On p Laplace Polynomial Solutions

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Abstract

We determine all real homogeneous polynomial solutions to the p Laplace equation, $-\infty < p < \infty, p \neq 1, 2$, of degree four in $\mathbb{R}^n, n \geq 3$, and show there are no degree five real homogeneous polynomial solutions in $\mathbb{R}^3$, when $-\infty < p < \infty, p \neq 1, 2$.
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Dedicated to Carl Minda on the occasion of his retirement from the University of Cincinnati

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1 Introduction

Let $x = (x_1, x_2, \ldots, x_n)$ denote a point in $\mathbb{R}^n$. Then $u : \mathbb{R}^n \to \mathbb{R}$ is said to be p harmonic in $\mathbb{R}^n, 1 < p < \infty$, if u satisfies the p Laplace equation :

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \text{ on } \mathbb{R}^n \tag{1.1}$$

in a certain weak Sobolev sense, where ∇u denotes the gradient of u and $\nabla\cdot$ is the divergence operator. It is well known (see for example ([D],[L1],[To],[W]) that weak solutions can be redefined on a set of Lebesgue measure zero in $\mathbb{R}^n$ to have Hölder continuous partial derivatives in $x_i, 1 \leq i \leq n$. Also if $\nabla u(x_0) \neq 0$, it follows from bootstrapping type arguments (see [GT, chapter 6]) that u is infinitely differentiable in a neighborhood of x_0. Finally one can apply a theorem, apparently originally proved by Höpf [H] (see also [F],[M]), to deduce that u is real analytic in a neighborhood of $x_0, 1 \leq i \leq n$. In [L] the first author asked if this result has a converse when $1 < p < \infty, p \neq 2$. That is if u is a real analytic solution to (1.1) in a neighborhood of x_0, when $1 < p < \infty, p \neq 2$, is it true that necessarily $\nabla u(x_0) \neq 0$. As noted in [L] , it follows from translation and dilation invariance of the p Laplacian that to prove this query it suffices to show there are no homogeneous polynomial solutions of degree ≥ 2 to (2.1) (in the next section) on $\mathbb{R}^n$ when $1 < p < \infty, p \neq 2$. In [L] this statement is proved when $n = 2$ and also that there are no second degree homogeneous polynomial solutions to the p Laplace equation in $\mathbb{R}^n, n \geq 2$, when $1 < p < \infty, p \neq 2$.

Recently in [T] Tkachev used nonassociative algebra arguments to prove

Theorem 1.1 *There are no homogeneous polynomial solutions to (2.1) in $\mathbb{R}^n$ of degree three when $-\infty < p < \infty, p \neq 1, 2$.*

In this note we use the 'magic point' introduced in [T] (see Lemma 2.1) and direct calculation to show :

Theorem 1.2 *All real homogeneous polynomials of degree 4 in $\mathbb{R}^n, n \geq 3$, which are solutions to (2.1) when $-\infty < p < \infty, p \neq 1, 2$, are of the form,*

$$u(x) = c\,(x_1^2 + x_2^2 + \cdots + x_k^2)^2, \text{ for } p = (4-k)/3, 2 \leq k \leq n.$$

Also there are no real homogeneous polynomials of degree 5 in $\mathbb{R}^3$ which are solutions to (2.1) when $-\infty < p < \infty, p \neq 1, 2$.

In the process of proving Theorem 1.2, we shall reprove Theorem 1.1. Our calculations were for us rather involved and so we have made frequent use of Maple-Mathematica to perform algebraic manipulations, as well as make exact evaluations of some one variable polynomial expressions. Thus the reader who wants to easily follow all our details, should have a computer algebra system at hand.

2 Preliminary Reductions

The proof of Theorem 1.2 is by contradiction. To begin, suppose u is a homogeneous polynomial solution to (1.1) on $\mathbb{R}^n$ of degree $m \geq 1$ for some $p, -\infty < p < \infty, p \neq 1, 2$, at points in $\mathbb{R}^n$ where

$\nabla u \neq 0$. We assume that u has a positive maximum on $\{x : |x| = 1\}$ at say y, otherwise consider $-u$. Using rotational invariance of the p Laplacian we may assume $y = e_n = (0, \dots, 0, 1)$. From Euler's equality for the partial derivatives of a homogeneous function, we deduce that $\nabla u(e_n) \neq 0$. Writing out (1.1) and dividing by $|\nabla u|^{p-4}$ it follows that in a neighborhood of e_n,

$$(p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \Delta u = 0. \tag{2.1}$$

For ease of calculation we change notation somewhat by putting $z = x_n$ and writing x for $(x_1, \dots, x_{n-1})$. We regard u as a polynomial in z with coefficients that are homogeneous polynomials in x. If $m = 3, 4, 5$ we claim that u can be written in the form,

$$u(x,z) = \frac{z^m}{m(m-1)} + (\sum_{i=1}^{n-1} a_i x_i^2) \frac{z^{m-2}}{2} + P(x) z^{m-3} + Q(x) z^{m-4} + R(x) z^{m-5} \tag{2.2}$$

where $a_i, 1 \leq i \leq n-1$, are real numbers while $Q, R \equiv 0$ if $m = 3$, and $R \equiv 0$ if $m = 4$. Also if $\not\equiv 0$, then P, Q, R, are homogeneous of degrees 3, 4, 5, in $x = (x_1, \dots, x_{n-1})$, respectively. To get (2.2) we may multiply u by a positive constant, if necessary, to obtain the first term on the righthand side in the expansion of u. The coefficient in z^{m-1} vanishes since u has a maximum on the unit sphere at $z = 1, x_i = 0, 1 \leq i \leq n-1$. Finally performing a rotation in the x variable, if necessary, we may assume that the coefficient of z^{m-2} has the desired form for some choice of $a_1, \dots, a_{n-1}$. Next we use (2.2) to partially expand (2.1) as a polynomial of degree $3m-4$ in z with coefficients that are homogeneous polynomials in x. For ease of calculation we put

$$\begin{aligned} M &= \sum_{i,j=1}^{n-1} u_{x_i} u_{x_j} u_{x_i x_j} + 2 \sum_{i=1}^{n-1} u_{x_i} u_z u_{x_i z} + u_z^2 u_{zz} \\ &= M_1 + M_2 + M_3 \end{aligned} \tag{2.3}$$

We first calculate the coefficients of z^{3m-4} down to z^{3m-7}, in the expansion of M in descending powers of z. If $S_k = \sum_{i=1}^{n-1} a_i^k x_i^2$ and δ_{ij} denotes the Kronecker delta we calculate,

$$\begin{aligned} M_1 = & \sum_{i,j=1}^{n-1} [(z^{m-2} a_i x_i + z^{m-3} P_{x_i} + z^{m-4} Q_{x_i} + z^{m-5} R_{x_i}) \\ & \times (z^{m-2} a_j x_j + z^{m-3} P_{x_j} + z^{m-4} Q_{x_j} + z^{m-5} R_{x_j})(\delta_{ij} a_j z^{m-2} + z^{m-3} P_{x_i x_j} + z^{m-4} Q_{x_i x_j} + z^{m-5} R_{x_i x_j})] \\ & = z^{3m-6} S_3 + z^{3m-7} [\sum_{i,j=1}^{n-1} a_i x_i a_j x_j P_{x_i x_j} + \sum_{i=1}^{n-1} 2 a_i^2 x_i P_{x_i}] + \dots \end{aligned} \tag{2.4}$$

$$M_2 = 2\sum_{i,j=1}^{n-1} \left(z^{m-2}a_ix_i + z^{m-3}P_{x_i} + z^{m-4}Q_{x_i} + z^{m-5}R_{x_i}\right) \Big(\tfrac{1}{m-1}z^{m-1} + \tfrac{(m-2)}{2}S_1z^{m-3}$$

$$+(m-3)Pz^{m-4} + (m-4)Qz^{(m-5)} + (m-5)Rz^{(m-6)}\Big) \left(z^{m-3}(m-2)a_ix_i + (m-3)P_{x_i}z^{m-4} + \right.$$

$$\left.(m-4)Q_{x_i}z^{m-5} + (m-5)R_{x_i}z^{m-6}\right) = z^{3m-6}\ \left(\tfrac{2m-4}{m-1}\right) S_2 + z^{3m-7}\left(\tfrac{4m-10}{m-1}\right)\sum_{i=1}^{n-1} a_ix_iP_{x_i} + \dots \tag{2.5}$$

Also,

$$M_3 = \Big(\tfrac{1}{m-1}z^{m-1} + \tfrac{(m-2)}{2}z^{m-3}S_1 + (m-3)Pz^{m-4} + (m-4)Qz^{(m-5)} + (m-5)Rz^{(m-6)} \Big)^2$$

$$\times \Big(z^{m-2} + \tfrac{(m-2)(m-3)}{2}z^{m-4}S_1 + (m-3)(m-4)Pz^{m-5} + (m-4)(m-5)Qz^{(m-6)}$$

$$+(m-5)(m-6)Rz^{(m-7)}\Big) = \tfrac{1}{(m-1)^2}z^{3m-4} + [\tfrac{(m-2)(m-3)}{2(m-1)^2} + \tfrac{(m-2)}{m-1}]S_1z^{3m-6} +$$

$$(\tfrac{(m-3)(m-4)}{(m-1)^2} + \tfrac{2(m-3)}{m-1})Pz^{3m-7} + \dots \tag{2.6}$$

Adding (2.4) - (2.6) gives,

$$M = \tfrac{1}{(m-1)^2}z^{3m-4} + \{\tfrac{(3/2)m^2-(11/2)m+5}{(m-1)^2}S_1 + \tfrac{2m-4}{m-1}S_2 + S_3\}z^{3m-6} + \{\sum_{i,j=1}^{n-1} a_ix_ia_jx_jP_{x_ix_j} +$$

$$\sum_{i=1}^{n-1} 2a_i^2x_iP_{x_i} + \tfrac{4m-10}{m-1}\sum_{i=1}^{n-1} a_ix_iP_{x_i} + (\tfrac{(m-3)(m-4)}{(m-1)^2} + \tfrac{2(m-3)}{m-1})P\}z^{3m-7}\ + \dots \tag{2.7}$$

Likewise, if $\lambda = \sum_{i=1}^{n-1} a_i$, then at (x,z)

$$\left(\sum_{i=1}^{n-1} u_{x_i}^2\right)\Delta u = \sum_{i=1}^{n-1} \left(z^{m-2}a_ix_i + z^{m-3}P_{x_i} + z^{m-4}Q_{x_i} + z^{m-5}R_{x_i}\right)^2$$

$$\times \Big(z^{m-2}(1+\lambda) + z^{m-3}\Delta P + z^{m-4}(\Delta Q + \tfrac{(m-2)(m-3)}{2}S_1) + z^{m-5}(\Delta R + (m-3)(m-4)P)\Big)$$

$$= z^{3m-6}(1+\lambda)S_2 + z^{3m-7}\left[2(1+\lambda)(\sum_{i=1}^{n-1} a_ix_iP_{x_i}) + S_2\Delta P\right] + \dots \tag{2.8}$$

Moreover,

$$u_z^2\,\Delta u = \Big(\tfrac{1}{m-1}z^{m-1} + \tfrac{(m-2)}{2}z^{m-3}S_1 + (m-3)Pz^{m-4} + (m-4)Qz^{(m-5)} + (m-5)Rz^{(m-6)}\Big)^2$$

$$\times\Big(z^{m-2}(1+\lambda) + z^{m-3}\Delta P + z^{m-4}(\Delta Q + \tfrac{(m-2)(m-3)}{2}S_1) + z^{m-5}(\Delta R + (m-3)(m-4)P)\Big)$$

$$= \tfrac{1+\lambda}{(m-1)^2}z^{3m-4} + \tfrac{\Delta P}{(m-1)^2}z^{3m-5} + [(\Delta Q + \tfrac{(m-2)(m-3)}{2}S_1)\tfrac{1}{(m-1)^2} + (1+\lambda)\tfrac{(m-2)}{m-1}S_1]z^{3m-6}$$

$$+ [(\Delta R + (m-3)(m-4)P)\tfrac{1}{(m-1)^2} + \tfrac{2(1+\lambda)(m-3)}{m-1}P + \tfrac{(m-2)}{(m-1)}\Delta P S_1]z^{3m-7} + \dots \tag{2.9}$$

Adding (2.8), (2.9) yields

$$\begin{aligned}
|\nabla u|^2\,\Delta u &= \tfrac{1+\lambda}{(m-1)^2}z^{3m-4} + \tfrac{\Delta P}{(m-1)^2}z^{3m-5} + \\
&\{(\Delta Q + \tfrac{(m-2)(m-3)}{2}S_1)\tfrac{1}{(m-1)^2} + (1+\lambda)[\tfrac{(m-2)}{m-1}S_1 + S_2]\}z^{3m-6} + \\
&\{(\Delta R + (m-3)(m-4)P)\tfrac{1}{(m-1)^2} + \tfrac{(m-2)}{(m-1)}\Delta P S_1 + 2(1+\lambda)\tfrac{(m-3)}{(m-1)}P \\
&+2(1+\lambda)(\sum_{i=1}^{n-1} a_i x_i P_{x_i}) + S_2\Delta P\}z^{3m-7} + \dots
\end{aligned}\tag{2.10}$$

Multiplying M by $p-2$ in (2.7) and adding to (2.10) we obtain from (2.2) that the coefficients multiplying powers of z must vanish. From z^{3m-4}, z^{3m-5}, we deduce

$$\lambda = \sum_{i=1}^{n-1} a_i = 1-p \text{ and } \Delta P = 0. \tag{2.11}$$

Using (2.11) and putting the coefficients of $z^{3m-6}, z^{3m-7} = 0$ we find after some arithmetic that

$$\begin{aligned}
&(a)\quad \Delta Q + \tfrac{(m-2)(m-3)}{2}S_1 = -(p-2)(m-1)^2\left\{\tfrac{(m-2)(m-3)}{2(m-1)^2}S_1 + \tfrac{(m-3)}{m-1}S_2 + S_3\right\} \\
&(b)\quad \Delta R + (m-3)(m-4)P = -(p-2)(m-1)^2\Bigg[\sum_{i,j=1}^{n-1} a_i x_i a_j x_j P_{x_i x_j} + \\
&\sum_{i=1}^{n-1} 2a_i^2 x_i P_{x_i} + \tfrac{(2m-8)}{m-1}\sum_{i=1}^{n-1} a_i x_i P_{x_i} \;+\; \tfrac{(m-3)(m-4)}{(m-1)^2}\,P\Bigg]
\end{aligned}\tag{2.12}$$

For use in proving Theorem 1.2 we use the equation for $\Delta Q, \Delta R$, in (2.12) to rewrite the expansion of Δu as

$$\Delta u = -(p-2)\left(z^{m-2} + z^{m-4}(m-1)^2\left[\frac{(m-2)(m-3)}{2(m-1)^2}S_1 + \frac{(m-3)}{m-1}S_2 + S_3\right] + \right.$$

$$\left. z^{m-5}(m-1)^2\left[\sum_{i,j=1}^{n-1} a_i x_i a_j x_j P_{x_i x_j} + \sum_{i=1}^{n-1} 2a_i^2 x_i P_{x_i} + \tfrac{(2m-8)}{m-1}\sum_{i=1}^{n-1} a_i x_i P_{x_i} + \tfrac{(m-3)(m-4)}{(m-1)^2}P\right]\right) + \dots \tag{2.13}$$

Next we use (2.13) to calculate the coefficient of z^{3m-8} in the expansion of (2.1) in powers of z. We find that

$$(p-2)^{-1}\left(\sum_{i=1}^{n-1} u_{x_i}^2\right)\Delta u = \dots$$

$$-z^{3m-8}\left(S_2(m-1)^2\left[\frac{(m-2)(m-3)}{2(m-1)^2}S_1 + \frac{(m-3)}{m-1}S_2 + S_3\right] + \sum_{i=1}^{n-1}(2a_i x_i Q_{x_i} + P_{x_i}^2)\right) + \dots$$

$$(p-2)^{-1}\, u_z^2\, \Delta u = \dots$$

$$-z^{3m-8}\left((m-2)(m-1)S_1\left[\frac{(m-2)(m-3)}{2(m-1)^2}S_1 + \frac{(m-3)}{m-1}S_2 + S_3\right] + \tfrac{2m-8}{m-1}Q + \tfrac{(m-2)^2}{4}S_1^2\right) + \dots$$

$$M_1 = \dots + z^{3m-8}\left(\sum_{i,j=1}^{n-1}(a_i x_i a_j x_j Q_{x_i x_j} + 2a_i x_i P_{x_j} P_{x_i x_j}) + \sum_{i=1}^{n-1}(2a_i^2 x_i Q_{x_i} + a_i P_{x_i}^2)\right) + \dots$$

$$M_2 = \dots + z^{3m-8}\left((m-2)^2 S_1 S_2 + \sum_{i=1}^{n-1}\frac{(4m-12)}{(m-1)}a_i x_i Q_{x_i} + \frac{(2m-6)}{m-1}P_{x_i}^2\right) + \dots$$

$$M_3 = \dots + z^{3m-8}\left(\frac{(m-4)(3m-7)}{(m-1)^2}Q + (\frac{(m-2)^2(m-3)}{2(m-1)} + \frac{(m-2)^2}{4})S_1^2\right) + \dots \tag{2.14}$$

Adding the rows in (2.14) together and putting the resulting coefficient of $z^{3m-8} = 0$, we conclude that

$$\sum_{i,j=1}^{n-1}(a_i x_i a_j x_j Q_{x_i x_j} + 2a_i x_i P_{x_j} P_{x_i x_j}) + \sum_{i=1}^{n-1}[(2a_i^2 + 2\tfrac{(m-5)}{m-1}a_i)x_i Q_{x_i} + (a_i + \tfrac{(m-5)}{m-1})P_{x_i}^2] + \tfrac{(m-4)(m-5)}{(m-1)^2}Q$$

$$= \tfrac{1}{2}(m-2)(m-5)S_1 S_2 + (m-1)(m-2)S_1 S_3 + (m-1)(m-3)S_2^2 + (m-1)^2 S_2 S_3. \tag{2.15}$$

3 Proof of Theorem 1.1 and Theorem 1.2 when m = 4

3.1 Proof of Theorem 1.1

If $m = 3$, then $Q \equiv R \equiv 0$ so from (2.11) we have

$$\sum_{i=1}^{n-1} a_i = 1 - p \tag{3.1}$$

while from (2.12) (a) we get

$$S_3 = 0 \tag{3.2}$$

Note from (3.2) that necessarily,

$$a_i = 0 \text{ for } 1 \leq i \leq n-1. \tag{3.3}$$

Using (3.3) in (3.2) we get $p = 1$, a contradiction. Thus there are no real homogeneous third degree polynomials on $\mathbb{R}^n$ that satisfy the p Laplace equation when $-\infty < p < \infty, p \neq 1, 2$.

3.2 Proof of Theorem 1.2 when m = 4

If $m = 4$, then $R \equiv 0$, so (2.12) (b) and (2.15) become

$$\sum_{i=1}^{n-1} 2a_i^2 x_i P_{x_i} + \sum_{i,j=1}^{n-1} a_i x_i a_j x_j P_{x_i x_j} = 0 \tag{3.4}$$

$$\begin{aligned} &-(1/3)|\nabla P|^2 + \sum_{i,j=1}^{n-1} 2a_i x_i P_{x_j} P_{x_i x_j} + \sum_{i=1}^{n-1} a_i P_{x_i}^2 \\ &+ \sum_{i,j=1}^{n-1} a_i x_i a_j x_j Q_{x_i x_j} - (2/3) \sum_{i=1}^{n-1} a_i x_i Q_{x_i} + 2 \sum_{i=1}^{n-1} a_i^2 x_i Q_{x_i} \\ &= -S_1 S_2 + 9 S_3 S_2 + 3 S_2^2 + 6 S_1 S_3 \end{aligned} \tag{3.5}$$

where we have collected terms in P, Q from (2.15) to facilitate further computation.

We claim that

$$P \equiv 0. \tag{3.6}$$

To prove our claim we first take derivatives with respect to x_k, x_l, x_r on the functions in equation (3.4). We obtain

$$2(a_k^2 + a_l^2 + a_r^2 + a_k a_l + a_k a_r + a_l a_r) P_{x_k x_l x_r} = 0 \tag{3.7}$$

From (3.7) and Schwarz's inequality we see that $P_{x_k x_l x_r} = 0$ unless all of $a_l, a_r, a_k = 0$. Thus if $\Lambda = \{i : a_i \neq 0\}$, then P has no nonzero terms containing an x_i whenever $i \in \Lambda$. Using this fact we see that each of the four sums involving P in (3.4), (3.5) are zero. That is,

$$-(1/3)|\nabla P|^2+\sum_{i,j=1}^{n-1}a_ix_ia_jx_jQ_{x_ix_j}-(2/3)\sum_{i=1}^{n-1}a_ix_iQ_{x_i}$$
$$+2\sum_{i=1}^{n-1}a_i^2x_iQ_{x_i}=-S_1S_2+9S_3S_2+3S_2^2+6S_1S_3 \tag{3.8}$$

If $a_l=0$, so $l\notin\Lambda$, then differentiating both sides of the above equation four times with respect to x_l and using the fact that P is homogeneous of degree three while Q is homogeneous of degree four, we obtain

$$-2|\nabla P_{x_l^2}|^2=0. \tag{3.9}$$

Thus in this case P has no terms of the form $x_l^2x_k$. Next taking four derivatives in (3.8), two each with respect to $l,r\notin\Lambda$ and using (3.9) we obtain $|\nabla P_{x_rx_l}|^2\equiv 0$. Thus (3.6) is valid.

Using (2.13), $P,R\equiv 0$, we continue our calculation of the rest of the coefficients in (2.1) when $m=4$. We obtain

$$M_1=\cdots+z^2\left[\sum_{i,j=1}^{n-1}2a_ix_iQ_{x_j}Q_{x_ix_j}+\sum_{i=1}^{n-1}a_iQ_{x_i}^2\right]+\sum_{i,j=1}^{n-1}Q_{x_i}Q_{x_j}Q_{x_ix_j}$$

$$M_2+M_3=\cdots+z^2\left(S_1\sum_{i=1}^{n-1}4a_ix_iQ_{x_i}+S_1^3\right)$$

$$(p-2)^{-1}|\nabla u|^2\Delta u=\cdots-z^2\left[(9S_3+3S_2+S_1)\sum_{i=1}^{n-1}2a_ix_iQ_{x_i}+S_1^2(9S_3+3S_2+S_1)+|\nabla Q|^2\right].$$

$$-(9S_3+3S_2+S_1)|\nabla Q|^2 \tag{3.10}$$

Adding the rows of (3.10) together and putting the resulting coefficient of $z^2=0$, we find

$$\sum_{i,j=1}^{n-1}2Q_{x_i}a_jx_jQ_{x_ix_j}+\sum_{i=1}^{n-1}a_iQ_{x_i}^2-|\nabla Q|^2=S_1^2(9S_3+3S_2)+(9S_3+3S_2-S_1)\sum_{i=1}^{n-1}2a_ix_iQ_{x_i} \tag{3.11}$$

while putting the constant term $=0$, yields

$$\sum_{i,j=1}^{n-1}Q_{x_i}Q_{x_j}Q_{x_ix_j}=(9S_3+3S_2+S_1)|\nabla Q|^2 \tag{3.12}$$

If $1\le k,l,r,q\le n-1$ we obtain in view of (3.6), after taking four derivatives of the lefthand side of (3.8) with respect to x_k,x_l,x_r,x_q, that

$$Q_{x_kx_rx_lx_q}\{2(a_ka_l+a_ka_r+a_ka_q+a_la_r+a_la_q+a_ra_q)+2(a_k^2+a_l^2+a_r^2+a_q^2)$$
$$-(2/3)(a_k+a_l+a_r+a_q)\}=\phi(k,l,r,q) \tag{3.13}$$

where $\phi(k,l,r,q)$ denotes the derivative of the righthand side of (3.8) with respect to x_k, x_l, x_r, x_q and so $=0$, unless there are two pairs of the four integers with each pair equal to a positive integer.

From (2.11) we see that if $a_i = 0, 1 \le i \le n-1$, then $p=1$, a contradiction. Thus we assume $a_k \neq 0$. Then In (3.13) we first take $r, q = k$ and $l \neq k$. We get

$$(2/3)(9a_k a_l + 18a_k^2 + 3a_l^2 - 3a_k - a_l)Q_{x_l x_k^3} = 0. \tag{3.14}$$

Also taking $r, q, l = k$ in (3.13) we arrive at

$$(2/3)(30a_k^2 - 4a_k)Q_{x_k^4} = 24a_k^3(9a_k^2 + 9a_k - 1).$$

Clearly $a_k \neq \frac{2}{15}$. Since $a_k \neq 0$, it follows that

$$Q_{x_k^4} = 72\, a_k^2 \left(\frac{9a_k^2 + 9a_k - 1}{60\, a_k - 8}\right) = 18\, a_k^2 \left(\frac{9a_k^2 + 9a_k - 1}{15\, a_k - 2}\right). \tag{3.15}$$

Next for ease of notation we put $y = Q_{x_k^4}, a_k = a, C = \sum_{i=1, i\neq k}^{n-1} (a_i - 1 + 6a)Q^2_{x_i x_k^3}$, and take six derivatives of (3.11) with respect to x_k. Dividing the resulting equality by 20, we obtain

$$y^2(7a-1) + 12a(a - 9a^3 - 3a^2)y - 36(9a^5 + 3a^4) + C = 0. \tag{3.16}$$

We note that if $Q_{x_i x_k^3} \neq 0$, then from (3.14) with $l = i$ and the quadratic formula, we have

$$a_i = \tfrac{1 - 9a \pm \sqrt{(9a-1)^2 - 12(18a^2 - 3a)}}{6}. \tag{3.17}$$

We now let $a_k = a$ be the smallest of $\{a_i \neq 0 : 1 \le i \le n-1\}$ in (3.17). We claim that each term in the sum defining C is ≤ 0 and < 0, unless either $a_i = 0, a_k = 1/6$, or $Q_{x_i x_k^3} = 0$. If $a_i = 0$, for some i then from (3.17) we see that $a = 1/6$ and thereupon from the minimality of a that either $a_l = 0, l \neq k$, or $Q_{x_l x_k^3} = 0$. If $a_i \neq 0$ and $Q_{x_i x_k^3} \neq 0$, then since $a \le a_i$ it follows from (3.17) that $a < 2/15$. Using this inequality and (3.17) once again we find that

$$a_i - 1 + 6a < 0 \text{ if } -864(a - 1/6)^2 < 0.$$

From minimality, $a \neq 1/6$, so our claim is proved.

Next we note from (3.15) that $y = \dfrac{2\,(81a^4 + 81a^3 - 9a^2)}{15a - 2}$. Using this value for y in (3.16) and multiplying by $(15a-2)^2$ we obtain after dividing out a term in a^4 that

$$\begin{aligned} &4(81a^2 + 81a - 9)^2(7a-1) + 24(1 - 9a^2 - 3a)(15a-2)\times \\ &(81a^2 + 81a - 9) - 36(9a+3)(15a-2)^2 + a^{-4}(15a-2)^2 C = 0 \end{aligned} \tag{3.18}$$

Using Maple-Mathematica we calculated that the polynomial, say F, in a which one gets from removing the term involving C in (3.18) simplifies to

$$F(a) = -78732a^5 + 26244a^4 + 34992a^3 - 23328a^2 +$$

$$4860a - 324 = 324(3a-1)^3(-9a^2 - 6a + 1)$$

so by the quadratic formula, F has real roots at

$$-1/3 \pm (1/3)\sqrt{2} \text{ and } 1/3 \text{ three times .}$$

Since $C = 0$ when $a = 1/6$ and $F(1/6) \neq 0$, we see that $a \neq 1/6$ in (3.18). Note from the above zeros of F that F is nonnegative only on $(-\infty, -(1+\sqrt{2})/3]$ and $[\sqrt{2}-1)/3, 1/3]$. Since $C \leq 0$, we conclude from (3.18) that if u exists, then necessarily a lies in one of the above intervals. From (3.17) and the quadratic formula we deduce that if $Q_{x_i x_k^3} \neq 0, a \neq 1/6$, then $a_i \neq 0$ and in order for a_i to be real we must have $-135a^2 + 18a + 1 > 0$ so $a \notin (-\infty, -(1+\sqrt{2})/3]$. Also from minimality, $a < 2/15$ so $a \notin [(\sqrt{2}-1)/3, 1/3]$. Thus

$$Q_{x_i x_k^3} = 0, \text{ for } i \neq k. \tag{3.19}$$

Armed with (3.19) we now note that $C = 0$ and then from (3.18) that the only possible values for a are the zeros of F. Next we differentiate both sides of (3.12) eight times with respect to x_k. Using (3.19) we obtain after division by 560,

$$Q^3_{x_k^4} = 2(9a^3 + 3a^2 + a)Q^2_{x_k^4} \tag{3.20}$$

We consider two cases, if $Q_{x_k^4} = 0$ then from (3.15) and the quadratic formula we deduce that $a = \frac{-9-\sqrt{117}}{18}$ which is not a zero of F and so leads to a contradiction. Thus $Q_{x_k^4} \neq 0$ and from (3.15), (3.20) we have

$$18\, a^2 \left(\frac{9a^2 + 9a - 1}{15\, a - 2} \right) = Q_{x_k^4} = 2(9a^3 + 3a^2 + a)$$

Multiplying both sides of this equation by $(15a - 2)$ and doing some arithmetic we find that

$$0 = -108a^4 + 108a^3 - 36a^2 + 4a = -4a(3a-1)^3.$$

Since $a \neq 0$, we must have $a = a_k = 1/3$.

Now suppose there exists $a_i \neq 0$ with $i \neq k$. Let a_l be the smallest $a_i \neq 0$ with $i \neq k$. Then from (3.17) with $a_i = 1/3$ and a replaced by a_l we see that necessarily $Q_{x_k x_l^3} = 0$ since otherwise $a_l = 0$. Using this fact we can essentially repeat the argument given for a_k verbatim with $a = a_l$ to get $a_l = 1/3$. Continuing this argument we obtain that either $a_i = 1/3$ when $i \in \Lambda \neq \emptyset$ or $a_i = 0 \notin \Lambda$.

Next we show Q so also u has no dependence on x_r when $r \notin \Lambda$. Indeed in this case we observe from (3.16) with $a = a_r$ and the definition of y, C that $Q_{x_i x_r^3} = 0$, when $1 \leq i \leq n-1$. Using (3.11) and repeating the argument after (3.9) it follows that $|\nabla Q_{x_l x_r x_q}| = 0$ when $r, l, q \notin \Lambda$. Also from (3.13) we see that $Q_{x_k x_r x_l x_q} = 0$ whenever two of the indices are in Λ and two not in Λ. From these facts and Euler's formula we see that $Q_{x_i x_r x_q} = 0$ whenever $x_r, x_q \notin \Lambda$. Finally using this equality and taking two derivatives with respect to $x_r \notin \Lambda$ on the functions in (3.11) we get

$$(4/3) \sum_{i=1}^{n-1} Q^2_{x_i x_r} = 0 \tag{3.21}$$

To get this equality we have used $a_i = 1/3, i \in \Lambda$ and Euler's formula to deduce that after taking two derivatives with respect to x_r on the lefthand sum in (3.11) we have

$$\sum_{i,j=1}^{n-1} 4Q_{x_i x_r} a_j x_j Q_{x_r x_i x_j} = (4/3) \sum_{i,j=1}^{n-1} Q_{x_i x_r} x_j Q_{x_r x_i x_j} = (8/3) \sum_{i=1}^{n-1} Q^2_{x_i x_r}.$$

From (3.21) and our earlier observations we conclude that u has no terms with an x_r in it when $r \notin \Lambda$.

From this fact and an induction type argument we see that to complete the proof of Theorem 1.2 when $m = 4$ we may as well assume that $a_i = 1/3$ for $1 \le i \le n-1$. Then from (2.11) we see that $p = (4-n)/3$. Continuing under this assumption we get from (3.15) that $Q_{x_k^4} = 2$ while from (3.13) we calculate, $Q_{x_k^2 x_l^2} = 2/3$ for $1 \le k, l \le n-1$. Also from earlier work we find that $Q_{x_k x_l x_r x_q} = 0$ otherwise. We conclude from Taylor's theorem that

$$u(x,z) = \frac{1}{12}(z^2 + x_1^2 + \ldots x_{n-1}^2)^2.$$

This completes the proof of Theorem 1.2 when $m = 4$.

4 Proof of Theorem 1.2 when m = 5, n = 3.

Putting $m = 5, n = 3$, in (2.15) we get after a slight rearrangement of terms,

$$\begin{aligned}&\sum_{i,j=1}^{2} 2a_i x_i P_{x_j} P_{x_i x_j} + \sum_{i,j=1}^{2} a_i P^2_{x_i} + \sum_{i=1}^{2} 2a_i^2 x_i Q_{x_i} + \sum_{i,j=1}^{2} a_i x_i a_j x_j Q_{x_i x_j} \\ &= 16 S_2 S_3 + 8 S_2^2 + 12 S_1 S_3 = \sum_{i,j=1}^{2} (16 a_i^2 a_j^3 + 8 a_i^2 a_j^2 + 12 a_i a_j^3)\, x_i^2\, x_j^2\end{aligned} \tag{4.1}$$

Taking three derivatives in x_k and one in $x_l, l \ne k$, on the functions in (4.1) we obtain

$$\sum_{i=1}^{2} (6a_l + 18a_k + 6a_i) P_{x_i x_k^2} P_{x_i x_k x_l} + (2a_l^2 + 12a_k^2 + 6a_k a_l) Q_{x_l x_k^3} = 0 \tag{4.2}$$

Now from (2.11), (2.12) (a), and the fact that $n = 3, k \ne l$, we deduce that

$$Q_{x_l x_k^3} = -Q_{x_l^3 x_k}, \text{ and } P_{x_i x_k^2} = -P_{x_i x_l^2}. \tag{4.3}$$

Using (4.3) in (4.2) we obtain

$$\sum_{i=1}^{2} (6a_l + 18a_k + 6a_i) P_{x_i x_l^2} P_{x_i x_k x_l} + (2a_l^2 + 12a_k^2 + 6a_k a_l) Q_{x_k x_l^3} = 0. \tag{4.4}$$

On the other hand, we can take three derivatives in x_l and one in x_k in order to get (4.2) with x_l, x_k interchanged. Doing this and subtracting from (4.4) it follows that

$$0 = 10(a_k + a_l)(a_k - a_l) Q_{x_l^3 x_k}$$

so either $a_1 = a_2$ or $a_1 = -a_2$ or $Q_{x_l^3 x_k} = 0$. $a_1 = -a_2$ is not permitted as we see from (2.11). If $a_1 = a_2$, then from (4.3) we deduce that the sum in (4.2) involving partial derivatives of P, vanishes. Thus from (4.2), $20a_k^2 Q_{x_k^3 x_l} = 0$. Now $a_1 = 0 = a_2$ is not allowed so in either case,

$$Q_{x_1^3 x_2} = Q_{x_2^3 x_1} = 0. \tag{4.5}$$

We continue under the assumption that

$$a_1 \neq a_2. \tag{4.6}$$

This assumption will be dealt with later. Then from harmonicity of P, (4.2), (4.3), (4.5), (4.6), we see that either $P_{x_2 x_1^2} = -P_{x_2^3} = 0$ or $P_{x_1 x_2^2} = -P_{x_1^3} = 0$. Without loss of generality, suppose that

$$P_{x_2^3} = -P_{x_2 x_1^2} = 0. \tag{4.7}$$

Using this equality in (2.12) (b) and taking third partials on ΔR we obtain

$$R_{x_2^5} = -R_{x_2^3 x_1^2} = R_{x_2 x_1^4}\,. \tag{4.8}$$

Taking four derivatives in x_1 in (4.1) and using (4.5), (4.7), we see that

$$30a_1 P_{x_1^3}^2 + 20a_1^2 Q_{x_1^4} - 2(192a_1^5 + 240a_1^4) = 0. \tag{4.9}$$

Next we compute the coefficient of z^6 in (2.1). In this calculation and subsequent calculations we put

$$V = 16S_3 + 8S_2 + 3S_1 \text{ and } W = \sum_{i,j=1}^{2} 16a_i x_i a_j x_j P_{x_i x_j} \; + \; \sum_{i=1}^{2} (8a_i + 32a_i^2)\, x_i P_{x_i} \; + \; 2P \; .$$

Using (2.4)-(2.6) and (2.13) with $m = 5, n = 3$, we calculate

$$\begin{aligned}
M_1 &= \cdots + z^6 \Bigg(2\sum_{i=1}^{2} a_i^2 x_i R_{x_i} \; + \; \sum_{i,j=1}^{2} a_i a_j x_i x_j R_{x_i x_j} \; + \; \sum_{i=1}^{2} 2a_i P_{x_i} Q_{x_i} \\
&+ \sum_{i,j=1}^{2} [(P_{x_i} P_{x_j} + 2a_i x_i Q_{x_j}) P_{x_i x_j} + 2a_i x_i P_{x_j} Q_{x_i x_j}] \Bigg) + \ldots \\
M_2 + M_3 &= \cdots + z^6 \left(\sum_{i=1}^{2} [\tfrac{3}{2} a_i x_i R_{x_i} + 15S_1 a_i x_i P_{x_i} + \tfrac{3}{2} P_{x_i} Q_{x_i}] \; + \; P\,(12S_2 + \tfrac{21}{2} S_1) \right) + \ldots \\
(p-2)^{-1} (\sum_{i=1}^{2} u_{x_i}^2) \Delta u &= \cdots - z^6 \left(\sum_{i=1}^{2} 2a_i x_i R_{x_i} + \sum_{i=1}^{2} (2a_i x_i V + 2Q_{x_i}\,)\, P_{x_i} + W S_2 \right) + \ldots \\
(p-2)^{-1}\, u_z^2 \Delta u &= \cdots - z^6 \left(\tfrac{3}{4} S_1 W \; + \; P(16S_3 + 8S_2 + 9S_1) \right) + \ldots
\end{aligned} \tag{4.10}$$

Adding the rows in (4.10) and putting the resulting coefficient of $z^6 = 0$, we find that

$$\sum_{i=1}^{2}[2a_i^2 x_i R_{x_i} - \tfrac{1}{2}a_i x_i R_{x_i}] + \sum_{i,j=1}^{2} a_i x_i a_j x_j R_{x_i x_j} + \sum_{i,j=1}^{2} 2a_j x_j (Q_{x_i} P_{x_i x_j} + P_{x_i} Q_{x_i x_j})$$

$$+ \sum_{i,j=1}^{2} [P_{x_i} P_{x_j} - (16S_2 + 12S_1)\, a_i x_i a_j x_j]\, P_{x_i x_j} + \sum_{i=1}^{2} (2a_i - \tfrac{1}{2}) P_{x_i} Q_{x_i}$$

$$+ (2S_2 - 16S_3)P + (3S_1 - 24S_2 - 32S_3)(\sum_{i=1}^{2} a_i x_i P_{x_i}) - (24S_1 + 32S_2)(\sum_{i=1}^{2} a_i^2 x_i P_{x_i}) = 0 \tag{4.11}$$

Taking five derivatives in x_2 on the functions in (4.11) and using (4.5), (4.7), we find

$$(30a_2^2 - \tfrac{5}{2}a_2) R_{x_2^5} = 0 \tag{4.12}$$

Also taking two derivatives in x_1, and three in x_2, we obtain in view of (4.5), (4.7), that

$$(12a_2^2 + 6a_1^2 + 12a_1 a_2 - \tfrac{3}{2}a_2 - a_1) R_{x_2^3 x_1^2} = 0 \tag{4.13a}$$

while taking four derivatives in x_1 and one in x_2 we get

$$(20a_1^2 - 2a_1 + 2a_2^2 - a_2/2 + 8a_1 a_2) R_{x_1^4 x_2} = 0 \tag{4.13b}$$

We use (4.12), (4.13), and (4.8) to show that

$$R_{x_2^5} = -R_{x_2^3 x_1^2} = R_{x_2 x_1^4} = 0. \tag{4.14}$$

Indeed if any of the terms in (4.14) are zero, then all are zero thanks to (4.8), so it suffices to consider the case when the expressions in parentheses in (4.12),(4.13) all vanish. In this case from (4.12) we have $a_2 = 0$ or $\frac{1}{12}$. If $a_2 = 0$ then from (4.13a) we find that either $a_1 = 0$ or $\frac{1}{6}$. $a_1 = a_2 = 0$ is not allowed since $p \neq 1$. Also from (4.13b) it follows that $a_1 = \frac{1}{10} \neq \frac{1}{6}$ when $a_2 = 0$. So $a_2 = \frac{1}{12}$ and using this value in the equation resulting from (4.13a) we conclude $a_1 = \pm\frac{1}{12}$. Neither value is permissible, since $a_1 \neq a_2$ and $p \neq 1$. Thus (4.14) is true. Finally we take five derivatives in x_1 on the functions in (4.11) and use (4.5), (4.7), to get

$$0 = (30a_1^2 - \tfrac{5}{2}a_1) R_{x_1^5} + (-5760a_1^4 - 4640a_1^3 + 220a_1^2) P_{x_1^3} + 30P_{x_1^3}^3 + (120a_1 - 5) P_{x_1^3} Q_{x_1^4}. \tag{4.15}$$

Next we compute the coefficient of z^5 in (2.1). We get

$$M_1 = \cdots + z^5 \left(\sum_{i,j=1}^{2} [2a_i x_i P_{x_j} R_{x_i x_j} + 2(a_i x_i R_{x_j} + P_{x_i} Q_{x_j}) P_{x_i x_j}] \right.$$

$$\left. + \sum_{i,j=1}^{2} (2a_i x_i Q_{x_j} + P_{x_i} P_{x_j}) Q_{x_i x_j} + \sum_{i=1}^{2} (2a_i P_{x_i} R_{x_i} + a_i Q_{x_i}^2) \right) + \dots$$

$$M_2 + M_3 = \cdots + z^5 \{ \sum_{i=1}^{2} [P_{x_i} R_{x_i} + 6S_1 P_{x_i}^2 + 12 S_1 a_i x_i Q_{x_i} + 20 P P_{x_i} a_i x_i + \tfrac{1}{2} Q_{x_i}^2]$$

$$+ (6S_2 + \tfrac{9}{2} S_1) Q + 6P^2 + \tfrac{27}{4} S_1^3 \} + \dots$$

$$(p-2)^{-1} (\sum_{i=1}^{2} u_{x_i}^2) \Delta u = \cdots - z^5 \left(\sum_{i=1}^{2} [(2W a_i x_i + 2R_{x_i} + V P_{x_i}) P_{x_i} + (Q_{x_i} + 2a_i x_i V S) Q_{x_i}] \right) + \dots . \tag{4.16}$$

$$(p-2)^{-1} u_z^2 \, \Delta u = \cdots - z^5 \left(PW + 4P^2 + 3S_1 Q + (\tfrac{9}{4} S_1^2 + \tfrac{1}{2} Q) V \right) + \dots$$

Adding the rows of (4.16) and then putting the coefficient of $z^5 = 0$ it follows that

$$0 = \sum_{i=1}^{2} (2a_i - 1) P_{x_i} R_{x_i} + 2 \sum_{i,j=1}^{2} a_j x_j R_{x_i} P_{x_i x_j} + 2 \sum_{i,j=1}^{2} a_i x_i P_{x_j} R_{x_i x_j} + \sum_{i=1}^{2} (a_i - \tfrac{1}{2}) Q_{x_i}^2$$

$$+ 2 \sum_{i,j=1}^{2} a_j x_j Q_{x_i} Q_{x_i x_j} + (-32 S_3 - 16 S_2 + 6 S_1) \sum_{i=1}^{2} a_i x_i Q_{x_i} + (2S_2 - 8S_3) Q$$

$$+ \sum_{i,j=1}^{2} 2 P_{x_j} Q_{x_i} P_{x_i x_j} + \sum_{i,j=1}^{2} P_{x_j} P_{x_i} Q_{x_i x_j} + \sum_{i=1}^{2} (3S_1 - 16 S_3 - 8 S_2) P_{x_i^2} + P \sum_{i=1}^{2} (8a_i - 32 a_i^2) x_i P_{x_i}$$

$$- (\sum_{i=1}^{2} a_i x_i P_{x_i}) \left[32 \sum_{i,j=1}^{2} a_i x_i a_j x_j P_{x_i x_j} + \sum_{i=1}^{2} (64 a_i^2 + 16 a_i) x_i P_{x_i} \right]$$

$$- \sum_{i,j=1}^{2} 16 P a_i x_i a_j x_j P_{x_i x_j} - 36 S_1^2 S_3 - 18 S_1^2 S_2. \tag{4.17}$$

Taking six derivatives in x_1 on the functions in (4.17) and using (4.5), (4.7), we have

$$0 = 15(14a_1 - 1) P_{x_1^3} R_{x_1^5} + (140 a_1 - 10) Q_{x_1^4}^2 + (780 a_1^2 - 2160 a_1^3 - 3840 a_1^4) Q_{x_1^4} + 210 P_{x_1^3}^2 Q_{x_1^4}$$

$$+ (1020 a_1 - 8160 a_1^2 - 25920 a_1^3) P_{x_1^3}^2 - (25920 a_1^5 + 12960 a_1^4). \tag{4.18}$$

Next we compute the coefficient of z^4 in (2.1). We calculate

$$M_1 = \cdots + z^4 \left(\sum_{i,j=1}^{2} (2a_i x_i Q_{x_j} + P_{x_i} P_{x_j}) R_{x_i x_j} + \sum_{i=1}^{2} 2a_i R_{x_i} Q_{x_i} \right.$$

$$\left. + \sum_{i,j=1}^{2} [(2P_{x_i} Q_{x_j} + 2a_i x_i R_{x_j}) Q_{x_i x_j} + Q_{x_i} Q_{x_j} P_{x_i x_j} + 2P_{x_i} R_{x_j} P_{x_i x_j}] \right) + \dots$$

$$M_2 + M_3 = \cdots + z_1^4 \{ \sum_{i=1}^{2} [(\tfrac{1}{2} Q_{x_i} + 9S_1 a_i x_i) R_{x_i} + (9S_1 P_{x_i} + 16 P a_i x_i) Q_{x_i} + 8PP_{x_i}^2 + 10 a_i x_i P_{x_i} Q]$$

$$+ (5Q + \tfrac{45}{2} S_1^2) P \} + \dots$$

$$(p-2)^{-1} (\sum_{i=1}^{2} u_{x_i}^2) \Delta u = \cdots - z^4 \left(\sum_{i=1}^{2} [(2Q_{x_i} + 2V a_i x_i) R_{x_i} + 2V P_{x_i} Q_{x_i}] + \sum_{i=1}^{2} (P_{x_i}^2 + 2a_i Q_{x_i}) W \right) + \dots$$

$$(p-2)^{-1} u_z^2 \Delta u = \cdots - z^4 \left((\tfrac{1}{2} Q + \tfrac{9}{4} S_1^2) W + 4PQ + 6PS_1 V \right) + \dots \tag{4.19}$$

Adding the rows in (4.19) and putting the resulting coefficient of $z^4 = 0$, we get

$$\sum_{i,j=1}^{2} (2Q_{x_i} a_j x_j + P_{x_i} P_{x_j}) R_{x_i x_j} + \sum_{i,j=1}^{2} (2P_{x_i} P_{x_i x_j} + 2a_i x_i Q_{x_i x_j}) R_{x_j} + \sum_{i=1}^{2} (2a_i - \tfrac{3}{2}) \, Q_{x_i} R_{x_i} +$$

$$(3S_1 - 16S_2 - 32S_3) \sum_{i=1}^{2} a_i x_i R_{x_i} + \sum_{i,j=1}^{2} [2P_{x_i} Q_{x_j} Q_{x_i x_j} + \{Q_{x_i} Q_{x_j} - (8Q + 36S_1^2) a_i x_i a_j x_j\} P_{x_i x_j}]$$

$$+ \sum_{i=1}^{2} (3S_1 - 32S_3 - 16S_2) P_{x_i} Q_{x_i} + \sum_{i=1}^{2} (12 P a_i x_i Q_{x_i} + 6 a_i x_i Q P_{x_i} - 16 Q a_i^2 x_i P_{x_i}) + \sum_{i=1}^{2} 6PP_{x_i}^2$$

$$- S_1^2 \sum_{i=1}^{2} (18 a_i + 72 a_i^2) x_i P_{x_i} - (\sum_{i=1}^{2} P_{x_i}^2 + 2a_i x_i Q_{x_i}) \left[\sum_{i,j=1}^{2} 16 a_i x_i a_j x_j P_{x_i x_j} + \sum_{i=1}^{2} (32 a_i^2 + 8a_i) x_i P_{x_i} \right]$$

$$- (96 S_3 + 48 S_2) S_1 P = 0. \tag{4.20}$$

Taking 7 derivatives with respect to x_1 in (4.20) and using (4.5), (4.7), we obtain

$$0 = 420 P_{x_1^3}^2 R_{x_k^5} + (560 a_1 - \tfrac{105}{2}) \, Q_{x_1^4} R_{x_1^5} + 210 (3a_1^2 - 16a_1^3 - 32a_1^4) R_{x_k^5} + 560 P_{x_1^3} Q_{x_1^4}^2$$

$$- (67200 a_1^3 + 16800 a_1^2 - 3570 a_1) P_{x_1^3} Q_{x_1^4} + \tfrac{70}{3} (54 - 216 a_1 - 1728 a_1^2) P_{x_1^3}^3 - 840 (528 a_1^4 + 102 a_1^3) P_{x_1^3}. \tag{4.21}$$

Next we show that (4.9), (4.15), (4.18), (4.21), imply

$$P_{x_1^3} = 0. \tag{4.22}$$

To make our notation less bulky we put $a_1 = a, P_{x_1^3} = 6A, Q_{x_1^4} = 24C, R_{x_1^4} = 120G$ and rewrite the above equations in A, C, G. We then divide the equation corresponding to (4.9) by $4\,! = 24$, the equation corresponding to (4.15) by $5! = 120$, the equation corresponding to (4.18) by $6! = 720$, and the one corresponding to (4.21) by $7! = 5040$. We deduce the following equations listed in the same order as the corresponding display:

$$\begin{aligned}
&(a) \quad 45aA^2 + 20a^2C - 16a^5 - 20a^4 = 0 \\
&(b) \quad (30a^2 - \tfrac{5}{2}a)G + (-288a^4 - 232a^3 + 11a^2)A + 54A^3 + (144a - 6)AC = 0 \\
&(c) \quad (210a - 15)AG + (112a - 8)C^2 + (26a^2 - 72a^3 - 128a^4)C + 252A^2C \\
&\qquad + (51a - 408a^2 - 1296a^3)A^2 - (36a^5 + 18a^4) = 0 \\
&(d) \quad 360A^2G + (320a - 30)\,CG + (15a^2 - 80a^3 - 160a^4)G + 384AC^2 \\
&\qquad -(1920a^3 + 480a^2 - 102a)AC + (54 - 216a - 1728a^2)A^3 - (528a^4 + 102a^3)A = 0.
\end{aligned} \tag{4.23}$$

To begin the proof of (4.22) first suppose $a = 0$ and $A = P_{x_1^3}/6 \neq 0$. Then from (4.23) (b) we see that $C = 9A^2$. Using this equality in (4.23) (c) we deduce that $G = 108A^3$. Substituting for G, C in (4.23) (d) we find that $40824A^5 + 54A^3 = 0$, which implies $A^2 < 0$, a contradiction. Thus in proving (4.22) we may assume, $a \neq 0$.

Using (a) of (4.23) and $a \neq 0$, it follows that

$$A^2 = -\tfrac{4}{9}aC + \tfrac{16}{45}a^4 + \tfrac{4}{9}a^3 \text{ so } aC \leq \tfrac{4}{5}a^4 + a^3 \tag{4.24}$$

Using (4.24) in (4.23) (b) we find that

$$\begin{aligned}
(30a^2 - \tfrac{5}{2}a)G &= -54A(-\tfrac{4}{9}aC + \tfrac{16}{45}a^4 + \tfrac{4}{9}a^3) \\
&\quad + A(288a^4 + 232a^3 - 11a^2) - (144a - 6)AC \\
&= A(\tfrac{1344}{5}a^4 + 208a^3 - 11a^2) - (120a - 6)AC\,.
\end{aligned} \tag{4.25}$$

Multiplying (4.23)(c) by $30a^2 - \frac{5}{2}a$ and using (4.25) to substitute for G we see that

$$\begin{aligned}
0 &= \{(210a - 15)A^2((\tfrac{1344}{5}a^4 + 208a^3 - 11a^2) - (120a - 6)C)\} + \{(30a^2 - \tfrac{5}{2}a)[(112a - 8)C^2 \\
&\quad + (26a^2 - 72a^3 - 128a^4)C + 252A^2C + (51a - 408a^2 - 1296a^3)A^2 - (36a^5 + 18a^4)]\} \\
&= T_1 + T_2.
\end{aligned} \tag{4.26}$$

Substituting for A^2 from (4.24) in T_1 and multiplying out the resulting expression we have

$$\begin{aligned}
T_1 &= (40a - 1360a^2 + 11200a^3)C^2 + (-34048a^6 - \tfrac{83200}{3}a^5 + \tfrac{11224}{3}a^4 - \tfrac{340}{3}a^3)C \\
&\quad + \tfrac{100352}{5}a^9 + \tfrac{587776}{15}a^8 + \tfrac{47072}{3}a^7 - \tfrac{7064}{3}a^6 + \tfrac{220}{3}a^5.
\end{aligned} \tag{4.27}$$

Also, substituting for A^2 from (4.24) we find that

$$
\begin{aligned}
T_2 &= (30a^2 - \tfrac{5}{2}a)[(112a-8)C^2 + (26a^2 - 72a^3 - 128a^4)C+ \\
&(252C + 51a - 408a^2 - 1296a^3)(-\tfrac{4}{9}aC + \tfrac{16}{45}a^4 + \tfrac{4}{9}a^3) - (36a^5 + 18a^4)] \\
&= (-240a^2 + 20a)C^2 + (16128a^6 + 5296a^5 - \tfrac{1360}{3}a^4 - \tfrac{25}{3}a^3)C \\
&-13824a^9 - 20480a^8 - \tfrac{12520}{3}a^7 + 638a^6 - \tfrac{35}{3}a^5.
\end{aligned}
\tag{4.28}
$$

Adding (4.27), (4.28) it follows from (4.26) that

$$
\begin{aligned}
0 &= (11200a^3 - 1600a^2 + 60a)C^2 + (-17920a^6 - \tfrac{67312}{3}a^5 + 3288a^4 - \tfrac{365}{3}a^3)C \\
&+\tfrac{31232}{5}a^9 + \tfrac{280576}{15}a^8 + \tfrac{34552}{3}a^7 - \tfrac{5150}{3}a^6 + \tfrac{185}{3}a^5.
\end{aligned}
\tag{4.29}
$$

Multiplying (4.23) (d) by $(30a^2 - \frac{5}{2}a)$, dividing out an A and using (4.24), (4.25) as in the derivation of (4.29) yields

$$
\begin{aligned}
&0 = [360(-\tfrac{4}{9}aC + \tfrac{16}{45}a^4 + \tfrac{4}{9}a^3) + (320a - 30)C + (15a^2 - 80a^3 - 160a^4)] \\
&\times[\tfrac{1344}{5}a^4 + 208a^3 - 11a^2 - (120a - 6)C] + (30a^2 - \tfrac{5}{2}a)[384C^2 - (1920a^3 + 480a^2 - 102a)C+ \\
&(54 - 216a - 1728a^2)(-\tfrac{4}{9}aC + \tfrac{16}{45}a^4 + \tfrac{4}{9}a^3) - (528a^4 + 102a^3)] = \\
&-\tfrac{135168}{5}a^8 - 8960a^7 + 4992a^6 + 1412a^5 + 30a^4 \\
&(+12288a^5 + 6784a^4 - 6020a^3 + 225a^2)C + (-7680a^2 + 3600a - 180)C^2
\end{aligned}
\tag{4.30}
$$

Note that (4.29), (4.30) give us two quadratic equations in the variable C with coefficients that are polynomials in a. To study these equations, let

$$
\begin{aligned}
&h_1 = 371589120a^6 + 55296000a^5 - 109541120a^4 + 26469120a^3 - 2185184a^2 + 59280a + 25 \\
&g_1 = 53760a^3 + 67312a^2 - 9864a + 365) \\
&f_1 = 120(560a^2 - 80a + 3) \\
&h_2 = -679477248a^6 + 280756224a^5 + 160989184a^4 - 111109120a^3 + 23476240a^2 \\
&-2124360a + 72225 \\
&g_2 = 12288a^3 + 6784a^2 - 6020a + 225 \\
&f_2 = 120(128a^2 - 60a + 3) \\
&\hat{e}_{k,1} = a^2(g_k + \sqrt{h_k})/f_k, \quad \hat{e}_{k.2} = a^2(g_k - \sqrt{h_k})/f_k, \text{ for } \ k = 1, 2.
\end{aligned}
\tag{4.31}
$$

We note from the quadratic formula and (4.29), (4.30) that $\hat{e}_{k,1}$, $\hat{e}_{k,2}$, are the solutions to (4.29), when $k=1$ and to (4.30) when $k=2$. We shall prove for $A, a \neq 0$ that the inequalities,

$$\hat{e}_{1,l} = \hat{e}_{2,m}, \qquad a\,\hat{e}_{1,l} \leq a^3 + (4/5)a^4 \text{ for some } m, l \in \{1,2\} \tag{4.32}$$

have no real solutions. To see this we first show that

$$h_2 < 0 \text{ for } a \geq \frac{4304}{10000} \text{ and } \quad a \leq \frac{-60521}{100000}. \tag{4.33}$$

In fact the fifth derivative of h_2 is easily seen to be negative on $(\frac{4}{10}, \infty)$. Working backward using Maple to calculate derivatives exactly at $\frac{4}{10}$ and elementary calculus we get that the fourth - first derivatives of h_2 are negative on $[\frac{4}{10}, \infty)$. Thus h_2 is decreasing on $(\frac{4}{10}, \infty)$ and using Maple we found $h_2(\frac{4304}{10000}) < 0$ (exactly) so (4.33) is valid for $a \geq \frac{4304}{10000}$. We can use the same argument to prove (4.33) for $a \in (-\infty, -.60521]$, only now successive derivatives alternate in sign. We omit the details.

Next we show that

$$h_1 < 0 \text{ on } (-.7, -.0004154). \tag{4.34}$$

To see this note that the fifth derivative of h_1 is negative on some $[-.7, x_0), x_0 < 0$, and positive on $(x_0, 0]$. Since $\frac{d^4h_1}{da^4}(-.7) > 0$ while $\frac{d^4h_1}{da^4}(0) < 0$ it follows from elementary calculus that $\frac{d^4h_1}{da^4} > 0$ on some $[-.7, x_1), x_1 < x_0$ and < 0 on $(x_1, 0)$. Likewise, $\frac{d^3h_1}{da^3}(-.7) < 0, \frac{d^3h_1}{da^3}(0) > 0$, so $\frac{d^3h_1}{da^3} < 0$ on some $(-.7, x_2), x_2 < x_1$, and > 0 on $(x_2, 0)$. Continuing this argument we find that $\frac{d^2h_1}{da^2}(-.7) < 0$, $\frac{d^2h_1}{da^2}(0) > 0$ and thereupon that $\frac{d^2h_1}{da^2} < 0$ on some $[-.7, x_3), x_3 < x_2$, and > 0 on $(x_3, 0]$. Finally since $\frac{dh_1}{da}(-.7) < 0, \frac{dh_1}{da}(0) > 0$, we see that h_1 decreases on some $[-.7, x_4], x_4 < x_3$ and increases on $[x_4, 0]$ From this fact and $h_1(-.7) < h_1(-.0004154) < -.003$ (evaluated exactly), we conclude that (4.34) is true.

Note that (4.33), (4.34), show for the intervals listed that at least one of the equations in (4.33), (4.34) has only complex solutions. Consequently the equations in (4.32) have no real solutions on the intervals listed in (4.33), (4.34).

To continue the proof of nonexistence for the inequalities in (4.32) we now show that if $e_{k.l} = \hat{e}_{k,l}/a^2$, $a \neq 0$, $1 \leq k, l \leq 2$, then

$$e_{1,1} > e_{1,2} > 1 + \tfrac{4}{5}a \text{ at real values when } \frac{1}{12} < a \leq \frac{45}{100} \tag{4.35}$$

To prove (4.35) we first note from (4.31) that $f_1 > 0$ for all a, so clearly $e_{1,1} > e_{1,2}$ at real values. Second we note again from (4.31) that $g_1 - (1 + \frac{4}{5}\,a)f_1 = 7792a^2 - 552a + 5 > 0$ for all $a > .07$ since this quadratic has zeros at $\frac{69}{1948} \pm \frac{1}{1948}\sqrt{2326} < .07$. So the righthand inequaliy in (4.35) is equivalent to showing that

$$k_1 = h_1 - (g_1 - (1 + \tfrac{4}{5}a)f_1)^2 = 864a(12a-1)(64a^2 + 24a - 25)(3 + 560a^2 - 80a) < 0 \tag{4.36}$$

on $(\frac{1}{12}, \frac{45}{100}]$, which is easily checked.

Next we show that

$$e_{2,1} > 1 + \tfrac{4}{5}a \text{ at real values when } 0 \leq a \leq \tfrac{1}{10} \text{ and } e_{2,2} > e_{2,1} \text{ at real values in } [\tfrac{15-\sqrt{129}}{64}, \tfrac{1}{10}] \tag{4.37}$$

Indeed it follows from the quadratic formula that $f_2 = 0$ at $\frac{15-\sqrt{129}}{64} \approx .056909$ and $\frac{15+\sqrt{129}}{64} \approx .4118409$. Using this fact, (4.31), and ballpark estimates we deduce first that $f_2 < 0$ on $[\frac{15-\sqrt{129}}{64}, \frac{1}{10}]$. Second, arguing as in the proof of (4.33)- (4.35), we see that $g_2 < 0$ on $[\frac{15-\sqrt{129}}{64}, \frac{1}{10}]$. Thus the second inequality in (4.37) is valid. We divide the proof of the first inequality in (4.37) into the subcases (a) $[\frac{15-\sqrt{129}}{64}, \frac{1}{10}]$ and (b) $[0, \frac{15-\sqrt{129}}{64}]$. To prove subcase (a) first observe that $(1+\frac{4}{5}a)f_2 - g_2 = 2816a^2 - 892a + 135 > 0$ for all a so since $f_2 < 0$, on $[\frac{15-\sqrt{129}}{64}, \frac{1}{10}]$, it suffices to show in subcase (a) that

$$\begin{aligned} j_2 = h_2 - [(1+\tfrac{4}{5}a)f_2 - g_2]^2 = \\ -144(128a^2 - 60a + 3)(36864a^4 + 2048a^3 - 8208a^2 + 1860a - 125) < 0 \end{aligned} \tag{4.38}$$

on this interval or in view of the above remark that

$$j = 36864a^4 + 2048a^3 - 8208a^2 + 1860a - 125 < 0 \text{ on } [0, \tfrac{1}{10}], \tag{4.39}$$

which is easily shown. On the other hand if $a \in [0, (15-\sqrt{129})/64]$. then $f_2 > 0$ so it suffices in subcase (b) to show j_2 in (4.38) > 0 or that (4.39) holds on $[0, \frac{1}{10}]$ which we have already stated is true. Thus (4.37) is valid. Next we show that

$$e_{2,1} > 1.2 \text{ at real values in [-.0004154, 0] while } e_{1,2} < e_{1,1} \leq 1.1 \text{ at real values in this interval.} \tag{4.40}$$

To prove the first inequality we recall that $f_2 > 0$ for $a \leq 0$ and note that $(\frac{12}{10})f_2 - g_2 = -12288a^3 + 11648a^2 + 207 - 2620a > 0$ on $[-.0004154, 0)$. So it suffices to show that

$$h_2 - ((\tfrac{12}{10})f_2 - g_2)^2 = -192(128a^2 - 60a + 3))(33792a^4 - 7232a^3 - 2592a^2 + 785a - 51) > 0$$

on $[-.0004154, 0]$, which follows easily from ballpark type estimates. To handle the second inequality, we point out once again that $e_{1,2} < e_{1,1}$ at real values since $f_1 > 0$ for all a. Also $\frac{11}{10}f_1 - g_1 = 31 - 696a + 6608a^2 - 53760a^3 > 0$ on $[-.0004154, 0]$. Thus to prove the second inequality in (4.40) it suffices to show

$$h_1 - (\frac{11}{10}f_1 - g_1)^2 = -24(3 + 560a^2 - 80a))(187392a^4 - 30208a^3 + 11648a^2 - 1076a + 13) < 0$$

on $[-.0004154, 0]$, which is clearly true. Finally we show that

$$e_{2,2} < 0 < e_{1,2} < e_{1,1} \text{ at real values of alll three functions in } [-.0004154, \frac{15-\sqrt{129}}{64}] \tag{4.41}$$

Indeed since $f_2 > 0$ on $[-.0004154, \frac{15-\sqrt{129}}{64}]$ it suffices to show for the left hand inequality that

$$g_2^2 - h_2 = (96(22a+5)(128a^2 - 60a + 3)(3072a^3 + 320a^2 - 640a - 15) < 0$$

or just that $3072a^3 + 320a^2 - 640a - 15 < 0$ on $[-.0004154, \frac{15-\sqrt{129}}{64}]$. The latter inequality is easily proven so we omit the details. Finally, $e_{1,2} < e_{1,1}$ is now obvious and also $g_1 > 0$ on $[-.0004154, \frac{15-\sqrt{129}}{64}]$ is easily shown so to prove $e_{1,2} > 0$ on this interval it suffices to show that

$$g_1^2 - h_1 = 48(3 - 80a + 560a^2)(93696a^4 + 280576a^3 + 172760a^2 - 25750a + 925) > 0$$

or just that $q = 93696a^4 + 280576a^3 + 172760a^2 - 25750a + 925 > 0$ on $[-.0004154, \frac{15-\sqrt{129}}{64}]$. To prove this inequality we note that

$$q > 172760a^2 - 25750a + 925$$

and this quadratic has roots at $\frac{2575}{34552} \pm \frac{\sqrt{238505}}{34552} > \frac{15-\sqrt{129}}{64}$ so is positive on the given interval. Thus (4.41) is valiid.

From (4.33)-(4.41) we conclude that (4.32) has no real solutions when $A \neq 0$. Thus (4.22) is true under assumption (4.6).

To continue the proof of Theorem 1.2 under assumption (4.6) we observe from (4.22), (4.7), and $\Delta P = 0$ that $P \equiv 0$. Using this fact in (2.4)-(2.9) we calculate the coefficient of z^3 in the expansion of (2.1) in powers of z. We get

$$\begin{aligned} M_1 &= \dots + z^3 \left(\sum_{i,j=1}^{2} [2a_i x_i R_{x_j} R_{x_i x_j} + Q_{x_i} Q_{x_j} Q_{x_i x_j}] + \sum_{i=1}^{2} a_i R_{x_i}^2 \right) + \dots \\ M_2 + M_3 &= \dots + z^3 \left(\sum_{i=1}^{2} (3S_1 Q_{x_i}^2 + 8Q a_i x_i Q_{x_i} + 9QS_1^2) + Q^2 \right) + \dots \\ (p-2)^{-1} |\nabla u|^2 \Delta u &= \dots - z^3 \left(\sum_{i=1}^{2} (V Q_{x_i}^2 + R_{x_i}^2) + 3QS_1 V + Q^2 \right) + \dots \end{aligned} \tag{4.42}$$

Adding the rows in (4.42) and putting the resulting coefficient of $z^3 = 0$ we have

$$\begin{aligned} 0 = &\sum_{i,j=1}^{2} [2a_i x_i R_{x_j} R_{x_i x_j} + Q_{x_i} Q_{x_j} Q_{x_i x_j}] + \\ &+ \sum_{i=1}^{2} [(a_i - 1) R_{x_i}^2 - (16S_3 + 8S_2) Q_{x_i}^2 + 8Q a_i x_i Q_{x_i}] \\ &- QS_1(48S_3 + 24S_2) \end{aligned} \tag{4.43}$$

Taking eight derivatives on x_1 in (4.43) and using (4.5), (4.8), (4.14), (4.22), we get

$$(630a_1 - 70) R_{x_1^5}^2 + 560 Q_{x_1^4}^3 - 1120(16a_1^3 + 8a_1^2 - 2a_1) Q_{x_1^4}^2 - (1680)(48a_1^4 + 24a_1^3) Q_{x_1^4} = 0 \tag{4.44}$$

Using once again $R_{x_1^5} = 120G, Q_{x_1^4} = 24C, a_1 = a$, rewriting (4.44) in terms of G, C and dividing the resulting expression by $8! = 40,320$, we find that

$$(225a - 25)G^2 + 192C^3 - (256a^3 + 128a^2 - 32a)C^2 - (48a^4 + 24a^3)C = 0. \tag{4.45}$$

Armed with (4.45) we now consider several possibilities for a and eventually obtain a contradiction to our assumption that u exists. First suppose $a_1 \neq 0$. Then from (4.23) (b) with $A = 0$, we see that either $G = 0$ or $a = 1/12$. If $a = 1/12$, then from (4.32) (a) we have $C = \frac{4}{5}(\frac{1}{12})^3 + (\frac{1}{12})^2 = 1/135$. Using this value for C in (4.23) (c) we obtain a rational number

$\approx 1.77748 \neq 0$. Thus $G = 0$ when $a \neq 0$. If $a \neq 0, C \neq 0$, then we can divide (4.45) by C to get a quadratic in C. Using $C = \frac{4}{5}a^3 + a^2$ in the resulting equation and also in (4.23) (c) we obtain after dividing by a^3, a^5, respectively the equations,

$$\begin{aligned}
(\alpha) \quad & 192a(\tfrac{4}{5}a+1)^2 - (256a^2 + 128a - 32)(\tfrac{4}{5}a+1) - (48a+24) \\
& \qquad = \tfrac{1}{25}(-2048a^3 - 1280a^2 + 1040a + 200) = 0. \\
(\beta) \quad & a^{-1}[(112a-8)(\tfrac{4}{5}a+1)^2 + (26 - 72a - 128a^2)(\tfrac{4}{5}a+1) - (36a+18)] \\
& \qquad = \tfrac{1}{25}(-768a^2 - 288a + 300) = 0
\end{aligned} \tag{4.46}$$

Using the quadratic formula in (4.46) (β) we get $a = -\frac{3 \pm \sqrt{109}}{16}$. Putting these values of a into (4.46) (α) we obtain $\frac{-81 \pm 27\sqrt{109}}{25} = 0$, a contradiction. On the other hand if $a \neq 0, C = 0$, then from (4.23) (a) we have $a = -5/4$ while from (4.23) (c) we deduce $a = -1/2$. Hence $a = 0$ and from (4.23) (c) we deduce $C = 0$ while from (4.45) we have $G = 0$. From $A = C = G = 0$, the definition of these letters, (4.5), (4.8), (4.14), and (2.12) $(a), (b)$ with $m = 5$, we deduce that $P \equiv Q \equiv R \equiv 0$. From these equalities and (4.1) it follows that either $a_2 = 0$ or $a_2 = -\frac{5}{4}$. $a_2 = 0$ is not allowed since $p \neq -1$. Moreover from (4.17) we have either $a_2 = 0$ or $a_2 = -\frac{1}{2}$, which contradicts the above. We have now considered all possible cases and reached a contradiction in each case (when $a_1 \neq a_2$) to our assumption that u is a solution to (2.1).

It remains to remove assumption (4.6). If (4.6) holds, i.e, $a_1 = a_2$, then (2.1) is invariant under rotations in the x variable, so we can choose the x_1 axis in such a way that the maximum of P in $\{x : x_1^2 + x_2^2 \leq 1\}$ occurs at $x_1 = 1, x_2 = 0$. Then necessarily, $P(x) = c(x_1^3 - 3x_2^2 x_1)$ for some $c > 0$. Thus we do not need (4.6) to prove (4.7). The only other places we used $a_1 \neq a_2$ was in the derivation of (4.14) to rule out the possibility that $a_1 = a_2 = 1/12$ and in the use of (4.14) to derive (4.44). In fact all equations in (4.12), (4.13) reduce to $0 = 0$ when $a_1 = a_2 = 1/12$, so we cannot use (4.14). However, (4.23) $(a) - (d)$ are still valid. Using $a = 1/12$ in (4.23) (b) we see that if $A \neq 0$, then from (4.25) we have

$$C = (1/4)[(1344/5)(\frac{1}{12})^4 + 208(\frac{1}{12})^3 - 11(\frac{1}{12})^2] = \frac{41}{2880} \approx .014236\,.$$

On the other hand from (4.24) we have $C \leq (4/5)(\frac{1}{12})^3 + (\frac{1}{12})^2 = 1/135 \approx .007407$. Since the two inequalities contradict each other we conclude that $A = 0$. It follows that $C = 1/135$. Using $a = 1/12, C = 1/135$ in (4.23)(c) we arrive as earlier at a postive rational number $\approx 1.77748 = 0$, a contradiction to our assumption that $a_1 = a_2 = 1/12$. This concludes the proof of Theorem 1.2.

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