G_0$ to $\Omega^*$ and set

$$v(x,t) \;=\; (-t+4)^{-3/2}\,(t \,+\, x/m\,)\, \exp\left[ - \frac{(x+4m)^2}{4(-t+4)} \right],\;\; (x,t) \in \Omega^*.$$

Clearly, $v$ is a solution to the adjoint heat equation in $\Omega^*$ and $v(A_0(t),t) = 0$, when $t \in I^*$. Moreover, it is easily checked that

$$\exp\left[ - \frac{(-t_0+4)^2\, m^2}{4(-t_0+4)} \right] \approx m\, \tfrac{\partial}{\partial x}\, v(A_0(t),t) \;\approx\; m^2\, v(x^*,t^*)$$

when $t \in I^*$, where $(x^*,t^*) = \;(A(t_0) + 100 r_0^{1/2}, t_0 + 8r_0)$. From this note, the fact that $A_0$ is Hölder 1/2 with norm $\le 1$ on a scale of $1/m^2$, Lemma 9.2 with $u, v$ as above ($I^*$ replacing $B_\rho(x_0,t_0)$), and the Hopf boundary maximum principle we find that for $t \in I$,

$$\begin{aligned} \left| \tfrac{\partial}{\partial x} G_0 \right| (A_0(t),t) \;\;&\le\; c\, \left| \tfrac{\partial}{\partial x} v \right| (A_0(t),t)\;\; \tfrac{G_0(x_1(I'),t_1(I'))}{v(x^*,t^*)} \\ &\le\; cm\, G_0(x_1(I'),t_1(I')). \end{aligned} \tag{9.10}$$

Raising both sides of this inequality to the $p$ th power, integrating and summing over $I$, we obtain (9.9) after some juggling.

From (9.8), (9.9), we find first for some $c_4 \geq 2$ that

$$\sum_{I \in \mathcal{F}_1} \int_I \left| \tfrac{\partial}{\partial x} G_1 \right|^p (A_1(s), s) ds \;\geq\; c_4^{-1} \, m^{p-1} \sum_{I \in \mathcal{F}_0} \int_I \left| \tfrac{\partial}{\partial x} G_0 \right|^p (A_0(s), s) ds \tag{9.11}$$

while (9.11) and another application of Lemma 9.3 yield for $c_4$ sufficiently large that

$$\sum_{I \in \mathcal{F}_1} \int_I \left| \tfrac{\partial}{\partial x} G_1 \right|^p (A_1(s), s) \, ds \;\geq\; \left[ c_4^{-1} \, m^{p-1} \right]^2 . \tag{9.12}$$

We note that if $m$ is large then (9.11), (9.12), imply that the spikes added to $D_0$ to get $D_1$ can be used to significantly increase the integral of interest.

Proceeding by induction, suppose that $A_k, k \geq 1$, has been defined with $A_{j-1} \leq A_j$ for $j = 1, \ldots, k$ , as well as corresponding domains

$$D_j = \{(x, t) : A_j(t) < x < 2, \, -4 < t < 4\}$$

and adjoint Green's functions $(G_j)$ with pole at ( 1, 1). Suppose also that families of intervals $\mathcal{F}_j, 1 \leq j \leq k$, and $\mathcal{F}_j^*$, $1 \leq j \leq k-1$ have been defined where the intervals in $\mathcal{F}_j^*$ have length $2r_j$ for $0 \leq j \leq k-1$ . Moreover, for $0 \leq j \leq k-1$

$$\mathcal{F}_{j+1} \;=\; \{[t_0 + \frac{r_j}{8m^2}, \, t_0 + \frac{r_j}{16}] : I_{r_j}(t_0) \in \, \mathcal{F}_j^* \, \}.$$

We assume that if $B_j = A_j - A_{j-1}, 1 \leq j \leq k$, then

$$\begin{aligned} &(a) \qquad \operatorname{supp} B_j \subset \bigcup_{I \in \mathcal{F}_{j-1}^*} \tfrac{1}{2} I, \\ &(b) \qquad \|B_j'\|_\infty \;\leq\; 5m/\sqrt{r_{j-1}}, \\ &(c) \qquad \|B_j\|_\infty \;\leq\; c\, m \, \sqrt{r_{j-1}} \, . \end{aligned} \tag{9.13}$$

Here $\frac{1}{2}I$ denotes the interval with the same center as $I$ and $1/2$ the sidelength. If $k > 1$, we also assume that

$$r_j = \frac{r_{j-1}}{32m^2} \tag{9.14}$$

for $1 \le j \le k-1$. Since $A_k = A_0 + \sum_{j=1}^{k} B_j$, we see from (9.13), (9.14) that for $k > 1$

$$\|A_k'\|_\infty \le 10m/\sqrt{r_{k-1}}\,. \tag{9.15}$$

Clearly (9.15) also holds if $k = 1$. From (9.15) we observe that $A_k$ is Hölder $1/2$ with norm $\le 1$ on a scale of $\frac{r_{k-1}}{100m^2}$.

Next suppose that $I_{r_{k-1}}(t_0) \in \mathcal{F}^*_{k-1}$ so by definition, $[t_0 + \frac{r_{k-1}}{8m^2}, t_0 + \frac{r_{k-1}}{16}] \in \mathcal{F}_k$. We divide this interval into disjoint subintervals of length $2r_k = \frac{r_{k-1}}{16m^2}$. We do this for each interval in $\mathcal{F}_k$. Let $\mathcal{F}^*_k$ be the family of all such subintervals together with all intervals of the form $[\, t_0 + \frac{r_{k-1}}{16m^2}\,,\, t_0 + \frac{r_{k-1}}{8m^2}]$ where $I_{r_{k-1}}(t_0) \in \mathcal{F}^*_{k-1}$. To define $A_{k+1}$ we proceed as in the case $k = 0$. That is we replace the graph of $A_k$ over each $I \in \mathcal{F}^*_k$ by a certain sawtooth. In fact if $I_{r_k}(t_0) \in \mathcal{F}^*_k$ we define $A_{k+1}$ on $I_{r_k}(t_0)$ exactly as in the case $k = 0$ with $r_0, A_0$ replaced by $r_k, A_k$. Let $A_{k+1} = A_k$ on $\mathbb{R} \setminus \bigcup_{I \in \mathcal{F}^*_k} I$ and put

$$D_{k+1} = \{(x,t) : A_k(t) < x < 2,\ -4 < t < 4\}.$$

Let $G_{k+1}$ be the adjoint Green's function for $D_{k+1}$ with pole at (1,1) and set

$$\mathcal{F}_{k+1} = \{\hat{I} = [t_0 + \frac{r_k}{8m^2}\,,\, t_0 + \frac{r_k}{16}\,] : I_{r_k}(t_0) \in \mathcal{F}^*_k\,\}.$$

Proceeding as in the case $k = 0$, we deduce as in (9.5)-(9.8) that

$$\int_{\hat{I}} \left| \tfrac{\partial}{\partial x} G_{k+1} \right|^p (A_{k+1}(s), s)\, ds \ \ge\ c^{-1} r_k^{1-p/2}\, m^{p-1}\, G_k(x_1, t_1)^p \tag{9.16a}$$

where $I, \hat{I}$ are as in the definition of $\mathcal{F}_{k+1}$ and $x_1, t_1$ are defined relative to $I$ as in the case $k = 0$

with $r_0, A_0$, replaced by $r_k, A_k$ . Also using Lemma 9.2 as in the case $k=0$ we obtain

$$c \sum_{I \in \mathcal{F}_k^*} r_k^{1-p/2} G_k(x_1,t_1)^p \ \geq \sum_{I \in \mathcal{F}_k} \int_I \left| \frac{\partial}{\partial x} G_k \right|^p (A_k(s),s)\, ds \tag{9.16b}$$

which is just (9.9) with $G_0, \mathcal{F}_0, \mathcal{F}_0^*$ replaced by $G_k, \mathcal{F}_k, \mathcal{F}_k^*$. Using (9.16a), (9.16b), and Lemma 9.2 we obtain an inequality analogous to (9.11) :

$$\sum_{I \in \mathcal{F}_{k+1}} \int_I \left| \frac{\partial}{\partial x} G_{k+1} \right|^p (A_{k+1}(s),s)\, ds \ \geq \ c_4^{-1}\, m^{p-1} \sum_{I \in \mathcal{F}_k} \int_I \left| \frac{\partial}{\partial x} G_k \right|^p (A_k(s),s) ds\,. \tag{9.17}$$

Iterating this inequality and using (9.12) we get for $c_4$ large enough

$$\sum_{I \in \mathcal{F}_{k+1}} \int_I \left| \frac{\partial}{\partial x} G_{k+1} \right|^p (A_{k+1}(s),s) ds \ \geq \ \left[ c_4^{-1}\, m^{p-1} \right]^{k+2} . \tag{9.18}$$

By induction we get $(A_k)_0^\infty$, $(D_k)_0^\infty$, and $(G_k)_0^\infty$. Let $D = \bigcup D_k$ and let $G$ be Green's function for $D$ with pole at (1, 1). Since $D_k \subset D$ we see from the maximum principle for the adjoint heat equation that $G \geq G_k$. Let $A = A_0 + \sum_1^\infty B_k = \lim_{k \to \infty} A_k$. By construction we have $A = A_{k+1} = A_k$ on $I_{r_k}(t_0) \setminus I_{r_k/2}(t_0)$ whenever $I_{r_k}(t_0) \in \mathcal{F}_k^*$. Using this fact and the Hopf boundary maximum principle, it follows that $|\frac{\partial}{\partial x} G| \ \geq \ |\frac{\partial}{\partial x} G_{k+1}|$ on $I_{r_k}(t_0) \setminus I_{r_k/2}(t_0)$. This inequality, (9.18), and Lemma 9.2 imply for $m$ large enough,

$$\begin{aligned} \sum_{I_{r_k} \in \mathcal{F}_k^*} \int_{I_{r_k} \setminus I_{r_k/2}} \left| \frac{\partial}{\partial x} G \right|^p (A(s),s)\, ds \ &\geq \sum_{I_{r_k} \in \mathcal{F}_k^*} \int_{I_{r_k} \setminus I_{r_k/2}} \left| \frac{\partial}{\partial x} G_{k+1} \right|^p (A_{k+1}(s),s) ds \\ &\geq c^{-1} \sum_{I \in \mathcal{F}_k} \int_I \left| \frac{\partial}{\partial x} G_k \right|^p (A_k(s),s) ds \\ &\geq c^{-1} \left[ c_4^{-1}\, m^{p-1} \right]^{k+1} \ \geq \ 2^k\,. \end{aligned} \tag{9.19}$$

We note that the second line in (9.19) is proved using Lemma 9.2 as in the proof of (9.17). That is we first use Lemma 9.2 to estimate the integral involving $\frac{\partial G_{k+1}}{\partial x}$ below by a sum of values of $G_{k+1}$ (see (9.16a)). Next we may replace $G_{k+1}$ by $G_k$ in this sum (since $G_{k+1} \geq G_k$) and then once again use Lemma 9.2 as in (9.16b) to get the third inequality in (9.19). Since $k$ is arbitrary in (9.19), we conclude that

$$\int_{-1}^{1} \left|\frac{\partial}{\partial x} G\right|^{p} (A(s), s)\, ds \;=\; \infty.$$

To complete the proof of Theorem 1.17 we show using a condition of Strichartz [Stz] that $\|D_{1/2}A\|_{*} \;\leq\; c\,m$. Indeed by construction we have $|\hat{I}| \leq \frac{1}{16}|I|$ whenever $\hat{I} \in \mathcal{F}_{k+1}, I \in \mathcal{F}_{k}^{*}$, as we see from the definition of $\mathcal{F}_{k+1}$ above (9.13). It follows from this fact that if $J$ is an interval of length $|J|$ with $r_{l+1} \leq |J| \leq r_{l}$, then for $k \geq l+1$,

$$\sum_{\{I \in \mathcal{F}_{k}^{*}:\, I \cap J \neq \emptyset\}} |I| \;\leq\; c\,(\frac{1}{16})^{l+1-k}\,|J|\,. \tag{9.20}$$

If $J$ is as above, then

$$A = A_{l+1} + \sum_{k=l+1} B_{k+1} \;=\; A_{l+1} \;+\; B$$

and by (9.15) we deduce

$$\int_{J}\int_{J} \frac{(\,A_{l+1}(s) - A_{l+1}(t)\,)^{2}}{(s-t)^{2}}\, ds\, dt \;\leq\; c\, m^{2}\, r_{l}^{-1}\, |J|^{2} \;\leq\; cm^{2}\, |J| \tag{9.21}$$

while from the triangle inequality we have

$$\int_{J}\int_{J} \frac{(\,B(s) - B(t)\,)^{2}}{(s-t)^{2}}\, ds\, dt \;\leq\; c \left[\sum_{k=l+1}^{\infty} \left(\int_{J}\int_{J} \frac{(\,B_{k+1}(s) - B_{k+1}(t)\,)^{2}}{(s-t)^{2}}\, ds\, dt\right)^{1/2}\right]^{2}. \tag{9.22}$$

From (9.13) we find that if $E = \bigcup_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} I$ for some fixed $k \geq l+1$, then

$$\begin{aligned}
\int_J\int_J \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt \;&\leq\; \int_E\int_E \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt + 2\int_E\int_{J\setminus E} \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt \\
&\leq c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} \int_I\int_I \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt \;+\; c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} \int_I\int_{\mathbb{R}\setminus I} \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt \\
&\leq c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} \int_{2I}\int_{2I} \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt \;+\; c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} \int_I\int_{\mathbb{R}\setminus 2I} \tfrac{(B_{k+1}(s)-B_{k+1}(t))^2}{(s-t)^2}\, ds\, dt \\
&\leq c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} m^2\,|I| \;+\; c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} \int_I \left( \int_{\mathbb{R}\setminus 2I} m^2 |I|\, (s-t)^{-2}\, ds \right) dt \\
&\leq c \sum_{\{I\in\mathcal{F}_k^*:\, I\cap J\neq\emptyset\}} m^2 |I| \;\leq\; cm^2\, (\tfrac{1}{16})^{l+1-k}\, |J|,
\end{aligned} \tag{9.23}$$

thanks to (9.20). Summing (9.23) and using (9.22) we obtain

$$\int_J\int_J \frac{(\,B(s)-B(t)\,)^2}{(s-t)^2}\, ds\, dt \;\leq\; c\, m^2\, |J|\,. \tag{9.24}$$

Combining (9.24) with (9.21) we conclude that

$$\int_J\int_J \frac{(\,A(s)-A(t)\,)^2}{(s-t)^2}\, ds\, dt \;\leq\; cm^2\, |J|. \tag{9.25}$$

If $|J| \geq r_0$ we can repeat the above argument with $l+1$ replaced by 0 to obtain that (9.25) is still true. Hence (9.25) holds whenever $J \subset \mathbb{R}$ is an interval. From (9.25) and a theorem of Strichartz[Stz] we find that $\|D_{1/2}\, A\|_* \;\leq\; cm$. The proof of Theorem 1.17 is now complete. $\Box$

**Proof of Corollaries 1.18, 1.19 :** For given $p, 1 < p < \infty$, let $A, G, D$ be as in Theorem 1.17 . Let $A^*(x,t) = A(t)\psi(|x|)\psi(t)$ , $(x,t) \in \mathbb{R}^n$, where $\psi \in C_0^\infty(\mathbb{R})$ is even with $\psi \equiv 1$ on ( - 10, 10 ). We note from Theorem 1.17 and (0.19) that $\|A\|_{\text{comm}} < \infty$. Let

$$\Omega = \{(\lambda, x, t) : \lambda > A^*(x,t),\; (x,t) \in \mathbb{R}^n\,\}$$

and let $g$ be Green's function for the adjoint heat equation in $\Omega$ with pole at $(1, 0, \ldots, 0, 1)$. Extend $G$ to a subdomain of $\mathbb{R}^{n+1}$ by defining $G(\lambda, x, t) = G(\lambda, t)$ when $(\lambda, t) \in D$ and $x \in \mathbb{R}^{n-1}$. Then $G, g$ both vanish on $\partial\Omega \cap B_1(0)$ and are adjoint caloric in $\Omega \cap B_1(0)$ so we can once again use Lemma 9.2 and the Hopf boundary maximum principle to conclude that

$$\left|\frac{\partial}{\partial x} G\right| \leq k \left|\frac{\partial}{\partial x} g\right| \text{ on } \partial\Omega \cap B_{3/4}(0)$$

where $k$ depends on the value of $G$ and $g$ at certain points in $\Omega$. Raising both sides of this inequality to the $p$ th power and integrating we obtain

$$\int_{\partial\Omega \cap B(0,3/4)} \left|\frac{\partial}{\partial x} g\right|^p d\sigma = +\infty$$

where $\sigma$ is defined relative to $A^*$ as in section 0. To complete the proof of Corollary 1.18 we note from [LM, ch 3] that the Radon-Nikodym derivative of parabolic measure at $(1, 0, \ldots, 0, 1)$ relative to $\Omega$ is equal $\sigma$ a.e. to $\frac{\partial}{\partial x} g$ on $\partial\Omega$. $\square$

To prove Corollary 1.19 we note as in [LS] that if the $L^p(\partial\Omega)$ Dirichlet problem can be solved for some $p, 1 < p < \infty$, then $\frac{\partial g}{\partial x} \in L^{p/(p-1)}(\partial\Omega)$ where again $g$ is the Green's function for the adjoint heat equation in $\Omega$ with pole at a certain point of $\Omega$. Thus the example in Corollary 1.18 for $p/(p-1)$ shows in view of (0.19) that Theorem 1.13 can be false for $\|\mathbb{D}_n A\|_*$ large enough. Next we note as in section 5 that if the $L^p(\partial\Omega)$ Neumann problem can be solved for some $p, 1 < p < \infty$, by way of layer potentials (in the sense of Theorem 1.14), then the $L^{p/(p-1)}(\partial\Omega)$ Dirichlet problem for the adjoint heat equation can also be solved by way of layer potentials. Thus using the transformation $t \to -t$ we can again use Corollary 1.18 to show that the analogue of Theorem 1.14 for any fixed $p, 1 < p < \infty$, need not be true if the smallness restriction on $\|\mathbb{D}_n A\|_*$ is removed. Finally, if Theorem 1.15 is valid for some $p, 1 < p < \infty$, then we can apply this theorem with $g$ equal to the fundamental solution to the heat equation with pole at a certain point of $\Omega$. We obtain that the nontangential maximal function of the gradient of the Green's function with respect to our chosen point is in $L^p(\partial\Omega)$. Using a limiting argument as in section 7 we then conclude that the Radon- Nikodym derivative of adjoint parabolic measure with respect to the given point is locally in $L^p(\partial\Omega)$. Thus once again we can use the transformation $t \to -t$ and Corollary 1.18 to construct examples of domains where $\|A\|_{\text{comm}} < \infty$ but Theorem 1.15 is false. $\square$

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