

Regularity and Free Boundary Regularity for the p -Laplace Operator in Reifenberg Flat and Ahlfors Regular Domains

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Abstract

In this paper we solve several problems concerning regularity and free boundary regularity, below the continuous threshold, for positive solutions to the p -Laplace equation, $1 < p < \infty$, vanishing on a portion of the boundary of an Ahlfors regular NTA-domain. In Theorem 1 of our paper we show that if $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is an Ahlfors regular NTA-domain and u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, with continuous boundary value 0 on $\partial\Omega \cap B(w, 4r)$, then $\nabla u(x) \rightarrow \nabla u(y)$ non-tangentially as $x \rightarrow y \in \partial\Omega \cap B(w, 4r)$, almost everywhere with respect to surface area, σ , on $\partial\Omega \cap B(w, 4r)$. Moreover, $\log |\nabla u|$ is of bounded mean oscillation on $\partial\Omega \cap B(w, r)$ with $\|\log |\nabla u|\|_{\text{BMO}(\partial\Omega \cap B(w, r))} \leq c$. If, in addition, Ω is Reifenberg flat with vanishing constant and $n \in \text{VMO}(\partial\Omega \cap B(w, 4r))$, where n denotes the unit inner normal to $\partial\Omega$ in the measure theoretic sense, then in Theorem 2 we prove that $\log |\nabla u| \in \text{VMO}(\partial\Omega \cap B(w, r))$. In Theorem 3 we prove the following converse to Theorem 2. Suppose u is as in Theorem 1, $\log |\nabla u| \in \text{VMO}(\partial\Omega \cap B(w, r))$, and that $\partial\Omega \cap B(w, r)$ is (δ, r_0) -Reifenberg flat. Then there exists $\bar{\delta} = \bar{\delta}(p, n)$ such that if $0 < \delta \leq \bar{\delta}$, then $\partial\Omega \cap B(w, r/2)$ is Reifenberg flat with vanishing constant and $n \in \text{VMO}(\partial\Omega \cap B(w, r/2))$. Finally, in Theorem 4 we establish a two-phase version of Theorem 3 without the smallness assumption on δ .

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1 Introduction

Recall that a Jordan domain $\Omega \subset \mathbf{R}^2$ is called a chord arc domain if $\partial\Omega$ is locally rectifiable and if there exists a constant λ , $0 < \lambda < \infty$, such that

$$\sigma(\gamma(w_1, w_2)) \leq \lambda|w_1 - w_2| \text{ for all } w_1, w_2 \in \partial\Omega, w_1 \neq w_2,$$

where $\gamma(w_1, w_2)$ is the shortest arc in the boundary which joins w_1 and w_2 and $\sigma(\gamma(w_1, w_2))$ is its length. Ω is called a vanishing chord arc domain if, in addition,

$$\frac{\sigma(\gamma(w_1, w_2))}{|w_1 - w_2|} = 1 + o(1) \text{ uniformly on compact subset of } \partial\Omega \text{ as } |w_1 - w_2| \rightarrow 0.$$

Let $\omega(\cdot) = \omega(\cdot, x)$ denote the harmonic measure associated to the Laplace operator and defined with respect to Ω and $x \in \Omega$. A classical result concerning the harmonic measure, due to Lavrentiev [L], states that if $\Omega \subset \mathbf{R}^2$ is a chord arc domain, then ω is mutually absolutely continuous with respect to σ , i.e., $d\omega = kd\sigma$ where k is the associated Poisson kernel. Moreover, Lavrentiev [L] proved that $\log k$ is in the space of functions of bounded mean oscillation, defined with respect to σ , on $\partial\Omega$. Later Pommerenke [P] proved that Ω is vanishing chord arc if and only if $\log k$ is in the space of functions of vanishing mean oscillation, defined with respect to σ , on $\partial\Omega$. Thus Pommerenke's theorem gives a characterization of the set of all planar vanishing chord arc domains in terms of the behaviour of the Poisson kernel. Concerning higher dimensional analogues of the results of Lavrentiev and Pommerenke, such results have recently been obtained by Kenig and Toro in a sequence of papers, see [KT,KT1-KT3]. In these papers the authors establish a number of results concerning the regularity and free boundary regularity, below the continuous threshold, for the Laplace equation in Reifenberg flat and Ahlfors regular domains. In particular, as an analogue of Pommerenke's result the authors obtain a characterization, in terms of the behaviour of the Poisson kernel, of what they refer to as 'chord arc domains with vanishing constant.'

The purpose of this paper is to establish appropriate versions, valid for the p -Laplace equation, $1 < p < \infty$, of the results in [KT,KT1-KT3]. While the results in [KT,KT1-KT3] concern harmonic functions and harmonic measure, i.e., the case $p = 2$, the results proved in this paper are valid for $1 < p < \infty$ and our results are completely new when $p \neq 2$. Consequently we also establish versions, valid in all dimensions, for the p -Laplace equation, $1 < p < \infty$, of the classical results of Lavrentiev [L] and Pommerenke [P] mentioned above.

The results in this paper build on the techniques and results established in [LN,LN1-LN6]. In these papers we study the regularity and free boundary regularity of p -harmonic functions, $p \neq 2$, $1 < p < \infty$, in Lipschitz domains and in domains which are well approximated by Lipschitz domains in the Hausdorff distance sense. To briefly outline these results we note that in [LN] we established the boundary Harnack inequality for positive p -harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$ and we carried out an in depth analysis of p -capacitary functions in starlike Lipschitz ring domains. The study in [LN] was continued in [LN1] where we proved Hölder continuity for ratios of positive p -harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$. In [LN1] we also studied the Martin boundary problem for p -harmonic functions in Lipschitz domains. In [LN], [LN2] we established, in the setting of Lipschitz and C^1 -domains the ' p -harmonic' analogues, $1 < p < \infty$, of theorems proved for harmonic functions

in [D], [JK], [J], [KT], [KT1] and [KT2] on regularity and free boundary regularity, below the continuous threshold, for the Poisson kernel associated to the Laplace operator. The results in this paper can be viewed as an extension of the results in [LN2] to Reifenberg flat and Ahlfors regular domains. We refer to [LN3] for a survey of the results in [LN, LN1, LN2]. In [LN4] we proved the boundary Harnack inequality and Hölder continuity for ratios of p -harmonic functions vanishing on a portion of certain Reifenberg flat and Ahlfors regular NTA-domains and we gave applications to the p -Martin boundary problem for these domains. Finally, in [LN5, LN6] we generalized the results in [C, C1] concerning general two-phase free boundary problems for the Laplace operator to the p -Laplace operator, $1 < p < \infty$. In [LN5] we also gave an application of our results to the free boundary-inverse type problem studied in [LN2]. Moreover, while the analysis in [LN2] is closely linked to [LN, LN1], and [LN5], this paper is more in the flavour of [LN4] and [LN6].

To properly state the results in this paper we need to introduce some notation. Points in Euclidean n -space \mathbf{R}^n are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. We let $\bar{E}, \partial E, \text{diam } E$, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$ and we define $d(y, E)$ to equal the distance from $y \in \mathbf{R}^n$ to E . $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbf{R}^n and we let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x . $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ is defined whenever $x \in \mathbf{R}^n, r > 0$, and dx denotes Lebesgue n -measure on \mathbf{R}^n . Let

$$h(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$$

be the Hausdorff distance between the sets $E, F \subset \mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O . Let $\|f\|_{1,q} = \|f\|_q + \|\nabla f\|_q$ be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue q norm in O . Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given a bounded domain G , i.e., a connected open set, and $1 < p < \infty$, we say that u is p -harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that if u is smooth and $\nabla u \neq 0$ in G , then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \text{ in } G \tag{1.2}$$

and u is a classical solution in G to the p -Laplace partial differential equation. Here, as in the sequel, $\nabla \cdot$ is the divergence operator.

In the following the notion of NTA-domains introduced in [JK1] will be important. We note that (iii) in our definition of NTA-domains below is different but equivalent to the usual Harnack chain condition given in [JK1] (see [BL], Lemma 2.5). We choose this definition in order to emphasize the dependence of Ω on M . M, r_0 , will be called the NTA-constants of Ω .

Definition 1.1 A domain Ω is called *non-tangentially accessible (NTA)* if there exist $M \geq 2$ and r_0 such that the following are fulfilled:

- (i) *corkscrew condition: for any $w \in \partial\Omega$, $0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying $M^{-1}r < |a_r(w) - w| < r$, $d(a_r(w), \partial\Omega) > M^{-1}r$,*
- (ii) $\mathbf{R}^n \setminus \bar{\Omega}$ *satisfies the corkscrew condition,*
- (iii) *uniform condition: if $w \in \partial\Omega$, $0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$, then there exists a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = w_1$, $\gamma(1) = w_2$, and such that*
 - (a) $H^1(\gamma) \leq M |w_1 - w_2|$,
 - (b) $\min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq M d(\gamma(t), \partial\Omega)$.

Given a domain $\Omega \subset \mathbf{R}^n$, $w \in \partial\Omega$, $0 < r < \infty$, we let $\Delta(w, r) = \partial\Omega \cap B(w, r)$.

Definition 1.2 Let $\Omega \subset \mathbf{R}^n$ be a domain and $r_0, \delta > 0$. Then Ω and $\partial\Omega$ are said to be (δ, r_0) -Reifenberg flat provided that there exists, whenever $w \in \partial\Omega$ and $0 < r < r_0$, a hyperplane $P = P(w, r)$ containing w such that

- (a) $h(\Delta(w, r), P \cap B(w, r)) \leq \delta r$
- (b) $\{x \in \Omega \cap B(w, r/2) : d(x, \partial\Omega) \geq 2\delta r\} \subset \text{one component of } \mathbf{R}^n \setminus P$.

For short we say that Ω and $\partial\Omega$ are δ -Reifenberg flat if Ω and $\partial\Omega$ are (δ, r_0) -Reifenberg flat for some $r_0 > 0$. We note that an equivalent definition of Reifenberg flat domains is given in [KT]. As in [KT] one can show that a δ -Reifenberg flat domain is a NTA-domain with constant $M = M(n)$, provided $0 < \delta < \hat{\delta}$, and $\hat{\delta} = \hat{\delta}(n)$ is small enough.

Definition 1.3 Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0) -Reifenberg flat domain for some (δ, r_0) , $0 < \delta < \hat{\delta}$, $r_0 > 0$, and let $w \in \partial\Omega$, $0 < r < r_0$. We say that $\Delta(w, r)$ is *Reifenberg flat with vanishing constant* if there exists, if for each $\epsilon > 0$, $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $x \in \Delta(w, r)$ and $0 < \rho < \tilde{r}$, then there exists a plane $P' = P'(x, \rho)$ containing x such that

$$h(\Delta(x, \rho), P' \cap B(x, \rho)) \leq \epsilon \rho.$$

Given a bounded domain Ω we let σ denote the restriction of the $(n - 1)$ -dimensional Hausdorff measure to $\partial\Omega$.

Definition 1.4 Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. We say that Ω and $\partial\Omega$ are *Ahlfors regular* provided that there exist $r_0 > 0, C \geq 1$, such that

$$C^{-1} \leq \frac{\sigma(\Delta(w, r))}{r^{n-1}} \leq C$$

whenever $w \in \partial\Omega$, $0 < r < r_0$.

We note that the left hand inequality in the above display is always valid in a NTA-domain where now $C = C(M)$ as follows easily from (i), (ii) of Definition 1.1 and the fact that Hausdorff measure decreases under a projection.

Next let $\Omega \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain and let $w \in \partial\Omega, 0 < r < r_0$. For $0 < b < 1, y \in \partial\Omega$ we let

$$\Gamma(y) = \Gamma_b(y) = \{x \in \Omega : d(x, \partial\Omega) > b|x - y|\}. \quad (1.3)$$

Given a measurable function k on $\cup_{y \in \Delta(w, 2r)} \Gamma(y) \cap B(w, 4r)$ we define the non-tangential maximal function $N(k) : \Delta(w, 2r) \rightarrow \mathbf{R}$ for k as

$$N(k)(y) = \sup_{x \in \Gamma(y) \cap B(w, 4r)} |k|(x) \text{ whenever } y \in \Delta(w, 2r). \quad (1.4)$$

We let $L^q(\Delta(w, 2r)), 1 \leq q \leq \infty$, be the space of functions which are integrable, with respect to the surface measure, σ , to the power q on $\Delta(w, 2r)$. Furthermore, given a measurable function f on $\Delta(w, 2r)$ we say that f is of bounded mean oscillation on $\Delta(w, r), f \in \text{BMO}(\Delta(w, r))$, if there exists $A, 0 < A < \infty$, such that

$$\int_{\Delta(y, s)} |f - f_\Delta|^2 d\sigma \leq A^2 \sigma(\Delta(y, s)) \quad (1.5)$$

whenever $y \in \Delta(w, r)$ and $0 < s \leq r$. Here f_Δ denotes the average of f on $\Delta = \Delta(y, s)$ with respect to the surface measure σ . The least A for which (1.5) holds is denoted by $\|f\|_{\text{BMO}(\Delta(w, r))}$. If f is a vector valued function, $f = (f_1, \dots, f_n)$, then $f_\Delta = (f_{1,\Delta}, \dots, f_{n,\Delta})$ and the BMO-norm of f is defined as in (1.5) with $|f - f_\Delta|^2 = \langle f - f_\Delta, f - f_\Delta \rangle$. Also, we say that f is of vanishing mean oscillation on $\Delta(w, r), f \in \text{VMO}(\Delta(w, r))$, provided $f \in \text{BMO}(\Delta(w, r))$ and provided for each $\epsilon > 0$ there is a $\eta > 0$ such that (1.5) holds with A replaced by ϵ whenever $0 < s < \min(\eta, r)$ and $x \in \Delta(w, r)$. For more on BMO we refer to [S, chapter IV].

Let $\Omega \subset \mathbf{R}^n$ be an Ahlfors regular NTA-domain with constants M, r_0 , and C . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then there exists, see Lemma 2.5 in section 2, a unique locally finite positive Borel measure μ on \mathbf{R}^n , with support in $\Delta(w, 4r)$, such that

$$\int_{\mathbf{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle dx = - \int_{\mathbf{R}^n} \theta d\mu \quad (1.6)$$

whenever $\theta \in C_0^\infty(B(w, 4r))$. Moreover, using Lemma 2.5 and Harnack's inequality for p -harmonic functions we can conclude that μ is a doubling measure in the following sense. There exists $c = c(p, n, M), 1 \leq c < \infty$, such that

$$\mu(\Delta(z, 2s)) \leq c\mu(\Delta(z, s)) \text{ whenever } z \in \Delta(w, 3r), s \leq r/c. \quad (1.7)$$

Assuming that $\Omega \subset \mathbf{R}^n$ is an Ahlfors regular NTA-domain we say that μ is an A^∞ -measure with respect to σ on $\Delta(w, 2r), d\mu \in A^\infty(\Delta(w, 2r), d\sigma)$ for short, if for some $\gamma > 0$ there exists $\epsilon = \epsilon(\gamma) > 0$ with the property that if $z \in \Delta(w, 2r), 0 < s < r$ and if $E \subset \Delta(z, s)$, then

$$\frac{\sigma(E)}{\sigma(\Delta(z, s))} \geq \gamma \text{ implies that } \frac{\mu(E)}{\mu(\Delta(z, s))} \geq \epsilon. \quad (1.8)$$

In this paper we first prove the following two theorems.

Theorem 1 *Let $\Omega \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain with constants M, r_0, C . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$ and let μ be as in (1.6). Then μ is absolutely continuous with respect to σ on $\Delta(w, 4r)$ and $d\mu \in A^\infty(\Delta(w, 2r), d\sigma)$. Moreover,*

$$\lim_{x \in \Gamma(y) \cap B(w, 4r), x \rightarrow y} \nabla u(x) \stackrel{\text{def}}{=} \nabla u(y)$$

exists for σ almost every $y \in \Delta(w, 4r)$ and for $b, 0 < b < 1$, fixed in the definition of $\Gamma(y)$. Also there exists $q > p - 1$ and a constant $c, 1 \leq c < \infty$, both depending only on p, n, M and C , such that

- (i) $N(|\nabla u|) \in L^q(\Delta(w, 2r))$,
- (ii) $\int_{\Delta(w, 2r)} |\nabla u|^q d\sigma \leq cr^{(n-1)(\frac{p-1-q}{p-1})} \left(\int_{\Delta(w, 2r)} |\nabla u|^{p-1} d\sigma \right)^{q/(p-1)}$,
- (iii) $\log |\nabla u| \in BMO(\Delta(w, r))$, $\|\log |\nabla u|\|_{BMO(\Delta(w, r))} \leq c$,
- (iv) $d\mu = |\nabla u|^{p-1} d\sigma$ σ almost everywhere on $\Delta(w, 2r)$.

Finally, $\Delta(w, 4r)$ has a tangent plane at $y \in \Delta(w, r)$ for σ almost every y . If $n(y)$ denotes the unit normal to this tangent plane pointing into $\Omega \cap B(w, 4r)$, then $\nabla u(y) = |\nabla u(y)|n(y)$.

Theorem 2 *Let $\Omega \subset \mathbf{R}^n$ be a bounded (δ, r_0) -Reifenberg flat domain, $0 < \delta < \hat{\delta}(n)$, which is also Ahlfors regular. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Assume, in addition, that $\Delta(w, 4r)$ is Reifenberg flat with vanishing constant and that $n \in VMO(\Delta(w, 4r))$. Then*

$$\log |\nabla u| \in VMO(\Delta(w, r)).$$

Concerning converse results we first prove the following theorem.

Theorem 3 *Let $\Omega \subset \mathbf{R}^n$ be a bounded (δ, r_0) -Reifenberg flat domain, $0 < \delta < \hat{\delta}(n)$, which is also Ahlfors regular. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Assume, in addition, that*

$$\log |\nabla u| \in VMO(\Delta(w, r)).$$

There exists $\bar{\delta} = \bar{\delta}(p, n), 0 < \bar{\delta} \ll 1$, such that if $\delta < \min\{\hat{\delta}, \bar{\delta}\}$ then $\Delta(w, r/2)$ is Reifenberg flat with vanishing constant and $n \in VMO(\Delta(w, r/2))$.

Finally we formulate a two-phase version of Theorem 3 without the flatness condition imposed on Ω through $\bar{\delta}$. In particular, let $\Omega^1 \subset \mathbf{R}^n$ and $\Omega^2 \subset \mathbf{R}^n$ be two Ahlfors regular NTA-domains with constants M, r_0, C . Moreover, assume, for some $w \in \partial\Omega^1 \cap \partial\Omega^2$ and $0 < r < r_0$, that

$$\Omega^1 \cap \Omega^2 \cap B(w, 16r) = \emptyset, \quad \partial\Omega^1 \cap B(w, 16r) = \partial\Omega^2 \cap B(w, 16r). \quad (1.9)$$

If (1.9) holds then we let $\Delta(w, s) = \partial\Omega^1 \cap B(w, s) = \partial\Omega^2 \cap B(w, s)$ for all $0 < s \leq 16r$ and we let σ denote the restriction of the $(n-1)$ -dimensional Hausdorff measure to $\Delta(w, 16r)$. We can now state the fourth and last main theorem proved in this paper.

Theorem 4 *Let $\Omega^1, \Omega^2 \subset \mathbf{R}^n$ be bounded domains satisfying (1.9) for some $w \in \mathbf{R}^n$, $0 < r < r_0$. Assume that Ω^1, Ω^2 are Ahlfors regular NTA-domains. Let u^i , for $i \in \{1, 2\}$, denote a positive p -harmonic function in $\Omega^i \cap B(w, 16r)$, and assume that u^i is continuous in $\bar{\Omega}^i \cap B(w, 16r)$ with $u^i = 0$ on $\Delta(w, 16r)$. If $\log |\nabla u^i| \in VMO(\Delta(w, 4r))$ for $i = 1, 2$, then $\Delta(w, r/2)$ is Reifenberg flat with vanishing constant and $n \in VMO(\Delta(w, r/2))$.*

Next we briefly discuss the proofs of Theorem 1-Theorem 4 and we start by discussing an important and recurrent theme in several of our arguments. Assume that $\Omega \subset \mathbf{R}^n$ is a bounded NTA-domain with constants M, r_0 , and let $p, 1 < p < \infty$, be given. We say that Ω supports the ‘fundamental inequality’ if there exist constants \bar{c} and \bar{a} , $1 \leq \bar{c}, \bar{a} < \infty$, which only depend on p, n, M , such that the following holds whenever $w \in \partial\Omega$, $0 < r < r_0$. Suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0$ on $\Delta(w, 4r)$. Then

$$\bar{a}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{a} \frac{u(y)}{d(y, \partial\Omega)} \quad \text{whenever } y \in \Omega \cap B(w, r/\bar{c}). \quad (1.10)$$

(1.10) need not hold in an arbitrary NTA-domain but we have been able to establish (1.10) when Ω is a starlike Lipschitz ring domain, see [LN, Lemma 2.5], when Ω is a Lipschitz domain, see [LN1] and Theorem 2.7 in Section 2, and when Ω is a NTA-domain which can be uniformly approximated by Lipschitz graph domains, see [LN4]. The last class of domains includes the case of δ -Reifenberg flat domains, with δ sufficiently small, see Theorem 2.8 in Section 2. If (1.10) holds, then from Lemma 2.4 in Section 2 it follows that u is infinitely differentiable in $\Omega \cap B(w, r/\bar{c})$ and hence a strong solution to (1.2) in $\Omega \cap B(w, r/\bar{c})$. Differentiating (1.2), we find that if $\zeta = \langle \nabla u, \hat{\xi} \rangle$, for some $\hat{\xi} \in \mathbf{R}^n$, $|\hat{\xi}| = 1$, then ζ satisfies, at $y \in \Omega \cap B(w, r/\bar{c})$, the partial differential equation $L\zeta = 0$, where

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(b_{ij}(y) \frac{\partial}{\partial y_j} \right) \quad (1.11)$$

and

$$b_{ij}(y) = |\nabla u|^{p-4} [(p-2)u_{y_i}u_{y_j} + \delta_{ij}|\nabla u|^2](y), \quad 1 \leq i, j \leq n. \quad (1.12)$$

In (1.12) δ_{ij} denotes the Kronecker delta. Furthermore,

$$c^{-1}\lambda(y)|\xi|^2 \leq \sum_{i,j=1}^n b_{ij}(y)\xi_i\xi_j \leq c\lambda(y)|\xi|^2, \quad \lambda(y) = |\nabla u(y)|^{p-2}, \quad (1.13)$$

whenever $\xi \in \mathbf{R}^n \setminus \{0\}$. Note from (1.10)-(1.13), that L is locally uniformly elliptic with bounded measurable coefficients in $\Omega \cap B(w, r/\bar{c})$.

To outline the proof of Theorem 1 we first note that in [LN, Theorem 3] we proved Theorem 1 assuming that Ω is a starlike Lipschitz ring domain and in [LN2, Theorem 1] we proved Theorem 1 when Ω is a Lipschitz domain. Unfortunately the proof of non-tangential limits for ∇u in [LN, LN2] relies heavily on the fundamental inequality (1.10). Since this inequality does not necessarily hold in an Ahlfors regular NTA-domain we are forced to use an alternative approach based on a result from [LN4] (see Lemma 2.9 in Section 2). After proving Theorem 1 we in Remark 3.3 also discuss work of Badger [B] and its applications to absolute continuity of p -harmonic measure and surface area.

To prove Theorem 2 we follow the corresponding proofs in [LN, Theorem 4] and [LN2, Theorem 2] for starlike C^1 -domains and C^1 -domains, respectively. Our argument is necessarily much more involved compared to [LN, LN2] but we are able, assuming only that Ω is Reifenberg flat with vanishing constant and that $n \in \text{VMO}(\Delta(w, 4r))$, to make use of the existence of ‘very big pieces of Lipschitz graphs with small constant’ in $\Delta(w, 4r)$ in order to derive estimates. The proof of the existence of ‘very big pieces of Lipschitz graphs with small constant’ emanates in the work of Semmes [Se, Se1] and was elaborated on, in our setting, by Kenig and Toro [KT1]. To prove Theorem 2 we use this set of ideas and we argue by contradiction. Using a blow-up argument and taking a limit we eventually arrive at a situation where the boundary is a hyperplane. In this simple geometry we easily obtain a contradiction to our assumption that $\log |\nabla u| \notin \text{VMO}(\Delta(w, r))$.

To prove Theorem 3 and Theorem 4 we have to prove that

$$\begin{aligned} (i) \quad & \Delta(w, r/2) \text{ is Reifenberg flat with vanishing constant,} \\ (ii) \quad & \lim_{\tilde{r} \rightarrow 0} \sup_{\tilde{w} \in \Delta(w, r/2)} \|n\|_{\text{BMO}(\Delta(\tilde{w}, \tilde{r}))} = 0. \end{aligned} \tag{1.14}$$

To do this we attempt to follow the corresponding proof in [LN2], where Lipschitz domains with small Lipschitz constant were considered (see also [LN5, Section 5]), and we prove Lemma 5.2-Lemma 5.5 below for Ahlfors regular NTA-domains. However the proofs of these lemmas are now considerably more difficult again primarily because (1.10) need not hold. To briefly outline the argument we note that we prove both statements in (1.14) by contradiction and by performing essentially the same blow-up argument. If, for example, (ii) is false, then using a blow-up argument we in the blow-up limit get a positive p -harmonic function, u_∞ , in an Ahlfors regular and $(4\delta, \infty)$ -Reifenberg flat domain, Ω_∞ , and u_∞ has several important properties. In particular, u_∞ vanishes continuously on $\partial\Omega_\infty$ and from Theorem 1 and our blow-up argument we also get the existence of a measure, μ_∞ , corresponding to u_∞ with

$$\int_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla \theta \rangle dx = - \int_{\partial\Omega_\infty} \theta d\mu_\infty \tag{1.15}$$

whenever $\theta \in C_0^\infty(\mathbf{R}^n)$. Moreover,

$$\mu_\infty = \sigma_\infty \text{ on } \partial\Omega_\infty, \quad |\nabla u_\infty(z)| \leq 1 \text{ whenever } z \in \Omega_\infty. \tag{1.16}$$

In (1.15) σ_∞ is surface measure on $\partial\Omega_\infty$. The conclusions in (1.15) and (1.16) follow from Lemma 5.2-Lemma 5.5 of Section 5. In this situation we can, depending on how small we

assume $\delta > 0$ to be, use either results from [ACF] (see also [DP]) or [LN5, LN6] to deduce that (1.15), (1.16) imply

$$u_\infty(y) = \langle y, \nu \rangle \text{ and } \Omega_\infty = \{y \in \mathbf{R}^n : \langle y, \nu \rangle > 0\} \text{ for some } \nu \in \partial B(0, 1). \quad (1.17)$$

(1.17) is easily seen to contradict our assumption that (ii) in (1.14) is false. Theorem 4 follows easily from Lemma 5.2-Lemma 5.5 due to their generality.

Finally, we mention that our proofs of Theorems 1-4 for $1 < p < \infty$ are modeled (as is [LN2]) on the beautiful work of [KT,KT1-KT3] for $p = 2$. However we have had several years after the above publications to ‘turn the screw tighter.’ Thus this article may be of interest also to those who only study the Laplace equation (the case $p = 2$).

The rest of the paper is organized as follows. In Section 2 we first collect estimates for p -harmonic functions in NTA-domains. We then recall a number of the results proved in [LN, LN1, LN4] concerning the boundary behaviour for p -harmonic functions in domains which are either Lipschitz, Reifenberg flat or Ahlfors regular NTA-domains. In subsection 2.1 we state some results for a uniformly elliptic operator \hat{L} and its Green function. The proof of Theorem 3 will use these results applied to the operator L in (1.11)-(1.13). In Section 3 and Section 4 we prove Theorem 1 and Theorem 2. Section 5 and Section 6 are devoted to the proof of Theorem 3 and Theorem 4, respectively. In Section 6 we also make some closing remarks concerning work in [KPT] and [B1].

2 p -harmonic functions and elliptic equations

Let $1 < p < \infty$ and let $\Omega \subset \mathbf{R}^n$ be a bounded domain. In this section we state a number of estimates for non-negative p -harmonic functions defined in Ω assuming that Ω is either a NTA-domain with constants M, r_0 , a Lipschitz domain with constant M , a (δ, r_0) -Reifenberg flat domain, $0 < \delta < \hat{\delta}(n)$, or an Ahlfors regular NTA-domain with constants C, M, r_0 . Throughout this section and this paper, unless otherwise stated, c will denote a positive constant ≥ 1 , not necessarily the same at each occurrence, depending only on p, n, M, C . In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1 , which may depend only on p, n, M, C and a_1, \dots, a_m , and which is not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on p, n, M, C .

For references to proofs of the following lemmas, Lemma 2.1-Lemma 2.5, we refer to [LN].

Lemma 2.1 *Given $p, 1 < p < \infty$, let u be a positive p -harmonic function in $B(w, 2r)$. Then*

$$\begin{aligned} (i) \quad & r^{p-n} \int_{B(w, r/2)} |\nabla u|^p dx \leq c (\max_{B(w, r)} u)^p, \\ (ii) \quad & \max_{B(w, r)} u \leq c \min_{B(w, r)} u. \end{aligned}$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $x, y \in B(w, r)$, then

$$(iii) \quad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\alpha \max_{B(w, 2r)} u.$$

Lemma 2.2 *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and p fixed, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $u \geq 0$ is p -harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. Then*

$$(i) \quad r^{p-n} \int_{\Omega \cap B(w, r/2)} |\nabla u|^p dx \leq c \left(\max_{\Omega \cap B(w, r)} u \right)^p.$$

Furthermore, there exists $\alpha = \alpha(p, n, M) \in (0, 1)$ such that if $x, y \in \Omega \cap B(w, r)$, then

$$(ii) \quad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\alpha \max_{\Omega \cap B(w, 2r)} u.$$

Lemma 2.3 *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and p fixed, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $u \geq 0$ is p -harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. There exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then*

$$\max_{\Omega \cap B(w, \tilde{r})} u \leq c u(a_{\tilde{r}}(w)).$$

Lemma 2.4 *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and p fixed, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is p -harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 2r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 2r)$. In particular, there exists $\sigma \in (0, 1]$, depending only on p, n , such that if $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 2r)$, then*

$$c^{-1} |\nabla u(x) - \nabla u(y)| \leq (|x - y|/\hat{r})^\sigma \max_{B(\hat{w}, \hat{r})} |\nabla u| \leq c \hat{r}^{-1} (|x - y|/\hat{r})^\sigma \max_{B(\hat{w}, 2\hat{r})} u.$$

u is infinitely differentiable in $\Omega \cap B(w, 2r) \cap \{x : |\nabla u(x)| > 0\}$ and

$$\int_{B(\hat{w}, \hat{r}/2) \cap \{|\nabla u| > 0\}} |\nabla u|^{p-2} \sum_{i,j=1}^n u_{x_i x_j}^2 dx \leq c r^{n-2} \max_{B(\hat{w}, \hat{r})} |\nabla u|^p$$

Lemma 2.5 *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and p fixed, $1 < p < \infty$. Given $w \in \partial\Omega$, $0 < r < r_0$, suppose that $u \geq 0$ is p -harmonic in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. There exists a unique locally finite positive Borel measure μ on $\Delta(w, 2r)$, such that whenever $\theta \in C_0^\infty(B(w, 2r))$, then*

$$(i) \quad \int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle dx = - \int \theta d\mu.$$

Moreover, there exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$(ii) \quad c^{-1} r^{p-n} \mu(\Delta(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c r^{p-n} \mu(\Delta(w, \tilde{r}/2)).$$

We say that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain if there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i ,

$$\begin{aligned}\Omega \cap B(x_i, 4r_i) &= \{y = (y', y_n) \in \mathbf{R}^n : y_n > \phi_i(y')\} \cap B(x_i, 4r_i), \\ \partial\Omega \cap B(x_i, 4r_i) &= \{y = (y', y_n) \in \mathbf{R}^n : y_n = \phi_i(y')\} \cap B(x_i, 4r_i),\end{aligned}\tag{2.1}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i . The Lipschitz constants of Ω are defined to be $M = \max_i \|\nabla\phi_i\|_\infty$ and $r_0 = \min_i r_i$. Moreover, a bounded domain $\tilde{\Omega} \subset \mathbf{R}^n$ is said to be starlike Lipschitz with respect to $\hat{x} \in \tilde{\Omega}$ provided

$$\partial\tilde{\Omega} = \{\hat{x} + R(\omega)\omega : \omega \in \partial B(0, 1)\} \text{ where } \log R : \partial B(0, 1) \rightarrow \mathbf{R} \text{ is Lipschitz on } \partial B(0, 1).$$

We will refer to $\|\log R\|_{\partial B(0,1)}$ as the Lipschitz constant for $\tilde{\Omega}$. Observe that this constant is invariant under scaling about \hat{x} .

We next collect a number of results proved in [LN, LN1, LN2, LN4].

Theorem 2.6 ([LN1, Theorem 2]) *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constants M, r_0 . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u, v are positive p -harmonic functions in $\Omega \cap B(w, 4r)$, that u, v are continuous in $\tilde{\Omega} \cap B(w, 4r)$ and $u = 0 = v$ on $\Delta(w, 4r)$. There exists $c_1 = c_1(p, n, M) \geq 1$ and $\alpha = \alpha(p, n, M), \alpha \in (0, 1)$, such that*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c_1 \left(\frac{|y_1 - y_2|}{r} \right)^\alpha$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_1)$.

Theorem 2.7 ([LN1, Lemma 4.28]) *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constants M, r_0 . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\tilde{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. There exists $c_2 = c_2(p, n, M) \geq 1$ and $\bar{\lambda} = \bar{\lambda}(p, n, M) \geq 1$ such that*

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap B(w, r/c_2)$.

Theorem 2.8 ([LN4, Lemma 3.8]) *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0) -Reifenberg flat domain. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\tilde{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Then there exist $\bar{\delta}_1 = \bar{\delta}_1(p, n) > 0, \bar{c}_1 = \bar{c}_1(p, n)$ and $\bar{\lambda} = \bar{\lambda}(p, n), 1 \leq \bar{c}_1 < \infty, 1 \leq \bar{\lambda} < \infty$, such that if $0 < \delta < \bar{\delta}_1$, then*

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap B(w, r/\bar{c}_1)$.

Lemma 2.9 ([LN4, Lemma 4.14]) *Let $\Omega \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain with constants, r_0, M, C . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. There exists a constant $c = c(p, n, M, C)$, $1 \leq c < \infty$, such that the following is true. There is a starlike Lipschitz domain $\tilde{\Omega} \subset \Omega \cap B(w, r)$, with center at a point \tilde{w} , $d(\tilde{w}, \partial\tilde{\Omega}) \geq c^{-1}r$, and with Lipschitz constant bounded by c , such that*

$$\frac{\sigma(\partial\tilde{\Omega} \cap \Delta(w, r))}{\sigma(\Delta(w, r))} \geq c^{-1}.$$

Moreover, if $y \in \tilde{\Omega}$ then

$$c^{-1}r^{-1}u(\tilde{w}) \leq u(y)/d(y, \partial\Omega) \leq cr^{-1}u(\tilde{w}).$$

2.1 Elliptic Equations

In this subsection we state some results for divergence form elliptic PDE which will be used in the proof of Theorem 3. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and suppose that

$$\hat{L} = \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(\hat{b}_{ij}(y) \frac{\partial}{\partial y_j} \right) \quad (2.2)$$

in Ω . We assume that $\{\hat{b}_{ij}(\cdot)\}$ are in $C^\infty(\Omega)$, $\hat{b}_{ij} = \hat{b}_{ji}$ for all $i, j \in \{1, \dots, n\}$, and

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^n \hat{b}_{ij}(y)\xi_i\xi_j \leq c|\xi|^2\lambda, \quad (2.3)$$

for some $\lambda \geq 1$. The following two lemmas, Lemma 2.10 and 2.11, follow from results and arguments in [CFMS], [JK1], and Schauder theory.

Lemma 2.10 *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let \hat{L} as in (2.2), (2.3) Then there exists a Green's function $g(\cdot, \cdot) : \Omega \times \Omega \rightarrow \mathbf{R} \cup \infty$ with the following properties.*

- (a') $g(x, y) = g(y, x)$ when $x \neq y, x, y \in \Omega$.
- (b') $\zeta g(\cdot, y) \in C^\infty(\Omega \setminus \bar{B}(y, \epsilon)) \cap W_0^{1,2}[\Omega \setminus \bar{B}(y, \epsilon)]$ when $\epsilon > 0, y \in \Omega, \zeta \in C_0^\infty(\mathbf{R}^n \setminus \bar{B}(y, \epsilon))$.
- (c') If $\theta \in C_0^\infty(\Omega)$, then $\theta(x) = \int \sum_{i,j=1}^n \hat{b}_{ij}(y) \theta_{y_i}(y) g_{y_j}(x, y) dy$ when $x \in \Omega$,
- (d') If $n > 2$, then $g(x, y) \leq c(\lambda, n)|x - y|^{2-n}$ when $x \neq y, x, y \in \Omega$,
if $n = 2$, then $g(x, y) \leq c(\lambda) \log\left(\frac{2 \text{diam } \Omega}{|x - y|}\right)$ when $x \neq y, x, y \in \Omega$.

The next lemma follows from an iterative type argument using the maximum principle, Harnack's inequality, and barrier type estimates for solutions to (2.2), (2.3).

Lemma 2.11 *Let Ω , \hat{L} , $g(\cdot, \cdot)$ be as in Lemma 2.10. Also assume that $x \in \partial\Omega$ and $B(x, r) \setminus \bar{\Omega}$ satisfies the corkscrew condition in Definition 1.1 (ii) relative to x , for $0 < r \leq r_0$. If $y, z \in \Omega$, such that $0 < |x - z| < |x - y|/2 < r_0/4$, then there exists $\beta \in (0, 1)$ depending on λ, n , and M in Definition 1.1 with*

$$g(z, y) \leq c(\lambda, n) \left(\frac{|x - z|}{|x - y|} \right)^\beta |z - y|^{2-n} \text{ when } n > 2,$$

$$g(z, y) \leq c(\lambda) \left(\frac{|x - z|}{|x - y|} \right)^\beta \log \left(\frac{2 \text{diam } \Omega}{|z - y|} \right) \text{ when } n = 2.$$

3 Proof of Theorem 1

The purpose of this section is to prove Theorem 1 and we divide the proof into the following lemmas.

Lemma 3.1 *Let $\Omega \subset R^n$, M, C, p, w, r_0, r, u and μ be as in the statement of Theorem 1. Then μ is absolutely continuous with respect to σ on $\Delta(w, 4r)$ and $d\mu \in A^\infty(\Delta(w, 2r), d\sigma)$. Let k^{p-1} denote the Radon-Nikodym derivative of μ with respect to σ on $\Delta(w, 4r)$, i.e., $d\mu/d\sigma = k^{p-1} \sigma$ almost everywhere on $\Delta(w, 4r)$. Then there exist $q > p - 1$ and a constant c , $1 \leq c < \infty$, which both only depend on p, n, M and C , such that*

$$N(|\nabla u|) \in L^q(\Delta(w, 2r)), \quad (3.1)$$

where $N(\cdot)$ is the non-tangential maximal function introduced in (1.4) relative to a fixed $b, 0 < b < 1$, and

$$\int_{\Delta(w, 2r)} k^q d\sigma \leq cr^{(n-1)(\frac{p-1-q}{p-1})} \left(\int_{\Delta(w, 2r)} k^{p-1} d\sigma \right)^{q/(p-1)}. \quad (3.2)$$

Lemma 3.2 *Let $\Omega \subset R^n$, $M, C, p, w, r_0, r, u, \mu$, be as in the statement of Theorem 1 and let k be as in Lemma 3.1. Then*

$$\lim_{x \in \Gamma(y) \cap B(w, 4r), x \rightarrow y} \nabla u(x) \stackrel{\text{def}}{=} \nabla u(y) \text{ exists for } \sigma\text{-a.e } y \in \Delta(w, 4r), \quad (3.3)$$

where $\Gamma(y)$ is the non-tangential cone at y introduced in (1.3) relative to a fixed $b, 0 < b < 1$. Moreover $\Delta(w, 4r)$ has a tangent plane at σ almost every $y \in \Delta(w, 4r)$. If $n(y)$ denotes the unit normal to this tangent plane pointing into $\Omega \cap B(w, 4r)$, then

$$k(y) = |\nabla u(y)| \text{ and } \nabla u(y) = |\nabla u(y)|n(y) \sigma \text{ almost every on } \Delta(w, 4r). \quad (3.4)$$

Proof of Theorem 1. It is easily seen that the above two lemmas imply Theorem 1. However it should be pointed out that (ii) of Theorem 1 implies (iii) of this theorem and also that $d\mu \in A^\infty(\Delta(w, 2r), d\sigma)$ (see [CF]). Thus the proof of Theorem 1 will be complete once we prove Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1. Recall that u is extended to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$ and that μ is the measure associated to u as in the statement of Lemma 2.5. Let $z \in \Delta(w, 2r)$ and $0 < s < r/c'$, where c' denotes the constant in Lemma 2.3, and let $E \subset \Delta(z, s)$ be a Borel set. Next let $0 < \gamma < 1$ be a degree of freedom to be fixed below, assume $\sigma(E) \geq \gamma\sigma(\Delta(z, s))$, and let $\tilde{\Omega}$ be as in Lemma 2.9 with w, r, \tilde{w} replaced by z, s, \tilde{z} . Also let \tilde{c} be the constant appearing in Lemma 2.9 and let \tilde{u} be the p -capacitary function for the ring domain $\tilde{\Omega} \setminus \bar{B}(\tilde{z}, \frac{s}{2\tilde{c}})$. Extend \tilde{u} to \mathbf{R}^n by defining $\tilde{u} \equiv 0$ on the complement of $\tilde{\Omega}$ and let $\tilde{\mu}$ be the measure associated to \tilde{u} , in the sense of Lemma 2.5, with support in $\partial\tilde{\Omega}$. Then, using the version of Theorem 1 established in [LN], valid in starlike Lipschitz rings, we can conclude that $\tilde{\mu}$ is an A^∞ -measure on $\partial\tilde{\Omega}$, with respect to surface measure on $\partial\tilde{\Omega}$. Using this conclusion and Lemma 2.9 we see that if γ is sufficiently near one and $F = \partial\tilde{\Omega} \cap E$, then $\sigma(F) \geq s^{n-1}/c_+$ and hence $\tilde{\mu}(F) \geq c_+^{-1}$, for some $c_+, 1 \leq c_+ < \infty$, independent of z and s . Next, using Harnack's inequality and the maximum principle we find that $cu \geq u(\tilde{z})\tilde{u}$ in $\tilde{\Omega} \setminus \bar{B}(\tilde{z}, \frac{s}{2\tilde{c}})$. From this inequality and Lemma 2.5 applied to $u/u(\tilde{z}), \tilde{u}$, we deduce for some $\hat{c} \geq 1$, depending only on p, n, M, C , that

$$\tilde{\mu}(B(y, t)) \leq \hat{c} \frac{\mu(B(y, t))}{\mu(B(z, s))} \text{ whenever } y \in F \text{ and } 0 < t < s/\hat{c}.$$

Using this inequality, the fact that $\mu, \tilde{\mu}$ are regular Borel measures, and a Vitali type covering type argument it follows that

$$\frac{\mu(E)}{\mu(\Delta(z, s))} \geq \frac{\mu(F)}{\mu(\Delta(z, s))} \geq \hat{c}^{-1} \tilde{\mu}(F) \geq (c_+ \hat{c})^{-1}. \quad (3.5)$$

From (3.5) and arbitrariness of s, z we conclude that $d\mu \in A^\infty(\Delta(w, 2r), d\sigma)$. Using results from [CF] it now follows that μ is absolutely continuous with respect to σ on $\Delta(w, 4r)$. Moreover, if $k^{p-1} = d\mu/d\sigma$ denotes the Radon - Nikodym derivative of μ on $\Delta(w, 4r)$ then there exists $\bar{c} = \bar{c}(p, n, M, C)$ and $q' > p - 1$, $q' = q'(p, n, M, C)$, such that if $y \in \Delta(w, 2r)$, $t > 0$, $\Delta(y, t) \subset \Delta(w, 2r)$, then

$$\int_{\Delta(y, t)} k^{q'} d\sigma \leq ct^{(n-1)\frac{(p-1-q')}{p-1}} \left(\int_{\Delta(y, t)} k^{p-1} d\sigma \right)^{q'/(p-1)}. \quad (3.6)$$

From (3.6) we see that (3.2) holds for $q = (q' + p - 1)/2$. Next we prove (3.1) for this q . To prove (3.1) we note that we can assume, without loss of generality, that

$$\max_{\Omega \cap B(w, 4r)} u = 1. \quad (3.7)$$

Let $\Gamma(y)$ be defined as in (1.3) for a fixed $b, 0 < b < 1$. Let $y \in \Delta(w, 2r)$ and let $z \in \Gamma(y) \cap B(y, r/8)$. Then, using Lemma 2.4 and Lemma 2.5 we obtain, with $s = |z - y|$,

$$\begin{aligned} |\nabla u(z)| &\leq c \frac{u(z)}{s} \leq c^2 s^{-1} \left(s^{p-n} \mu(\Delta(y, s)) \right)^{1/(p-1)} \\ &= c^2 \left(s^{1-n} \int_{\Delta(y, s)} k^{p-1} d\sigma \right)^{1/(p-1)} \leq c^2 (M(k^{p-1})(y))^{1/(p-1)}, \end{aligned} \quad (3.8)$$

where $c \geq 1$ depends on p, n, M, C, b , and

$$M(f)(y) = \sup_{0 < s < r/4} s^{1-n} \int_{\Delta(y,s)} f d\sigma$$

whenever f is an integrable function on $\Delta(w, 3r)$. Next we define

$$N_\rho(|\nabla u|)(y) = \sup_{\Gamma(y) \cap B(w, 4r) \cap \{x \in \Omega : d(x, \partial\Omega) \leq \rho\}} |\nabla u| \text{ whenever } \rho \leq r/8 \text{ and } y \in \Delta(w, 2r). \quad (3.9)$$

Using (3.6)-(3.8) and the Hardy-Littlewood maximal theorem we see that

$$\begin{aligned} \int_{\Delta(w, 2r)} N_{r/8}(|\nabla u|)^q d\sigma &\leq c \int_{\Delta(w, 2r)} (M(k^{p-1}))^{q/(p-1)} d\sigma \\ &\leq c^2 r^{-\frac{(n-1)(q+1-p)}{p-1}} \left(\int_{\Delta(w, 2r)} k^{p-1} d\sigma \right)^{q/(p-1)}. \end{aligned} \quad (3.10)$$

From Lemma 2.4 and (3.7) we also find that $|\nabla u(x)| \leq cr^{-1}$ whenever $x \in (\Gamma(y) \cap B(w, 4r)) \setminus B(y, r/8)$ and $y \in \Delta(w, 2r)$. Thus $N(|\nabla u|)(y) \leq N_{r/8}(|\nabla u|)(y) + cr^{-1}$ whenever $y \in \Delta(w, 2r)$. Therefore, using (3.10) as well as Lemma 2.5 (ii) and (3.7), we can conclude that (3.2) holds. This concludes the proof of Lemma 3.1. \square

Proof of Lemma 3.2. To prove Lemma 3.2 we argue by contradiction. Suppose there exists a set $F \subset \Delta(w, 4r)$ with $\sigma(F) > 0$, such that Lemma 3.2 is false for each $y \in F$. Under this assumption we let $z \in F$ be a point of density for F with respect to σ . Then

$$\frac{\sigma(\Delta(z, t) \setminus F)}{\sigma(\Delta(z, t))} \rightarrow 0 \text{ as } t \rightarrow 0. \quad (3.11)$$

Hence, since Ω is Ahlfors regular there exists $c = c(p, n, M, C) \geq 1$ such that

$$c\sigma(\partial\tilde{\Omega} \cap \Delta(z, s) \cap F) \geq s^{n-1}, \quad (3.12)$$

if $s > 0$ is small enough, and where $\tilde{\Omega} \subset \Omega$ is the starlike Lipschitz domain defined in Lemma 2.9 with w, \tilde{w}, r replaced by z, \tilde{z}, s . To get a contradiction we show that

$$\text{Lemma 3.2 holds for } \sigma \text{ almost every } y \in \partial\tilde{\Omega} \cap \Delta(z, s). \quad (3.13)$$

Thus to prove Lemma 3.2 it suffices to prove (3.13) whenever $\Delta(z, s) \subset \Delta(w, 4r)$. Let $\tilde{\sigma}$ denote surface area on $\partial\tilde{\Omega}$ and let E be the set of all $y \in \partial\tilde{\Omega} \cap \Delta(z, s)$ which satisfies the following.

- (a) y is a point of density for E relative to $\sigma, \tilde{\sigma}, \mu$.
- (b) There is a tangent plane $T(y)$ to both $\partial\Omega, \partial\tilde{\Omega}$ at y .
- (c) $\lim_{t \rightarrow 0} t^{-n} \sigma(\partial\Omega \cap B(y, t)) = \lim_{t \rightarrow 0} t^{-n} \sigma(\partial\tilde{\Omega} \cap B(y, t)) = \hat{a}$.
- (d) $\lim_{t \rightarrow 0} t^{-n} \mu(\partial\Omega \cap B(y, t)) = \hat{a} k(y)^{p-1}$.

In (3.14), \hat{a} , denotes the Lebesgue $(n - 1)$ -measure of the unit ball in \mathbf{R}^{n-1} . We claim that

$$\sigma(\partial\tilde{\Omega} \cap \Delta(z, s) \setminus E) = 0. \quad (3.15)$$

Indeed (a) of (3.14) for σ almost every y is a consequence of the fact that $\sigma, \tilde{\sigma}$ are regular Borel measures and differentiation theory while (a) for μ and σ almost every y follows from the same observations and Lemma 3.1. To prove (b) of (3.14) we need to prove, for σ almost every y in $\partial\tilde{\Omega} \cap \Delta(z, s)$, that there exists a plane $T(y)$ with

$$\lim_{t \rightarrow 0} \frac{h(T(y) \cap B(y, t), \partial\hat{\Omega} \cap B(y, t))}{t} = 0 \quad (3.16)$$

where h denotes Hausdorff distance and $\hat{\Omega} \in \{\Omega, \tilde{\Omega}\}$. (3.16) for σ almost every $y \in \partial\tilde{\Omega} \cap \Delta(z, s)$ follows, for $\hat{\Omega} = \tilde{\Omega}$, essentially from Rademacher's theorem. Also if (3.16) holds for $\hat{\Omega} = \tilde{\Omega}$, then using the uniform condition in Definition 1.1 for NTA-domains, $\tilde{\Omega} \subset \Omega$, and (a) of (3.14) for $\tilde{\sigma}$, one sees that necessarily (3.16) is also valid when $\hat{\Omega} = \Omega$. (c) of (3.14) for σ almost every $y \in \partial\tilde{\Omega} \cap \Delta(z, s)$ follows for $\tilde{\sigma}$ from a well known formula for surface area of a Lipschitz graph and the Lebesgue differentiation theorem. Moreover (c) and (a) of this display for $\tilde{\sigma}$, as well as Ahlfors regularity of $\partial\Omega$, imply (c) for σ and σ almost every $y \in \partial\tilde{\Omega} \cap \Delta(z, s)$. One could also get (c) using geometric measure theory (see [EG]). To get (d) of (3.14) for σ almost every y , we use (c) of this display, Lemma 3.1, and once again the Lebesgue differentiation theorem. Thus (3.15) is true.

We now use a blow-up argument to complete the proof of Lemma 3.2. Let E, s be as in (3.14) and $y \in E$. Using invariance of the p -Laplace equation under rotations and dilations we may assume $y = 0$ and that $T = T(0) = \{x \in \mathbf{R}^n : x_n = 0\}$ where $T(0)$ is the tangent plane in (3.14). Let $\{t_m\}$ be a decreasing sequence of positive numbers with limit zero and $t_1 \ll s$. Let

$$\begin{aligned} \Omega_m &= \{x : t_m x \in \Omega \cap B(z, s)\} \\ \tilde{\Omega}_m &= \{x : t_m x \in \tilde{\Omega} \cap B(z, s)\} \\ v_m(x) &= t_m^{-1} u(t_m x), \quad x \in B(z, s). \end{aligned} \quad (3.17)$$

Fix $R \gg 1$. Then for m sufficiently large, say $m \geq m_0, m_0 = m_0(R)$, we note that v_m is p -harmonic in $\Omega_m \cap B(0, R)$ and continuous in $B(0, R)$ with $v_m \equiv 0$ on $B(0, R) \setminus \Omega_m$. Let $\rho S = \{\rho x : x \in S\}$ whenever $\rho > 0$ and $S \subset \mathbf{R}^n$. Define

$$\nu_m(G) = t_m^{1-n} \mu(t_m G), \text{ whenever } G \text{ is a Borel subset of } B(0, R). \quad (3.18)$$

Then ν_m is the measure corresponding to v_m as in Lemma 2.5 for $m \geq m_0$. Let $\eta = u(\tilde{z})/s$. From Lemma 2.4 and Lemma 2.9 we see that

$$|\nabla v_m| \leq c\eta \text{ on } \tilde{\Omega}_m. \quad (3.19)$$

Also from (3.19) and Lemma 2.2, Lemma 2.3, we deduce that

$$|v_m(x)| \leq c \left(\frac{d(x, \partial\Omega_m)}{R} \right)^\alpha \eta R, \quad x \in \Omega_m \cap B(0, R), \quad (3.20)$$

where α is the Hölder exponent in Lemma 2.2. We assume, as we may, that $H = \{x : x_n > 0\}$ contains \tilde{z} . Then since Ω is an NTA-domain and $\tilde{\Omega}$ is starlike Lipschitz with respect to \tilde{z} , we from (3.16) find that

$$h(\Omega_m \cap B(0, R), H \cap B(0, R)) + h(\tilde{\Omega}_m \cap B(0, R), H \cap B(0, R)) \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.21)$$

From (3.19)-(3.21) we see that a subsequence of $\{v_m\}$, denoted $\{v'_m\}$ converges uniformly on compact subsets of \mathbf{R}^n to a Hölder continuous function v with $v \equiv 0$ in $\mathbf{R}^n \setminus H$. Also $v \geq 0$ is p -harmonic in H . We now apply the boundary Hölder continuity estimate in Theorem 2.6 with Ω, u replaced by H, x_n , respectively. Letting $r \rightarrow \infty$ we get $v(x) = \alpha x_n^+$ for some $\alpha \geq 0$, where $x_n^+ = \max(x_n, 0)$. We assert that

$$\alpha = k(y) \tag{3.22}$$

where α is the Hölder exponent in Lemma 2.2. To prove (3.22) observe from Lemma 2.5 and (3.20) that the sequence of measures, (ν'_m) , corresponding to (v'_m) have uniformly bounded total masses on $B(0, R)$. Also from Lemma 2.1 and (3.20) we see that $\{v'_m\}$ is uniformly bounded in $W^{1,p}(B(0, R))$. Using these facts and Lemma 2.4, Lemma 2.5 (i), we obtain that ν'_m converges weakly to ν where ν is the measure associated with αx_n^+ . One easily computes that $\nu = \alpha^{p-1} \sigma_H$ where σ_H , denotes the $(n-1)$ -dimensional Hausdorff measure on H . Using this computation, weak convergence, (3.18), and (3.14) (d), we get

$$\alpha^{p-1} \hat{a} R^{n-1} = \lim_{m \rightarrow \infty} \nu'_m(B(0, R)) = \lim_{m \rightarrow \infty} s_m^{1-n} \mu(B(0, R s_m)) = \hat{a} R^{n-1} k^{p-1}(y).$$

Thus (3.22) is true. From (3.22) and our earlier observations we see that $x \rightarrow t^{-1}v(tx)$ converges uniformly as $t \rightarrow 0$ to αx_n^+ on compact subsets of \mathbf{R}^n and $x \rightarrow \nabla u(tx)$ converges uniformly to αe_n as $t \rightarrow 0$ when x lies in a compact subset of H . Given $0 < \beta < 1$, let $K_\beta = \{x \in H : x_n \geq \beta|x|\}$. In view of these remarks and (3.22) we conclude that

$$\lim_{t \rightarrow 0} \nabla u(t\omega) = k(y)e_n \tag{3.23}$$

whenever $0 < \beta < 1$ is fixed and $\omega \in K_\beta$ with $|\omega| = 1$. Finally, using the uniform condition in Definition 1.1 once again it is easily seen, for given $0 < b < 1$ and $t > 0$ small, that there exists $\beta > 0$ such that $\Gamma_b(0) \cap B(0, t) \subset K_\beta$. From this observation, (3.23), and our earlier reduction we conclude the validity of Lemma 3.2. \square

Armed with Lemma 3.1 and Lemma 3.2 we get Theorem 1 as pointed out after the statement of these lemmas. \square

Remark 3.3 *It is an interesting question as to what extent Theorem 1 remains valid when the Ahlfors regularity assumption is replaced by a weaker assumption. Regarding this question, we believe that an argument in Badger [B] together with Lemma 2.1-Lemma 2.5 can be used to prove the following proposition.*

Proposition 3.4 *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain with constants M, r_0 . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, let u, μ be as in Theorem 1. Set*

$$A = \{x \in \Delta(w, 4r) : \liminf_{\rho \rightarrow 0} \frac{\sigma(\Delta(x, \rho))}{\rho^{n-1}} < \infty\}.$$

Then σ and μ restricted to A are mutually absolutely continuous.

To briefly outline the proof of this proposition we note that in [B] it is shown that a theorem of David-Semmes, regarding Lipschitz approximation (on every scale) in bounded Ahlfors regular NTA-domains, can be proved under weaker assumptions. Using this result it

follows easily that if $\Delta(y, 4s) \subset \Delta(w, 4r)$ and $s^{1-n}\sigma(\Delta(y, 4s)) \leq \tau < \infty$, then Lemma 2.12 remains valid with w, r, \tilde{w} , replaced by y, s, \tilde{y} , where now c also depends on τ . This lemma and the same argument as in the proof of Lemma 3.1 imply as in [B, Proposition 4.5] that there exist $\epsilon, \delta > 0$, depending only on p, n, M, τ , such that for every Borel set $E \subset \Delta(y, s)$, the following is true.

$$\begin{aligned} (a) \quad & \text{If } \mu(E) \leq \delta\mu(\Delta(y, s)) \text{ then } \sigma(E) \leq \epsilon\sigma(\Delta(y, s)). \\ (b) \quad & \text{If } \sigma(E) \leq \delta\sigma(\Delta(y, s)) \text{ then } \mu(E) \leq \epsilon\mu(\Delta(y, s)). \end{aligned} \tag{3.24}$$

Proposition 3.4 is obtained from (3.24) and a Vitali covering argument (see [B,(4.18)]).

4 Proof of Theorem 2

The purpose of this section is to prove Theorem 2. Let u, Ω, r_0 , be as in Theorem 2. For $w \in \partial\Omega$, $0 < r < r_0$, we define

$$\hat{n}(w, r) = \frac{\tilde{n}(w, r)}{|\tilde{n}(w, r)|} \text{ where } \tilde{n}(w, r) = \frac{1}{\sigma(\Delta(w, r))} \int_{\Delta(w, r)} n(y) d\sigma(y) \tag{4.1}$$

where $n(y)$ is the inner unit normal at $y \in \partial\Omega$ guaranteed to exist, by Theorem 1, σ almost everywhere. Let $\hat{P}(w, r)$ be the hyperplane which is orthogonal to $\hat{n}(w, 2r)$ and which contains w . Using coordinates $x = (x', x_n)$, $x' = x - x_n\hat{n}(w, 2r) \in \hat{P}(w, r)$ we introduce the cylinder

$$C(w, r) = \{x = (x', x_n) = x' + x_n\hat{n}(w, 2r) \text{ such that } x' \in \hat{P}(w, r) \cap B(w, r), |x_n| \leq r\}. \tag{4.2}$$

Moreover, we let $\pi(w, r)(x)$ denote the orthogonal projection of $x \in \mathbf{R}^n$ onto $\hat{P}(w, r)$, i.e., $\pi(w, r)(x) = \pi(w, r)(x', x_n) = x'$. If $G \subset \mathbf{R}^n$ then we let $\pi(w, r)(G)$ denote the projection of G . In the proof of Theorem 2 we will use the following lemma.

Lemma 4.1 *Let $\Omega \subset \mathbf{R}^n$ be (δ, r_0) -Reifenberg flat and Ahlfors regular with constants r_0, C . Suppose also that $w \in \partial\Omega$, $0 < r < r_0/100$, and $\hat{P}(w, r)$ is as defined above. There exists $\tilde{\delta} < \hat{\delta}(n)$ such that if $0 < \delta < \tilde{\delta}$ and*

$$\|n\|_{BMO(\Delta(w, 100r))} \leq \delta^4, \tag{4.3}$$

then $\partial\Omega \cap C(w, r)$ contains very big pieces of a Lipschitz graph with small constant in the following sense. There is a constant $\tilde{c} = \tilde{c}(n, C)$, $1 \leq \tilde{c} < \infty$, and a Lipschitz function $\phi : \hat{P}(w, r) \rightarrow \mathbf{R}$ with $\|\nabla\phi\|_\infty \leq \tilde{c}\delta$, such that if $\Gamma = \{x = (x', x_n) \in \mathbf{R}^n : x_n = \phi(x')\}$, then

$$\begin{aligned} (i) \quad & \sigma(((\partial\Omega \setminus \Gamma) \cup (\Gamma \setminus \partial\Omega)) \cap C(w, r)) \leq e^{-1/(\tilde{c}\delta)} r^{n-1}, \\ (ii) \quad & \partial\Omega \cap C(w, r) = G \cup F, \quad G \subset \Gamma, \quad \sigma(F) \leq e^{-1/(\tilde{c}\delta)} r^{n-1}, \\ (iii) \quad & \text{if } x = (x', x_n) \in F \text{ then } |x_n - \phi(x')| \leq \tilde{c}\delta d(x', \pi(w, r)(G)). \end{aligned}$$

Furthermore, if $\phi^\pm(x') = \phi(x') \pm 2\tilde{c}\delta d(x', \pi(w, r)(G))$ and $\Omega^\pm = \{(x', x_n) \in \mathbf{R}^n : x_n > \phi^\pm(x')\}$, then $\|\nabla\phi^\pm\|_\infty \leq 3\tilde{c}\delta$ and

$$(iv) \quad \Omega^+ \cap C(w, r) \subset \Omega \cap C(w, r) \subset \Omega^- \cap C(w, r).$$

Proof. This result emanates in the work of Semmes [Se, Se1] but was elaborated on, in our setting, by Kenig and Toro [KT1]. In particular, for more on this, and the particular statements of Lemma 4.1, we refer to [CKL, p.64-71]. \square

Proof of Theorem 2. Let q be as in the statement of Theorem 1 and let $\tilde{q} = \min\{(q + p - 1)/2, p\}$. To prove Theorem 2 it suffices, by way of a lemma of Sarason (see [KT]), to prove that there exists $\epsilon_0 > 0$ and $r_0 = r_0(\epsilon)$, defined for $\epsilon \in (0, \epsilon_0)$, such that whenever $y \in \Delta(w, r)$ and $0 < s < r_0(\epsilon)$, then

$$\frac{1}{\sigma(\Delta(y, s))} \int_{\Delta(y, s)} |\nabla u|^{\tilde{q}} d\sigma \leq (1 + \epsilon) \left(\frac{1}{\sigma(\Delta(y, s))} \int_{\Delta(y, s)} |\nabla u|^{p-1} d\sigma \right)^{\tilde{q}/(p-1)}. \quad (4.4)$$

We prove (4.4) by contradiction. Indeed, if (4.4) is not true then there exist sequences $\{s_m\}_1^\infty, \{y_m\}_1^\infty$ satisfying $y_m \in \Delta(w, r), m = 1, 2, \dots, \lim_{m \rightarrow \infty} s_m = 0$, and for which (4.4) is false with y, s replaced by $y_m, s_m, m = 1, 2, \dots$. In the following we let, for $0 < \epsilon \ll 1$ fixed,

$$A = e^{1/\epsilon}, \quad \delta = \epsilon^2. \quad (4.5)$$

Moreover, with this choice of δ we let

$$\{\delta_m\} \text{ be a sequence such that } 0 < \delta_m \leq \delta \text{ and } \delta_m \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (4.6)$$

As usual $c \geq 1$ will denote a constant which may depend on p, n, M, C but which is independent of $\{y_m\}, \{s_m\}, \epsilon, \{\delta_m\}$ and m . Since $\Delta(w, 4r)$ is Reifenberg flat with vanishing constant, $n \in \text{VMO}(\Delta(w, 4r))$, and $s_m \rightarrow 0$ as $m \rightarrow \infty$, we may also assume, for $m = 1, 2, \dots$, that

$$\begin{aligned} (i) \quad & \Omega \text{ is } (\delta_m, \hat{r}_0)\text{-Reifenberg flat with } \hat{r}_0 = \hat{r}_0(\delta_m) > 100Ans_m, \\ (ii) \quad & \|n\|_{\text{BMO}(\Delta(y_m, 100As_m))} \leq \delta_m^4. \end{aligned} \quad (4.7)$$

Assuming (4.5), (4.6), and (4.7) we let

$$\hat{P}_m = \hat{P}(y_m, As_m), \quad \hat{n}_m = \hat{n}(y_m, As_m) \text{ and } C_m = C(y_m, As_m) \quad (4.8)$$

be defined as above Lemma 4.1 with δ, w, r replaced by δ_m, y_m, As_m . Using Lemma 4.1 we get for each $m = 1, 2, \dots$, a function $\phi_m : \hat{P}_m \rightarrow \mathbf{R}$, with $\|\nabla \phi_m\|_\infty \leq \tilde{c}\delta_m$, such that if $\Gamma_m = \{x = (x', x_n) \in \mathbf{R}^n : x_n = \phi_m(x')\}$, then,

$$\begin{aligned} (i) \quad & \sigma(((\partial\Omega \setminus \Gamma_m) \cup (\Gamma_m \setminus \partial\Omega)) \cap C_m) \leq e^{-1/(\tilde{c}\delta_m)}(As_m)^{n-1} \leq e^{-1/(\tilde{c}\delta)}(As_m)^{n-1}, \\ (ii) \quad & \partial\Omega \cap C_m = G_m \cup F_m, \quad G_m \subset \Gamma_m, \\ (iii) \quad & \sigma(F_m) \leq e^{-1/(\tilde{c}\delta_m)}(As_m)^{n-1} \leq e^{-1/(\tilde{c}\delta)}(As_m)^{n-1}. \end{aligned} \quad (4.9)$$

Moreover, if $x = (x', x_n) \in F_m$, then

$$\begin{aligned} & |x_n - \phi_m(\pi(y_m, As_m)(x))| \\ & \leq \tilde{c}\delta_m d(\pi_m(y_m, As_m)(x), \pi_m(y_m, As_m)(G_m)), \end{aligned} \quad (4.10)$$

where $\pi_m(y_m, As_m)(x)$ is the orthogonal projection of $x \in \mathbf{R}^n$ onto \hat{P}_m . Furthermore, if we let

$$\begin{aligned}\phi_m^\pm(x') &= \phi_m(x') \pm 2\tilde{c}\delta_m d(x', \pi(y_m, As_m)(G_m)), \\ \Omega_m^\pm &= \{(x', x_n) \in \mathbf{R}^n, x_n > \phi_m^\pm(x)\},\end{aligned}\tag{4.11}$$

then

$$\|\|\nabla\phi_m^\pm\|\|_\infty \leq 3\tilde{c}\delta_m \text{ and } \Omega_m^+ \cap C_m \subset \Omega \cap C_m \subset \Omega_m^- \cap C_m.\tag{4.12}$$

Let σ_m^\pm denote surface measure on $\partial\Omega_m^\pm$.

Next extend u to $B(w, 4r)$ by putting $u \equiv 0$ in $B(w, 4r) \setminus \bar{\Omega}$ and let μ be the measure associated to u in the sense of Lemma 2.5. Let u_m^- be the p -harmonic function in $\Omega_m^- \cap C_m$ which coincides with u on $\partial(\Omega_m^- \cap C_m)$ and let u_m^+ be a non-negative p -harmonic function in $\Omega_m^+ \cap C_m$ which is continuous in the closure of $\Omega_m^+ \cap C_m$, and satisfies $u_m^+ \leq u_m^-$ on $\partial(\Omega_m^+ \cap C_m)$, $u_m^+ = u_m^-$ on $\Omega_m^+ \cap \partial C_m \cap \{x \in \mathbf{R}^n : d(x, \partial\Omega_m^+) \geq \delta_m s_m\}$ and $u_m^+ \equiv 0$ on $\partial\Omega_m^+ \cap C_m$. Extend u_m^- and u_m^+ to C_m by putting $u_m^- \equiv 0$ on $C_m \setminus \Omega_m^-$, $u_m^+ \equiv 0$ on $C_m \setminus \Omega_m^+$, and let μ_m^\pm be the measures on $\partial\Omega_m^\pm \cap C_m$ associated to u_m^\pm in the sense of Lemma 2.5. In the following we let $y_m^\pm \in \partial\Omega_m^\pm$ be points with the same projection under $\pi(y_m, As_m)$ as y_m and we set $\Delta_m = \Delta(y_m, s_m)$, $\Delta_m^\pm = \partial\Omega_m^\pm \cap B(y_m^\pm, s_m)$. We note that

$$|y_m^\pm - y_m| \leq c'\delta_m As_m\tag{4.13}$$

as we see from (4.10). Furthermore, using (4.5), (4.7), and (4.9) we see, if ϵ is sufficiently small, that

$$\begin{aligned}(i) \quad (1 - \hat{\epsilon}) &\leq \frac{\sigma_m^\pm(\Delta_m^\pm)}{\sigma(\Delta_m)} \leq (1 + \hat{\epsilon}), \\ (ii) \quad (1 - \hat{\epsilon}) &\leq \frac{\sigma(\Delta_m \setminus \Delta_m^\pm)}{\sigma(\Delta_m)} \leq (1 + \hat{\epsilon}),\end{aligned}\tag{4.14}$$

where $\hat{\epsilon} = e^{-1/(c'\epsilon^2)}$ and $c' = c'(n, C)$. To continue our proof of Theorem 2 we introduce

$$A_m = \frac{1}{\sigma(\Delta_m)} \int_{\Delta_m} |\nabla u|^{\tilde{q}} d\sigma, \quad A_m^\pm = \frac{1}{\sigma_m^\pm(\Delta_m^\pm)} \int_{\Delta_m^\pm} |\nabla u_m^\pm|^{\tilde{q}} d\sigma_m^\pm,\tag{4.15}$$

and we shall prove the following lemmas.

Lemma 4.2 *There exists $c \geq 1$, independent of m , such that*

$$A_m \leq (1 + e^{-1/(c\epsilon)}) A_m^- \left(\frac{\mu(\Delta_m)}{\mu_m^+(\Delta_m^+)} \right)^{\tilde{q}/(p-1)}$$

whenever $m \in \{1, 2, \dots\}$.

Lemma 4.3 *There exists $c \geq 1$, independent of m , such that*

$$\limsup_{m \rightarrow \infty} A_m^- \left(\frac{\sigma_m^-(\Delta_m^-)}{\mu_m^-(\Delta_m^-)} \right)^{\tilde{q}/(p-1)} \leq (1 + e^{-1/(c\epsilon)}).$$

Lemma 4.4 *Let μ_m^\pm be the measures associated to u_m^\pm as above. Then,*

$$\limsup_{m \rightarrow \infty} \frac{\mu_m^+(\Delta_m^+)}{\mu_m^-(\Delta_m^-)} = 1.$$

We now complete the proof of Theorem 2 using Lemma 4.2-Lemma 4.4. Let \bar{c} be the largest of the constants in Lemma 4.2-Lemma 4.4. Assuming (4.4) to be false we see from these lemmas and (4.14) (i) that

$$\begin{aligned} 1 + \epsilon &\leq \limsup_{m \rightarrow \infty} A_m \left(\frac{\sigma(\Delta_m)}{\mu(\Delta_m)} \right)^{\bar{q}/(p-1)} \leq (1 + e^{-1/(\bar{c}\epsilon)}) \cdot \limsup_{m \rightarrow \infty} A_m^- \left(\frac{\sigma(\Delta_m)}{\mu_m^+(\Delta_m^+)} \right)^{\bar{q}/(p-1)} \\ &\leq (1 + e^{-1/(\bar{c}\epsilon)})^2 \cdot \limsup_{m \rightarrow \infty} \left(\frac{\mu_m^-(\Delta_m^-)}{\mu_m^+(\Delta_m^+)} \frac{\sigma(\Delta_m)}{\sigma_m^-(\Delta_m^-)} \right)^{\bar{q}/(p-1)} \\ &\leq 1 + e^{-1/(c\epsilon)} \end{aligned} \quad (4.16)$$

for some $c = c(p, n, C) \geq 1$, provided ϵ_0 is small enough. Choosing ϵ_0 still smaller if necessary, we see that (4.16) can not hold if $0 < \epsilon \leq \epsilon_0$. Hence (4.4) must be true.

Proof of Lemma 4.2. To start the proof we note that

$$A_m = A_m^{(1)} + A_m^{(2)} \quad (4.17)$$

where

$$A_m^{(1)} = \frac{1}{\sigma(\Delta_m)} \int_{G_m \cap \Delta_m^-} |\nabla u|^{\bar{q}} d\sigma, \quad A_m^{(2)} = \frac{1}{\sigma(\Delta_m)} \int_{\Delta_m \setminus (G_m \cap \Delta_m^-)} |\nabla u|^{\bar{q}} d\sigma \quad (4.18)$$

and where the set G_m was introduced in (4.9). Using the reverse Hölder inequality in Theorem 1 (ii), Hölder's inequality, (4.9) and (4.14) (ii) we see that

$$A_m^{(2)} \leq c \left(\frac{\sigma(\Delta_m \setminus (G_m \cap \Delta_m^-))}{\sigma(\Delta_m)} \right)^{1-\bar{q}/q} A_m \leq e^{-1/(c\epsilon)} A_m \quad (4.19)$$

provided c is large enough and ϵ_0 small enough.

To estimate $A_m^{(1)}$ we let E_m^- be the set of points $z \in G_m \cap \Delta_m^-$ such that

$$\lim_{\rho \rightarrow 0} \frac{\sigma(\Delta(z, \rho) \cap G_m \cap \Delta_m^-)}{\sigma(\Delta(z, \rho))} = 1 = \lim_{\rho \rightarrow 0} \frac{\sigma_m^-(\Delta_m^-(z, \rho) \cap G_m \cap \Delta_m^-)}{\sigma_m^-(\Delta_m^-(z, \rho))} \quad (4.20)$$

and such that $\nabla u(z)$ and $\nabla u_m^-(z)$ exist. Applying Theorem 1 to u and u_m^- and using the Lebesgue density theorem we see that $\sigma(E_m^-) = \sigma(G_m \cap \Delta_m^-)$, $\sigma_m^-(E_m^-) = \sigma_m^-(G_m \cap \Delta_m^-)$ and that

$$A_m^{(1)} = \frac{1}{\sigma(\Delta_m)} \int_{E_m^-} |\nabla u|^{\bar{q}} d\sigma. \quad (4.21)$$

Moreover, if $z \in E_m^-$, then $|\nabla u(z)| \leq |\nabla u_m^-(z)|$ as we see from the maximum principle and the definition of the set E_m^- . Furthermore, based on the definition of the set E_m^- in (4.20) we also find that

$$\lim_{\rho \rightarrow 0} \frac{\sigma(\Delta(z, \rho))}{\sigma_m^-(\Delta_m^-(z, \rho))} = 1 \text{ whenever } z \in E_m^-. \quad (4.22)$$

Using these facts, (4.14), Theorem 1, Hölder's inequality and standard arguments we get

$$\begin{aligned} A_m^{(1)} &\leq \frac{\sigma_m^-(\Delta_m^-)}{\sigma(\Delta_m)} \frac{1}{\sigma_m^-(\Delta_m^-)} \int_{G_m \cap \Delta_m^-} |\nabla u_m^-|^{\tilde{q}} d\sigma_m^- \\ &\leq (1 + c\hat{\epsilon}) \frac{1}{\sigma_m^-(\Delta_m^-)} \int_{G_m \cap \Delta_m^-} |\nabla u_m^-|^{\tilde{q}} d\sigma_m^- \leq (1 + c\hat{\epsilon}) A_m^-. \end{aligned} \quad (4.23)$$

Combining (4.17), (4.19) and (4.23) we deduce that

$$A_m \leq (1 + e^{-1/(c\hat{\epsilon})}) A_m^-. \quad (4.24)$$

Next let E_m^+ be the set of points $z \in G_m \cap \Delta_m^+$ such that (4.20) holds with $\Delta_m^-, \sigma_m^-, \Delta_m^-(z, \rho)$ replaced by $\Delta_m^+, \sigma_m^+, \Delta_m^+(z, \rho)$. From Theorem 1 and $u_m^+ \leq u_m$ we see that if $z \in E_m^+$, then $|\nabla u_m^+(z)| \leq |\nabla u(z)|$ and we can conclude that

$$\mu(\Delta_m) \geq \mu_m^+(G_m \cap \Delta_m^+) \geq (1 - e^{-1/(c\hat{\epsilon})}) \mu_m^+(\Delta_m^+),$$

where the last inequality follows from the same argument as in (4.19). Combining this inequality with (4.24) we conclude the validity of Lemma 4.2 for c large enough. \square

Proof of Lemma 4.3. Since $\partial\Omega_m^-$ is the graph of a Lipschitz function with Lipschitz norm $\rightarrow 0$ as $m \rightarrow \infty$, we can proceed along the lines of the proof of Theorem 4 in [LN] (see also Theorem 2 in [LN2]) to get Lemma 4.3. For the readers convenience we briefly outline this proof.

Let $\hat{y}_m = y_m^- + \frac{1}{10} A s_m \hat{n}_m$. We note that if ϵ is sufficiently small, then the domain D_m^- obtained by drawing all line segments from points in $B(\hat{y}_m, \frac{A s_m}{100})$ to points in $\partial\Omega_m^- \cap B(y_m^-, \frac{A s_m}{10})$, is starlike Lipschitz with respect to \hat{y}_m . Let $\hat{D}_m^- = D_m^- \setminus \bar{B}(\hat{y}_m, \frac{A s_m}{1000})$ and note that the Lipschitz constant of \hat{D}_m^- is $\leq c = c(n, C)$. Let \hat{u}_m^- be the p -capacitary function for \hat{D}_m^- , i.e., \hat{u}_m^- is non-negative, $\hat{u}_m^- = 0$ and $\hat{u}_m^- = 1$ continuously on ∂D_m^- and $\partial B(\hat{y}_m, \frac{A s_m}{1000})$, respectively, and \hat{u}_m^- is p -harmonic in \hat{D}_m^- . Extend \hat{u}_m^- to $\mathbf{R}^n \setminus D_m^-$ by setting $\hat{u}_m^- \equiv 0$ on $\mathbf{R}^n \setminus D_m^-$ and let $\hat{\mu}_m^-$ be the measure, with support on ∂D_m^- , corresponding to \hat{u}_m^- as in Lemma 2.5. Next suppose ϵ_0 is so small that $A/100 \geq 2c_1$, where c_1 is as in Theorem 2.6. Then, using Theorem 2.6 with r, w, u_1, u_2 replaced by $A s_m/100, y_m^-, u_m^-, \hat{u}_m^-$, we deduce, for ϵ small enough, that if $w_1, w_2 \in B(y_m^-, 2s_m) \cap \hat{D}_m^-$, then

$$\left| \log \left(\frac{\hat{u}_m^-(w_1)}{u_m^-(w_1)} \right) - \log \left(\frac{\hat{u}_m^-(w_2)}{u_m^-(w_2)} \right) \right| \leq c A^{-\alpha}, \quad (4.25)$$

where c, α are the constants in Theorem 2.6 and hence independent of m . Letting $w_1, w_2 \rightarrow z_1, z_2 \in \partial \hat{D}_m^- \cap B(y_m^-, 2s_m)$ in (4.25) and using Theorem 1 we see that

$$\left| \log \left(\frac{|\nabla \hat{u}_m^-(z_1)|}{|\nabla u_m^-(z_1)|} \right) - \log \left(\frac{|\nabla \hat{u}_m^-(z_2)|}{|\nabla u_m^-(z_2)|} \right) \right| \leq c A^{-\alpha} \quad (4.26)$$

for σ_m^- -almost all $z_1, z_2 \in \partial\hat{D}_m^- \cap B(y_m^-, 2s_m)$. From the inequality in (4.26) we deduce

$$(1 - cA^{-\alpha}) \frac{|\nabla\hat{u}_m^-(z_1)|}{|\nabla\hat{u}_m^-(z_2)|} \leq \frac{|\nabla u_m^-(z_1)|}{|\nabla u_m^-(z_2)|} \leq (1 + cA^{-\alpha}) \frac{|\nabla\hat{u}_m^-(z_1)|}{|\nabla\hat{u}_m^-(z_2)|}, \quad (4.27)$$

where $c = c(p, n, C)$. Let

$$\hat{A}_m^- := \frac{1}{\sigma_m^-(\Delta_m^-)} \int_{\Delta_m^-} |\nabla\hat{u}_m^-|^{\tilde{q}} d\sigma_m^-. \quad (4.28)$$

From (4.27) and Theorem 1 we see that

$$\frac{\hat{A}_m^-}{(\hat{\mu}_m^-(\Delta_m^-))^{\tilde{q}/(p-1)}} \geq (1 - cA^{-\alpha})^{\tilde{q}} \frac{A_m^-}{(\mu_m^-(\Delta_m^-))^{\tilde{q}/(p-1)}}. \quad (4.29)$$

One concludes from (4.29) and simple estimates that it suffices to prove Lemma 4.3 with μ_m^-, u_m^- , replaced by $\hat{\mu}_m^-, \hat{u}_m^-$. Thus one proves for $c = c(p, n, C)$ suitably large and ϵ_0 sufficiently small that

$$\limsup_{m \rightarrow \infty} \hat{A}_m^- \left(\frac{\sigma_m^-(\Delta_m^-)}{\hat{\mu}_m^-(\Delta_m^-)} \right)^{\tilde{q}/(p-1)} \leq 1 + e^{-1/(c\epsilon)}. \quad (4.30)$$

To prove (4.30), let T_m^- be a conformal affine mapping of \mathbf{R}^n which maps the plane W containing the origin and with normal e_n onto \hat{P}_m with $T_m^-(0) = y_m^-$ and $T_m^-(e_n) = \hat{y}_m^-$. Let $\tilde{D}_m^-, \tilde{u}_m^-$ be such that $T_m^-(\tilde{D}_m^-) = \hat{D}_m^-$ and $\hat{u}_m^-(T_m^-x) = \tilde{u}_m^-(x)$ whenever $x \in \tilde{D}_m^-$. Then, since the p -Laplace equation is invariant under translations, rotations, and dilations, we see that \tilde{u}_m^- is the p -capacitary function for \tilde{D}_m^- . Moreover, if $\tilde{\mu}_m^-$ corresponds to \tilde{u}_m^- as in Lemma 2.5 and $\tilde{\sigma}_m^-$ denotes surface measure on $\partial\tilde{D}_m^-$, then

$$\hat{A}_m^- \left(\frac{\sigma_m^-(\Delta_m^-)}{\hat{\mu}_m^-(\Delta_m^-)} \right)^{\tilde{q}/(p-1)} = \tilde{A}_m^- \left(\frac{\tilde{\sigma}_m^-(\partial\tilde{D}_m^- \cap B(0, 10/A))}{\tilde{\mu}_m^-(\partial\tilde{D}_m^- \cap B(0, 10/A))} \right)^{\tilde{q}/(p-1)}, \quad (4.31)$$

where

$$\tilde{A}_m^- := \frac{1}{\tilde{\sigma}_m^-(\partial\tilde{D}_m^- \cap B(0, 10/A))} \int_{\partial\tilde{D}_m^- \cap B(0, 10/A)} |\nabla\tilde{u}_m^-|^{\tilde{q}} d\tilde{\sigma}_m^-. \quad (4.32)$$

Letting $m \rightarrow \infty$ one deduces from Lemma 2.1 and Lemma 2.2, that \tilde{u}_m^- converges uniformly on \mathbf{R}^n to \tilde{u}^- where \tilde{u}^- is the p -capacitary function for the starlike Lipschitz ring domain, $\tilde{D}^- = \hat{D} \setminus B(e_n, 1/100)$. Also \hat{D} is obtained by drawing all line segments connecting points in $B(0, 1) \cap W$ to points in $B(e_n, 1/10)$. Using a Rellich type inequality and arguing as in [LN, displays (5.27)-(5.41)] it follows that

$$\begin{aligned} \limsup_{m \rightarrow \infty} (\tilde{A}_m^-)^{1/\tilde{q}} &\leq \left(\frac{1}{H^{n-1}(W \cap B(0, 10/A))} \int_{W \cap B(0, 10/A)} |\nabla\tilde{u}^-|^p dH^{n-1} \right)^{1/p} \\ \liminf_{m \rightarrow \infty} \frac{\tilde{\mu}_m^-(\partial\tilde{D}_m^- \cap B(0, 10/A))}{\tilde{\sigma}_m^-(\partial\tilde{D}_m^- \cap B(0, 10/A))} &\geq \frac{1}{H^{n-1}(W \cap B(0, 10/A))} \int_{W \cap B(0, 10/A)} |\nabla\tilde{u}^-|^{p-1} dH^{n-1} \end{aligned} \quad (4.33)$$

where H^{n-1} denotes $(n-1)$ -dimensional Lebesgue or Hausdorff measure on W . Assuming (4.33) we note from Schwarz reflection that \tilde{u}^- has a p -harmonic extension to $B(0, 1/2)$ with $\tilde{u}^- \equiv 0$ on $W \cap B(0, 1/2)$. From barrier estimates we have $c^{-1} \leq |\nabla \tilde{u}^-| \leq c$ on $B(0, 1/4)$ where c depends only on p, n , and from Lemma 2.4 we find that $|\nabla \tilde{u}^-|$ is Hölder continuous with exponent σ on $\bar{B}(0, 1/4) \cap W$. Using these facts we conclude first that there exist $\hat{z} \in \bar{B}(0, 10/A) \cap W$ and a constant c such that

$$(1 - cA^{-\sigma}) |\nabla \tilde{u}^-(\hat{z})| \leq |\nabla \tilde{u}^-(z)| \leq (1 + cA^{-\sigma}) |\nabla \tilde{u}^-(\hat{z})| \quad (4.34)$$

whenever $z \in B(0, 10/A) \cap W$. Combining (4.30), (4.31), (4.33) and (4.34) one deduces for c, ϵ_0^{-1} , sufficiently large in view of (4.5) that

$$\limsup_{m \rightarrow \infty} \hat{A}_m^- \left(\frac{\sigma_m^-(\Delta_m^-)}{\hat{\mu}_m^-(\Delta_m^-)} \right)^{\hat{q}/(p-1)} \leq (1 + cA^{-\sigma})^{\hat{q}} \leq 1 + e^{-1/(c\epsilon)}, \quad (4.35)$$

which is (4.30). This completes our outline of the proof of Lemma 4.3. \square

Proof of Lemma 4.4. We use the same notation as in Lemma 4.3. If Lemma 4.4 is false, then there exist subsequences of $\{u_m^+\}, \{u_m^-\}$, say $v_j^+ = u_{m_j}^+, v_j^- = u_{m_j}^-$, with corresponding measures, $\{\nu_j^+\}, \{\nu_j^-\}$, and surface balls $\tilde{\Delta}_j^+ = \Delta_{m_j}^+, \tilde{\Delta}_j^- = \Delta_{m_j}^-$, satisfying

$$\lim_{j \rightarrow \infty} \frac{\nu_j^+(\tilde{\Delta}_j^+)}{\nu_j^-(\tilde{\Delta}_j^-)} \text{ exists and is } \neq 1. \quad (4.36)$$

To get a contradiction we note from the definition of v_j^+, v_j^- , and Lemma 2.1-Lemma 2.3 that

$$v_j^+ \leq u \leq v_j^- \leq v_j^+ + c\tilde{\delta}_j^\alpha u_{m_j}^+(\hat{y}_{m_j}) \quad (4.37)$$

on $\partial(\Omega_{m_j}^+ \cap C_{m_j})$ where $\alpha \in (0, 1)$ is as in Lemma 2.2 and $\tilde{\delta}_j = \delta_{m_j}$. From the maximum principle for p -harmonic functions we see that (4.37) also holds in $\Omega_{m_j}^+ \cap C_{m_j}$. Let $\check{v}_j^+(x) = v_j^+(T_{m_j}^-(x))/v_j^+(\hat{y}_{m_j}), x \in B(0, 10)$, and $\check{v}_j^- = v_j^-(T_{m_j}^-(x))/v_j^-(\hat{y}_{m_j}), x \in B(0, 10)$, where $T_{m_j}^-$ is the conformal affine mapping in Lemma 4.3. Let $C(0, 1)$ be the cylinder with center at 0, axis parallel to e_n , and with both height and radius equal to 1. From Lemma 2.1-Lemma 2.4 we see that \check{v}_j^+ and \check{v}_j^- , for $j = 1, 2, \dots$, have the following properties.

- (a) \check{v}_j^+ and \check{v}_j^- are uniformly Hölder continuous of order α in $C(0, 1)$.
- (b) \check{v}_j^+ and \check{v}_j^- are uniformly bounded in the norm of $W^{1,p}(C(0, 1))$.
- (c) $\nabla \check{v}_j^+$ and $\nabla \check{v}_j^-$ are uniformly Hölder continuous of order σ on compact subsets of $(T_{m_j}^-)^{-1}(\Omega_{m_j}^+ \cap C_{m_j})$.

Using these facts and (4.37) we get subsequences of $\{\check{v}_j^+\}, \{\check{v}_j^-\}$, also denoted $\{\check{v}_j^+\}, \{\check{v}_j^-\}$, with $\check{v}_j^+, \check{v}_j^- \rightarrow v$ as $j \rightarrow \infty$, where v is continuous in $C(0, 1)$, p -harmonic in the component H of $C(0, 1) \setminus W$ containing e_n . Moreover $v \in W^{1,p}(C(0, 1)) \cap L^\infty(C(0, 1))$, and $\check{v}_j^+, \check{v}_j^- \rightarrow v$ uniformly on compact subsets of $C(0, 1)$ as $j \rightarrow \infty$ while $\nabla \check{v}_j^+, \nabla \check{v}_j^- \rightarrow \nabla v$ uniformly on compact subsets of H . Finally v_j^-, v_j^+ both converge weakly to v in $W^{1,p}(C(0, 1))$ and $j \rightarrow \infty$. Let $\check{\nu}_j^+, \check{\nu}_j^-, \nu$ be the

measures corresponding to $\check{\nu}_j^+, \check{\nu}_j^-, \check{\nu}$ as in Lemma 2.5. Then from the above results concerning the various convergence, and Lemma 2.5, we see that $\{\check{\nu}_j^+\}, \{\check{\nu}_j^-\}$, converge weakly to ν . Hence

$$1 = \lim_{j \rightarrow \infty} \frac{\check{\nu}_j^+(C(0, 10/A))}{\check{\nu}_j^-(C(0, 10/A))} = \lim_{j \rightarrow \infty} \frac{\nu_j^+(\tilde{\Delta}_j^+)}{\nu_j^-(\tilde{\Delta}_j^-)}. \quad (4.38)$$

This equality contradicts (4.36). Thus Lemma 4.4 is true. \square

5 Proof of Theorem 3

In the proof of Theorems 3 and 4 we shall need the following lemma.

Lemma 5.1 *Let $\Omega \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain with constants, r_0, M, C . Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Suppose also that $\log |\nabla u| \in VMO(\Delta(w, r))$. Given $\epsilon > 0$ there exist $\tilde{r} = \tilde{r}(\epsilon), 0 < \tilde{r} < r/4$, and $c = c(p, n, M, C), 1 \leq c < \infty$, such that the following is true whenever $0 < r' \leq \tilde{r}$. There exists a set $G \subset \Delta(w, r')$ such that*

- (a) $\frac{\sigma(G \cap \Delta(w, r'))}{\sigma(\Delta(w, r'))} \geq 1 - \epsilon,$
- (b) $(1 - \epsilon)b^{p-1} \leq \frac{\mu(\Delta(y, s))}{\sigma(\Delta(y, s))} \leq (1 + \epsilon)b^{p-1}$ whenever $0 < s < r', y \in G \cap \Delta(w, r')$.

Here μ is the measure associated with u as in Lemma 2.5 and $\log b$ is the average of $\log |\nabla u|$ on $\Delta(w, 4r')$.

Proof. This lemma was proved in [LN2, Lemma 4.1] assuming that Ω is a bounded Lipschitz domain. The proof is essentially unchanged when $\Omega \subset \mathbf{R}^n$ is a bounded Ahlfors regular NTA-domain. \square

We now begin the proof of Theorem 3. Let $\Omega \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain with constants, r_0, M, C and $w \in \partial\Omega, 0 < r < r_0/4$. Recall from Section 1 that to prove Theorem 3 it suffices to prove (i), (ii) of (1.14). We begin with (ii) and shall once again argue by contradiction using a blow-up argument. Hence we assume that

$$0 < \eta = \lim_{\tilde{r} \rightarrow 0} \sup_{\tilde{w} \in \Delta(w, r/2)} \|n\|_{\text{BMO}(\Delta(\tilde{w}, \tilde{r}))}.$$

Then there exist a sequence of points $\{w_j\}, w_j \in \Delta(w, r/2)$, and a sequence of scales $\{r_j\}$, with $2^j r_j \rightarrow 0$, such that

$$\eta = \lim_{j \rightarrow \infty} \left(\frac{1}{\sigma(\Delta(w_j, r_j))} \int_{\Delta(w_j, r_j)} |n - n_{\Delta(w_j, r_j)}|^2 d\sigma \right)^{1/2}. \quad (5.1)$$

For fixed $p, 1 < p < \infty$ suppose that u is a positive p -harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Suppose also that $\log |\nabla u| \in VMO(\Delta(w, r))$

where ∇u is defined by way of Theorem 1 on $\Delta(w, 4r)$, σ almost everywhere. We now apply Lemma 5.1 to u with w, r' replaced by $w_j, 2^j r_j$ and with $\epsilon = 2^{-j^2}$. Then for j large enough there exists a set $\tilde{G}_j \subset \Delta(w_j, 2^j r_j)$, such that

$$\begin{aligned} (a') \quad & \frac{\sigma(\tilde{G}_j \cap \Delta(w_j, 2^j r_j))}{\sigma(\Delta(w_j, 2^j r_j))} \geq 1 - 2^{-j^2}, \\ (b') \quad & (1 - 2^{-j^2})b_j^{p-1} \leq \frac{\mu(\Delta(y, s))}{\sigma(\Delta(y, s))} \leq (1 + 2^{-j^2})b_j^{p-1} \text{ whenever } 0 < s < 2^j r_j \\ & \text{and } y \in \tilde{G}_j \cap \Delta(w, 2^j r_j). \end{aligned} \tag{5.2}$$

In (5.2) (b'), $\log b_j$ denotes the average of $\log |\nabla u|$ on $\Delta(w_j, 2^{j+2} r_j)$ with respect to σ . Let $T_j(z) = w_j + r_j z$ and put, for $j = 1, 2, \dots$,

$$\begin{aligned} \Omega_j &= T_j^{-1}(\Omega) = \{r_j^{-1}(x - w_j) : x \in \Omega\}, \\ \lambda_j &= (r_j b_j)^{-1}, \\ u_j(z) &= \lambda_j u(T_j(z)) \text{ whenever } z \in T_j^{-1}(B(w, 4r)). \end{aligned} \tag{5.3}$$

From translation and dilation invariance of the p -Laplace equation we see that u_j is p -harmonic in $\Omega_j \cap T_j^{-1}(B(w, 4r))$ and continuous in $T_j^{-1}(B(w, 4r))$ with $u_j \equiv 0$ in $T_j^{-1}(B(w, 4r)) \setminus \Omega_j$. Moreover, let μ, μ_j be the measures associated with u, u_j as in Lemma 2.5, defined in $B(w, 4r), T_j^{-1}(B(w, 4r))$, and let σ, σ_j be the surface measures on $\partial\Omega, \partial\Omega_j$, respectively. Using Ahlfors regularity of Ω_j and Theorem 1 we see that if H_j is a Borel subset of $\partial\Omega_j \cap T_j^{-1}(B(w, 4r))$, then

$$\sigma_j(H_j) = r_j^{1-n} \sigma(T_j(H_j)), \quad \mu_j(H_j) = \lambda_j^{p-1} r_j^{p-n} \mu(T_j(H_j)). \tag{5.4}$$

From (5.2), we see that if $G_j = T_j^{-1}(\tilde{G}_j)$, then $G_j \subset \partial\Omega_j \cap B(0, 2^j)$ and

$$\begin{aligned} (\alpha) \quad & \frac{\sigma_j(G_j \cap \partial\Omega_j \cap B(0, 2^j))}{\sigma_j(\partial\Omega_j \cap B(0, 2^j))} \geq 1 - 2^{-j^2}, \\ (\beta) \quad & 1 - 2^{-j^2} \leq \frac{\mu_j(\partial\Omega_j \cap B(z, s))}{\sigma_j(\partial\Omega_j \cap B(z, s))} \leq 1 + 2^{-j^2} \text{ whenever } 0 < s < 2^j \\ & \text{and } z \in G_j \cap \partial\Omega_j. \end{aligned} \tag{5.5}$$

In fact, (5.5) (β) is a straightforward consequences of (5.2) (a'), (b'), (5.3), and (5.4). Using this notation we prove the following lemmas.

Lemma 5.2 *Let $\Omega \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain with constants M, r_0, C , and let $w, r, \{w_j\}, \{r_j\}, \{\Omega_j\}$ be as above. Then there exists a subsequence $k = k(j), j = 1, 2, \dots$, satisfying $\Omega_k \rightarrow \Omega_\infty$ and $\partial\Omega_k \rightarrow \partial\Omega_\infty$ in the Hausdorff distance sense, uniformly on compact subsets of \mathbf{R}^n , as $j \rightarrow \infty$. Moreover, Ω_∞ is an Ahlfors regular NTA-domain with constants, $M, \infty, C' = C'(M, C)$. Also if $F = (F_1, F_2, \dots, F_n)$, where $F_i, 1 \leq i \leq n$, is continuous on \mathbf{R}^n with compact support, then*

$$\int_{\partial\Omega_k} \langle n_k, F \rangle d\sigma_k \rightarrow \int_{\partial\Omega_\infty} \langle n_\infty, F \rangle d\sigma_\infty \text{ as } j \rightarrow \infty$$

where n_k, n_∞ denote the inner unit normals to $\partial\Omega_k, \partial\Omega_\infty$, respectively, and σ_k, σ_∞ are the surface measures on $\partial\Omega_k, \partial\Omega_\infty$, respectively. Finally, if Ω is (δ, r_0) -Reifenberg flat, $0 < \delta < \hat{\delta}(n)$, then we can also choose the subsequence so that Ω_∞ is a 4δ -Reifenberg flat domain.

Lemma 5.3 *Let $\Omega \subset \mathbf{R}^n$ be an Ahlfors regular NTA-domain, with constants M, r_0, C , and let $w, r, \{w_j\}, \{r_j\}, \{\Omega_j\}, \{u_j\}, \{\mu_j\}$, and $k = k(j)$, be as above. If $\{\Omega_k\}, \Omega_\infty$ are as in Lemma 5.2, then there exist subsequences of $\{u_k\}, \{\mu_k\}$ (also denoted $\{u_k\}, \{\mu_k\}$), with $u_k \rightarrow u_\infty$, uniformly on compact subsets of \mathbf{R}^n and where u_∞ is a non-negative p -harmonic function in Ω_∞ and u_∞ is Hölder continuous on \mathbf{R}^n with $u_\infty \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\infty$. Moreover, if μ_∞ denotes the measure corresponding to u_∞ , then $\mu_k \rightarrow \mu_\infty$ weakly and $|\nabla u_\infty| \geq 1$ holds σ_∞ almost everywhere on $\partial\Omega_\infty$.*

Lemma 5.4 *Under the same scenario as in Lemma 5.3 we have*

$$|\nabla u_\infty(z)| \leq 1 \text{ for all } z \in \Omega_\infty.$$

Lemma 5.5 *Under the assumptions in Lemma 5.3 it is true that $\sigma_k \rightarrow \sigma_\infty$ weakly as Radon measures.*

Lemma 5.6 *Assume that Ω_∞ is a 4δ -Reifenberg flat domain and that Ω_∞ is Ahlfors regular. Moreover, assume that $\Omega_\infty, u_\infty, \mu_\infty, \sigma_\infty$ are as in Lemma 5.2-Lemma 5.5. There exists $\bar{\delta} = \bar{\delta}(p, n)$ small such that if $\delta < \bar{\delta}$, then Ω_∞ is a half-space and $u_\infty(y', y_n) = y_n$ in an appropriate coordinate system.*

Before proving Lemma 5.1-Lemma 5.6 we show that these lemmas imply (5.1). Indeed, from Lemma 5.2 and Lemma 5.6 we see that $\langle n_k, \cdot \rangle d\sigma_k \rightarrow \langle e_n, \cdot \rangle d\sigma_\infty$ as $j \rightarrow \infty$, weakly as measures in an appropriate coordinate system. Thus, for $1 \leq m \leq n$,

$$\lim_{j \rightarrow \infty} \int_{\partial\Omega_k \cap B(0,1)} \langle n_k, e_m \rangle d\sigma_k = \delta_{mn} \sigma_\infty(\partial\Omega_\infty \cap B(0,1)), \quad (5.6)$$

where n_k is the inner unit normal to $\partial\Omega_k$ and δ_{mn} is the Kronecker delta. Let a_k denote the average of n_k on $\partial\Omega_k \cap B(0,1)$ with respect to σ_k . From (5.6), Lemma 5.5, and the fact that (5.1) is scale invariant we first see that

$$\lim_{j \rightarrow \infty} \left(\frac{1}{\sigma_k(\partial\Omega_k \cap B(0,1))} \int_{\partial\Omega_k \cap B(0,1)} |e_n - a_k|^2 d\sigma_k \right)^{1/2} = 0. \quad (5.7)$$

Second from (5.1), (5.7), the triangle inequality, Lemma 5.5, and (5.6) we obtain

$$\begin{aligned} 0 < \eta &= \lim_{j \rightarrow \infty} \left(\frac{1}{\sigma_k(\partial\Omega_k \cap B(0,1))} \int_{\partial\Omega_k \cap B(0,1)} |n_k - a_k|^2 d\sigma_k \right)^{1/2} \\ &\leq \limsup_{j \rightarrow \infty} \left(\frac{1}{\sigma_k(\partial\Omega_k \cap B(0,1))} \int_{\partial\Omega_k \cap B(0,1)} |n_k - e_n|^2 d\sigma_k \right)^{1/2} \\ &= \limsup_{j \rightarrow \infty} \left(\frac{1}{\sigma_k(\partial\Omega_k \cap B(0,1))} \int_{\partial\Omega_k \cap B(0,1)} 2(1 - \langle n_k, e_n \rangle) d\sigma_k \right)^{1/2} = 0. \end{aligned}$$

We have reached a contradiction. Thus (ii) in (1.14) is true once we prove Lemma 5.2-Lemma 5.6.

5.1 Proof of Lemma 5.2-Lemma 5.6

Proof of Lemma 5.2. For convergence of $\{\Omega_k\}$ to Ω_∞ in the sense of Hausdorff distance, and the statement about Ω_∞ , see [KT, Theorem 4.1]. Using these facts, the display in Lemma 5.2 follows from the Gauss-Green theorem and geometric measure theory, see [EG, sec 5.8]. \square

Proof of Lemma 5.3. Let $\{\Omega_k\}$ be as in Lemma 5.2 and let $\{u_k\}, \{\mu_k\}$ be subsequences of $\{u_j\}, \{\mu_j\}$, corresponding to $\{\Omega_k\}$. Then from Lemma 2.1-Lemma 2.5 applied to u_k and (5.5) (β) we deduce that u_k is bounded, Hölder continuous, and locally in $W^{1,p}$ on compact subsets of \mathbf{R}^n with the norms of all functions bounded above by constants which are independent of j . Also, if $B(x, 2\rho) \subset \Omega_\infty$, then for large j we see from Lemma 5.1 and Lemma 2.4 that ∇u_k is Hölder continuous and bounded on compact subsets of $\Omega_k \cap B(x, \rho)$ again with constants independent of j for j large enough. Thus we assume, as we may, that $\{u_k\}$ converges uniformly and weakly in $W^{1,p}$ on compact subsets of \mathbf{R}^n to u_∞ and that $\{\nabla u_k\}$ converges uniformly to ∇u_∞ on compact subsets of Ω_∞ . Also, $u_\infty > 0$ in Ω_∞ and u_∞ is p -harmonic in Ω_∞ and continuous on \mathbf{R}^n , with $u_\infty \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\infty$. Using these facts we deduce that if μ_∞ denotes the measure associated with u_∞ as in Lemma 2.5, and $\theta \in C_0^\infty(\mathbf{R}^n)$, then

$$\begin{aligned} -\int_{\mathbf{R}^n} \theta d\mu_\infty &= \int_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla \theta \rangle dx \\ &= \lim_{j \rightarrow \infty} \int_{\mathbf{R}^n} |\nabla u_k|^{p-2} \langle \nabla u_k, \nabla \theta \rangle dx = -\lim_{j \rightarrow \infty} \int_{\mathbf{R}^n} \theta d\mu_k. \end{aligned} \quad (5.8)$$

Thus $\{\mu_k\}$ converges weakly to μ_∞ .

To show that $|\nabla u_\infty| \geq 1$ σ_∞ almost everywhere on $\partial\Omega_\infty$ we observe from Theorem 1 and Lemma 5.2 that it suffices to prove

$$\sigma_\infty \leq \mu_\infty. \quad (5.9)$$

To prove (5.9) we first observe from Theorem 1 that $d\mu_k = |\nabla u_k|^{p-1} d\sigma_k$ on $\partial\Omega_k \cap (T_k)^{-1}(B(w, 4r))$. Second, using this observation, (5.5) (β) , and differentiation theory we see that

$$1 - 2^{-k^2} \leq |\nabla u_k|^{p-1} \leq 1 + 2^{-k^2} \quad (5.10)$$

σ_k almost everywhere on $\partial\Omega_k \cap G_k \cap B(0, 2^k)$. Let $0 \leq \theta \in C_0^\infty(\mathbf{R}^n)$. Using what we have already proved in Lemma 5.2, (5.5) (α) , and (5.10), we deduce that

$$\begin{aligned} \int \theta d\mu_\infty &= \lim_{j \rightarrow \infty} \int_{\partial\Omega_k} \theta |\nabla u_k|^{p-1} d\sigma_k \geq \liminf_{j \rightarrow \infty} \int_{G_k \cap \partial\Omega_k} \theta |\nabla u_k|^{p-1} d\sigma_k \\ &\geq \liminf_{j \rightarrow \infty} (1 - 2^{-k^2}) \int_{G_k \cap \partial\Omega_k} \theta d\sigma_k = \liminf_{j \rightarrow \infty} \int_{G_k \cap \partial\Omega_k} \theta d\sigma_k. \end{aligned} \quad (5.11)$$

Furthermore,

$$\int_{G_k \cap \partial\Omega_k} \theta d\sigma_k = \int_{\partial\Omega_k} \theta d\sigma_k + A_k, \quad (5.12)$$

where, for fixed θ and for j large enough, we have, using (5.5) (α) and Ahlfors regularity of Ω_k , that

$$|A_k| \leq \|\theta\|_\infty 2^{-k^2} \sigma_k(\partial\Omega_k \cap B(0, 2^k)) \leq c \|\theta\|_\infty 2^{-k^2} 2^{(n-1)k}. \quad (5.13)$$

From (5.12), (5.13) and lower semicontinuity type arguments we also deduce that

$$\int \theta d\sigma_\infty \leq \liminf_{j \rightarrow \infty} \int \theta d\sigma_k = \liminf_{j \rightarrow \infty} \int_{G_k \cap \partial\Omega_k} \theta d\sigma_k. \quad (5.14)$$

Using (5.14) in (5.11) we get (5.9). The proof of Lemma 5.3 is now complete. \square

Proof of Lemma 5.4. We temporarily fix j large and put $\tilde{\xi} = 2^{1-k^2}$ where $k = k(j)$ is as in Lemma 5.3 and Lemma 5.4. We note from Theorem 1, with $b = 100^{-100n^2}$, $b' = b^{100}$, (5.10), and Egoroff's theorem that there exists a compact set $K = K(k) \subset G_k \cap B(0, 2^{k-4})$ and $\rho > 0$ such that

$$\begin{aligned} (i) \quad & \sigma_k(G_k \cap B(0, 2^{k-4}) \setminus K) \leq \tilde{\xi} \sigma_k(\partial\Omega_k \cap B(0, 2^k)), \\ (ii) \quad & 1 - \tilde{\xi} \leq |\nabla u_k(y)|^{p-1} \leq 1 + \tilde{\xi} \text{ if } y \in B(x, \rho) \cap \Gamma_{b'}(x) \text{ and } x \in K. \end{aligned} \quad (5.15)$$

Here, $\Gamma_{b'}(x) = \{z \in \Omega_k : d(z, \partial\Omega_k) > b'|x - z|\}$. Let $Q(w, s) = \{y : |y_j - w_j| \leq s, 1 \leq j \leq n\}$ and let $\{Q_i = Q_i(\hat{y}_i, r_i)\}$ be a Whitney cube decomposition of Ω_k satisfying

$$\begin{aligned} (a) \quad & Q_i \cap Q_m = \emptyset, i \neq m, \\ (b) \quad & 10^{-6n} d(Q_i, \partial\Omega_k) \leq r_i \leq 10^{-4n} d(Q_i, \partial\Omega_k), \\ (c) \quad & \bigcup_i \bar{Q}_i = \Omega_k. \end{aligned} \quad (5.16)$$

Next let $\{\eta_i\}$ be a partition of unity adapted to $\{Q_i\}$. That is

$$\begin{aligned} (i) \quad & \sum_i \eta_i \equiv 1. \\ (ii) \quad & \text{The support of } \eta_i \text{ is contained in the union of } \bar{Q}_i \text{ and } \bigcup \{Q_j : \bar{Q}_j \cap \bar{Q}_i \neq \emptyset\}. \\ (iii) \quad & \eta_i \geq 0 \text{ is infinitely differentiable with } \eta_i \geq c^{-1} \text{ on } Q_i \text{ and} \\ & \max\{r_i^{-1} |(\eta_i)_{x_l}|, r_i^{-2} |(\eta_i)_{x_l x_m}|\} \leq c, \text{ for } 1 \leq m, l \leq n. \end{aligned} \quad (5.17)$$

To prove Lemma 5.4 we will use an argument somewhat similar to the one in [LV]. Let

$$\hat{D} = \bigcup_{x \in K} \Gamma_b(x) \cap B(0, 2^k) \text{ and } D = \hat{D} \cap \{z \in \Omega_k : |\nabla u_k(z)|^{p-1} > 1 + \tilde{\xi}\}. \quad (5.18)$$

Using (5.5), Lemma 2.5, Lemma 2.4, we deduce that

$$|\nabla u_k(y)| \leq c \text{ and } c^{-1} \leq r_l^{-1} u_k(y) \leq c \text{ on } B(\hat{y}_l, \frac{1}{2}d(\hat{y}_l, \partial\Omega_k)) \quad (5.19)$$

whenever $\bar{Q}_l \cap \hat{D} \neq \emptyset$ for some $c = c(p, n)$. By definition we have $1 + \tilde{\xi} < |\nabla u_k|^{p-1}$ on D . As in (1.11) we put

$$L = \sum_{l,m=1}^n \frac{\partial}{\partial y_l} \left(b_{lm}(y) \frac{\partial}{\partial y_m} \right) \text{ for } y \in D, \quad (5.20)$$

where

$$b_{lm}(y) = b_{ml}(y) = |\nabla u_k|^{p-4} [(p-2)(u_k)_{y_l}(u_k)_{y_m} + \delta_{lm} |\nabla u_k|^2](y), 1 \leq l, m \leq n. \quad (5.21)$$

From (5.19) and (1.13) we deduce that there exists $\bar{c} = \bar{c}(p, n)$ with

$$\bar{c}^{-1} |\xi|^2 \leq \sum_{l,m=1}^n b_{lm}(y) \xi_l \xi_m \leq \bar{c} |\xi|^2, \quad (5.22)$$

whenever $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$ and $y \in D$. Observe from (5.22) and Lemma 2.4 that L in (5.20) is a uniformly elliptic operator in D with infinitely differentiable coefficients. Note also that $\zeta = u_k$ or $\zeta =$ a partial derivative of u_k is a solution to $L\zeta = 0$ in D . Let $a = \max(3, p)$ and put

$$v(y) = [\max(|\nabla u_k|^2 - (1 + 2\tilde{\xi})^{2/(p-1)}, 0)]^a(y) \text{ when } y \in \hat{D}.$$

By definition $v(y) > 0$ at $y \in \hat{D}$ only if $|\nabla u_k|^{p-1}(y) > (1 + 2\tilde{\xi})$, v has continuous partial derivatives of second order in \hat{D} and $Lv \geq 0$ in D . Also from (5.15) and the definition of v, D , we deduce that there is an open set N containing $K \cup (\partial D \cap \{x : |\nabla u_k(x)|^{p-1} = 1 + \tilde{\xi}\})$ with

$$v \equiv 0 \text{ on } D \cap N. \quad (5.23)$$

Next fix $z_0 \in \Omega_\infty$ with $|z_0| \leq 2^{k/4}$ and suppose $v(z_0) > 0$. We claim that

$$z_0 \in D \text{ and } d(z_0, \partial\Omega_k) \geq \max\left\{\frac{1}{k}, \frac{d(z_0, \partial\Omega_\infty)}{2}\right\}, \quad (5.24)$$

for j large enough. The second statement in (5.24) follows from Lemma 5.2. To prove the first claim it suffices to show that

$$h(\partial\Omega_k \cap B(0, 2^{1+k/2}), \partial\hat{D} \cap B(0, 2^{1+k/2})) \leq c \tilde{\xi}^{1/(2n)}. \quad (5.25)$$

Indeed if $y \in \partial\hat{D} \cap B(0, 2^{1+k/2}) \setminus K$, then there exists $w \in \partial\Omega_k \setminus K$ with $|x - y| \geq |y - w|/b$ whenever $x \in K$. Hence by the triangle inequality, $|x - w| \geq (1/b - 1)|y - w|$ whenever $x \in K$. Now from (5.15), (5.5), and Ahlfors regularity of $\partial\Omega_k$ we deduce that every ball about w of radius $c\tilde{\xi}^{1/(2n)}$ must, for k large enough, contain points of K . Thus every $y \in \partial\hat{D} \cap B(0, 2^{1+k/2})$ lies within $c\tilde{\xi}^{1/(2n)}$ of a point of $\partial\Omega_k \cap B(0, 2^{1+k/2})$. Here c depends only on p, n , and the NTA and Ahlfors regularity constants for Ω . The opposite inequality with these sets interchanged follows easily from (5.15), (5.5). Thus (5.25) is true.

Let D_1 be the component of D containing z_0 and let $g(\cdot, \cdot)$ be Green's function for L relative to D_1 . Extend $g(z_0, \cdot)$ to \hat{D} by putting $g(z_0, \cdot) \equiv 0$ on $\hat{D} \setminus D_1$. Then from (b') of Lemma 2.10 we see that $g(z_0, \cdot) \in W^{1,2}(\hat{D} \setminus B(z_0, \epsilon))$ for each $\epsilon > 0$. Let $\hat{v}(x) = v(x)$ when $x \in D_1$ and $\hat{v} = 0$ elsewhere in \hat{D} . Next consider fixed t with $2^{k/2} \leq t \leq 2^{1+k/2}$ and put

$$\Lambda(t) = \{l : \text{supp } \eta_l \subset \hat{D} \cap B(0, t) \text{ and } r_l \geq \tau\} \quad (5.26)$$

where $\tau = \min(\rho^2, \tilde{\xi}^2)$ and where $\text{supp } \eta_l$ denotes the support of η_l . Recall that ρ was introduced in (5.15). If $i \in \Lambda(t)$, we observe from (5.23) and (5.26) that $\hat{v} \eta_i$ can be uniformly approximated

in the $C^2(\hat{D})$ norm by $C_0^\infty(D_1)$ functions. Using this fact, Lemma 2.14 (c'), and integrating by parts we see, if $\theta = \hat{v} \sum_{i \in \Lambda(t)} \eta_i$, that

$$\begin{aligned}
v(z_0) &= \sum_{i \in \Lambda(t)} \int \sum_{l,m=1}^n b_{lm} (\hat{v} \eta_i)_{y_l} g(z_0, \cdot)_{y_m} dy \\
&= -2 \int \sum_{i \in \Lambda(t)} \sum_{l,m=1}^n b_{lm} \hat{v}_{y_l} (\eta_i)_{y_m} g(z_0, \cdot) dy \\
&\quad - \int \sum_{i \in \Lambda(t)} \sum_{l,m=1}^n (b_{lm} \hat{v}_{y_l})_{y_m} \eta_i g(z_0, \cdot) dy \\
&\quad - \int \sum_{i \in \Lambda(t)} \hat{v} \sum_{l,m=1}^n (b_{lm} (\eta_i)_{y_l})_{y_m} g(z_0, \cdot) dy \\
&=: -I_1(t) - I_2(t) - I_3(t).
\end{aligned} \tag{5.27}$$

Using that $L\hat{v} \geq 0$ we deduce that $I_2(t) \geq 0$. Observe that if $\sum_{i \in \Lambda(t)} \eta_i \equiv 1$ on Q_l , then the integrands in the integrals defining $I_1(t), I_3(t)$ vanish on Q_l . Let $\Lambda_1(t)$ be the set of all l such that $v \neq 0$ on $\text{supp } \eta_l$ and such that there exists $m \notin \Lambda(t), i \in \Lambda(t)$, with $\text{supp } \eta_m \cap \text{supp } \eta_l \neq \emptyset$ while $\text{supp } \eta_i \cap \text{supp } \eta_l = \emptyset$. Using the above observation, Lemma 2.4, (5.17) (iii), and (5.19) we deduce that

$$|I_3|(t) + |I_1|(t) \leq c \sum_{m \in \Lambda_1(t)} \int_{Q_m} (r_m^{-1} |\nabla u_k|^{2(p-1)} \sum_{l,i=1}^n |(u_k)_{y_l y_i}| + r_m^{-2}) g(z_0, \cdot) dy. \tag{5.28}$$

Applying Hölder's inequality, (5.19), and Lemma 2.4 in (5.28) we find that

$$|I_3|(t) + |I_1|(t) \leq cJ(t), \quad J(t) := \sum_{m \in \Lambda_1(t)} r_m^{(n-1)/2} \left(\int_{Q_m} r_m^{-3} g^2(z_0, \cdot) dy \right)^{1/2}. \tag{5.29}$$

Summarizing we have

$$v(z_0) \leq cJ(t) \text{ whenever } t \text{ satisfies } 2^{k/2} \leq t \leq 2^{1+k/2}. \tag{5.30}$$

To estimate $J(t)$ we divide the indices in $\Lambda_1(t)$ into $\Lambda_{11}(t), \Lambda_{12}(t)$, where $\Lambda_{11}(t)$ consists of all i in $\Lambda_1(t)$ with $B(\hat{y}_i, \frac{d(\hat{y}_i, \partial\Omega_k)}{100}) \cap \partial B(0, t) \neq \emptyset$, while $\Lambda_{12}(t) = \Lambda_1(t) \setminus \Lambda_{11}(t)$. We write $J(t) = J_1(t) + J_2(t)$ where $J_1(t), J_2(t)$, are defined as in (5.29) with $\Lambda_{11}(t), \Lambda_{12}(t)$ replacing $\Lambda_1(t)$, respectively. We next estimate $J_1(t)$ and $J_2(t)$.

We first estimate $J_1(t)$ whenever t satisfies $2^{k/2} \leq t \leq 2^{1+k/2}$. To do this we first note, using Hölder's inequality, disjointness of $\{Q_m\}$, and the definition of $\Lambda_{11}(t)$, that

$$\begin{aligned}
J_1(t) &\leq c \left(\sum_{m \in \Lambda_{11}(t)} r_m^{n-1} \right)^{1/2} \left(\sum_{m \in \Lambda_{11}(t)} \int_{Q_m} r_m^{-3} g^2(z_0, \cdot) dy \right)^{1/2} \\
&\leq c^2 2^{k(n-1)/4} \left(\sum_{m \in \Lambda_{11}(t)} \int_{Q_m} r_m^{-3} g^2(z_0, \cdot) dy \right)^{1/2} =: c^2 2^{k(n-1)/4} J_{11}(t).
\end{aligned} \tag{5.31}$$

Let $P = B(0, 2^{2+k/2}) \setminus B(0, 2^{-1+k/2})$ and let Λ' denote all Whitney cubes that have a nonempty intersection with the closure of $\hat{D} \cap P$. Integrating J_1 over $t \in (2^{k/2}, 2^{k/2+1})$ we obtain from (5.31) that

$$\begin{aligned} \int_{2^{k/2}}^{2^{1+k/2}} J_1(t) dt &\leq c2^{kn/4} \left(\int_{2^{k/2}}^{2^{1+k/2}} J_{11}(t)^2 dt \right)^{1/2} \\ &\leq c2^{kn/4} \left(\int_P \sum_{m \in \Lambda'} r_m^{-2} \chi_{Q_m} g(z_0, \cdot)^2 dy \right)^{1/2}. \end{aligned} \quad (5.32)$$

To get the last inequality in (5.32) we interchanged the order of integration and used the fact that each Q_m has an index appearing in $\Lambda_{11}(t)$ for a set of t of at most length cr_m . We now claim that there exists $\bar{c} \geq 1$, depending only on p, n , and the NTA and Ahlfors regularity constants for Ω , such that if $V = \{x : \bar{c}^{-1}2^{k/2} \leq |x| \leq \bar{c}2^{k/2}\} \cap \hat{D}$, then

$$\int_P \sum_{m \in \Lambda'} r_m^{-2} \chi_{Q_m} g(z_0, \cdot)^2 dy \leq \bar{c} \int_V |\nabla g(z_0, \cdot)|^2 dy. \quad (5.33)$$

(5.33) follows easily from (ii) in Definition 1.1 and Theorem 2 in [Le]. For completeness we sketch a proof of (5.33). From Poincaré's inequality we see that

$$\int_{Q_m} r_m^{-2} \chi_{Q_m} g(z_0, \cdot)^2 dy \leq c \int_{Q_m} |\nabla g(z_0, \cdot)|^2 dy + cr_m^{n-2} g_{Q_m}^2 \quad (5.34)$$

where $g_{Q_m}(z_0, \cdot)$ is the average of $g(z_0, \cdot)$ on Q_m . Then since $g(z_0, \cdot) \in W_0^{1,2}(\hat{D} \setminus B(z_0, \epsilon))$ for each $\epsilon > 0$ and the complement of \hat{D} satisfies the corkscrew condition in (ii) of Definition 1.1, we deduce

$$|g_{Q_m}| \leq cr_m \inf_{Q_m} M(\chi_V |\nabla g(z_0, \cdot)|) \quad (5.35)$$

where once again Mf denotes the Hardy-Littlewood maximal function of f . Using (5.35) in (5.34), summing, and applying the Hardy-Littlewood maximal theorem we get (5.33). Armed with (5.33) we can now complete the estimate of $J_1(t)$. Indeed, using (5.33) in (5.32), and standard Caccioppoli type inequalities for solutions to L , we deduce that

$$\left(\int_{2^{k/2}}^{2^{1+k/2}} J_1(t) dt \right)^2 \leq c2^{kn/2} \int_V |\nabla g(z_0, \cdot)|^2 dy \leq c2^{k(n-2)/2} \int_W g(z_0, \cdot)^2 dy \quad (5.36)$$

where $W = \{x : \bar{c}^{-2}2^{k/2} \leq |x| \leq \bar{c}^2 2^{k/2}\} \cap \hat{D}$ and \bar{c} is as in the definition of V . Observe from (5.5) and the definition of \hat{D}, K , that there exists $x \in \partial \hat{D} \cap \partial \Omega_k$ with $|x| \leq 2^{-k}$ for k large. Using this value of x , as well as $z = z_0, y = z$, in Lemma 2.11 it follows that

$$g(z, z_0) \leq ck2^{k(1-n/2-\beta/4)} \text{ whenever } z \in W. \quad (5.37)$$

Using (5.37) in (5.36) we conclude that

$$\left(\int_{2^{k/2}}^{2^{1+k/2}} J_1(t) dt \right)^2 \leq ck^2 2^{k(1-\beta/2)}. \quad (5.38)$$

We now estimate $J_{12}(t)$ whenever $t \in (2^{k/2}, 2^{k/2+1})$. To do this we first note that if $i \in \Lambda_{12}(t)$, then

$$B(\hat{y}_i, \frac{d(\hat{y}_i, \partial\Omega_k)}{100}) \cap \partial\hat{D} \neq \emptyset. \quad (5.39)$$

In fact if (5.39) was false, then from the definition of $\Lambda_1(t), \Lambda_{11}(t)$, we see that $d(\hat{y}_i, \partial\Omega_k) \leq c\tau$. Hence, for k large enough it would follow from (5.15) and the definition of b, b', \hat{D}, τ, v , that $v \equiv 0$ on $\text{supp } \eta_i$, and this contradicts $i \in \Lambda_1(t)$. Furthermore, if $i \in \Lambda_{12}(t)$ and $z \in Q_i$, then from Lemma 2.10 (d'), Harnack's inequality for u_k , and the maximum principle for solutions to L we see that

$$g(z_0, z) \leq c\phi u_k(z) \text{ whenever } z \in Q_i \quad (5.40)$$

where $\phi := u_k(z_0)^{-1} d(z_0, \partial\Omega_\infty)^{2-n}$ if $n > 2$ and $\phi := u_k(z_0)^{-1} \log(k/d(z_0, \partial\Omega_\infty))$ if $n = 2$. Also from (5.19) and (5.39) we deduce that

$$u_k(z) \leq cr_i \text{ whenever } z \in Q_i \subset \Lambda_{12}. \quad (5.41)$$

Using (5.40), (5.41) we get

$$J_2(t) = \sum_{m \in \Lambda_{12}(t)} r_m^{(n-1)/2} \left(\int_{Q_m} r_m^{-3} g^2(z_0, \cdot) dy \right)^{1/2} \leq c\phi \sum_{m \in \Lambda_{12}(t)} r_m^{n-1}. \quad (5.42)$$

Our next task is to estimate the sum on the right hand side of (5.42). Observe from (5.39) and the same argument as in the proof of (5.25) that if $i \in \Lambda_{12}(t)$, then there exists $\tilde{c} \geq 1$ and $\tilde{y}_i \in \partial\Omega_k$ such that

$$B(\tilde{y}_i, 2r_i/\tilde{c}) \cap K = \emptyset, d(\tilde{y}_i, K) \leq \tilde{c}r_i, \text{ and } |\hat{y}_i - \tilde{y}_i| \leq \tilde{c}r_i. \quad (5.43)$$

From (5.43) and the fact that $\{Q_m\}$ are Whitney cubes we deduce that if $i \in \Lambda_{12}(t)$, then

$$B(\tilde{y}_i, r_i/\tilde{c}) \cap B(\tilde{y}_l, r_l/\tilde{c}) \neq \emptyset \text{ for at most } N \text{ indexes } l, \quad (5.44)$$

where N depends on the NTA, Ahlfors regularity constants for Ω, p, n, b , but is independent of k . Using Ahlfors regularity of $\partial\Omega_k$, (5.44), (5.15), and (5.5) (β) in (5.42), we find that

$$\begin{aligned} J_2(t) &\leq c\phi \sum_{m \in \Lambda_{12}(t)} \sigma_k(\partial\Omega_k \cap B(\tilde{y}_m, r_m/\tilde{c})) \\ &\leq c\phi \sigma_k(\partial\Omega_k \cap B(0, 2^{k-4}) \setminus K) \leq 2^{-k}\phi, \end{aligned} \quad (5.45)$$

whenever $k \geq k_0$ for some k_0 large enough.

To complete the proof of Lemma 5.4 we first deduce, from (5.38) and weak type estimates, the existence of $\hat{t} \in (2^{k/2}, 2^{k/2+1})$ with $J_1(\hat{t}) \leq ck2^{-\beta k/4}$. Using this inequality, (5.45) in (5.29) and then (5.27), we conclude that

$$v(z_0) \leq |I_1(\hat{t})| + |I_2(\hat{t})| \leq 2^{-k\beta/8} + 2^{-k}\phi$$

for k large enough. Thus $v(z_0) = v(z_0, k) \rightarrow 0$ as $k \rightarrow \infty$. Based on the definition of v we conclude that $|\nabla u_\infty|(z_0) \leq 1$ in Ω_∞ . Since $z_0 \in \Omega_\infty$ is arbitrary, the proof of Lemma 5.4 is complete. \square

Proof of Lemma 5.5. Using Lemma 5.3 and Lemma 5.4 we can conclude that $d\sigma_\infty = d\mu_\infty$. Hence, for $\theta \in C_0^\infty(\mathbf{R}^n)$ and the subsequence, $k(j), j = 1, 2, \dots$, we have

$$\begin{aligned} \int \theta d\sigma_\infty &= \lim_{j \rightarrow \infty} \int_{\partial\Omega_k} \theta |\nabla u_k|^{p-1} d\sigma_k \geq \limsup_{j \rightarrow \infty} \int_{G_k \cap \partial\Omega_k} \theta |\nabla u_k|^{p-1} d\sigma_k \\ &\geq \limsup_{j \rightarrow \infty} (1 - 2^{-k^2}) \int_{G_k \cap \partial\Omega_k} \theta d\sigma_k = \limsup_{j \rightarrow \infty} \int_{\partial\Omega_k} \theta d\sigma_k \end{aligned} \quad (5.46)$$

where the last equality follows as in (5.12) and (5.13). Moreover from (5.14) we have

$$\int_{\partial\Omega_\infty} \theta d\sigma_\infty \leq \liminf_{j \rightarrow \infty} \int_{\partial\Omega_k} \theta d\sigma_k \text{ whenever } 0 \leq \theta \in C_0^\infty(\mathbf{R}^n). \quad (5.47)$$

Combining (5.46) and (5.47) we see that

$$\sigma_k \rightarrow \sigma_\infty \text{ weakly as } j \rightarrow \infty. \quad (5.48)$$

This completes the proof of Lemma 5.5. \square

Proof of Lemma 5.6. As stated in the introduction we can use either the results in [ACF] (see also [DP]) or [LN5, LN6] to complete the proof of Lemma 5.6. Each argument makes rather subtle, somewhat different, smallness assumptions on δ . Thus it is not easy to tout the merits of one proof over the other. An outline of the proof using results from [ACF] is given after (4.42) in [LN2]. To outline the other proof one can first show that there exists $\delta_1 = \delta_1(p, n)$ small, such that if $\delta < \min\{\hat{\delta}, \delta_1\}$, then u_∞ is a weak solution in \mathbf{R}^n to the two-phase free boundary problem studied in [LN5, LN6] with $u_\infty^- \equiv 0$ and $G(s) = 1 + s, s \in [0, \infty)$. Then, using [LN6, Theorem 2] one can conclude that there exists $\delta_2 = \delta_2(p, n, M, C)$ such that if $0 < \delta < \delta_2$ then the following is true. There exist $N = N(p, n, M, C), 1 \leq N < \infty$, and a Lipschitz function $\phi_\infty : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$, with Lipschitz norm bounded by N , such that $\partial\Omega_\infty = \{(y', y_n) : y_n > \phi_\infty(y')\}$. In particular, $\partial\Omega_\infty$ is the graph of a Lipschitz function. Combining this conclusion with [LN5, Theorem 1] it follows that $\partial\Omega_\infty$ is in fact $C^{1,\gamma}$ -smooth for some $\gamma = \gamma(p, n, M, C) \in (0, 1)$. Using these conclusions one can then complete the proof of Lemma 5.6 as follows. Let $\bar{\delta} = \min\{\delta_1, \delta_2\}$, where δ_1, δ_2 are as above, and let $R \gg 1$ be an arbitrary real number. Then, applying [LN5, Theorem 1] with $u(y) = u_\infty(Ry)/R, y \in B(0, 2)$, one finds that

$$\sup_{\{y' : (y', \phi_\infty(y')) \in B(0, R/8)\}} |\nabla \phi_\infty(y') - \nabla \phi_\infty(0)| \leq c|y'|^\sigma R^{-\sigma} \quad (5.49)$$

where $c = c(p, n, M, C)$. Letting $R \rightarrow \infty$ one gets that $\nabla \phi_\infty \equiv \nabla \phi_\infty(0)$. Thus ϕ_∞ is linear and consequently Ω_∞ is a halfspace. This completes the proof of Lemma 5.6. \square

Proof of 1.14 (i) and Theorem 3. The proof of (i) of (1.14) is essentially the same as the proof of (1.14) (ii). Indeed, if $\Delta(w, r/2)$ is not Reifenberg flat with vanishing constant, there exist sequences $\{w_j\}$, $w_j \in \Delta(w, r/2)$, and $\{r_j\}$, $r_j \rightarrow 0$, such that

$$\inf_P \frac{1}{r_j} h(\Delta(w_j, r_j), P \cap B(w_j, r_j)) \geq \lambda, \quad (5.50)$$

for some $\lambda > 0$. The infimum in (5.50) is taken with respect to all $(n-1)$ -dimensional planes P containing w_j . We then argue as in (5.2)-(5.5) and Lemma 5.1 to get a subsequence $\{\Omega_k\}$, with $\partial\Omega_k \rightarrow \partial\Omega_\infty$ in the Hausdorff distance sense, uniformly on compact subsets of \mathbf{R}^n . Hence

$$\inf_{P'} h(\partial\Omega_\infty \cap B(0, 1), P' \cap B(0, 1)) \geq \lambda, \quad (5.51)$$

where now the infimum is taken with respect to all $(n-1)$ -dimensional planes P' containing 0. However, using Lemma 5.6 we see that if Ω is δ -Reifenberg flat and δ is small enough, then Ω_∞ is a halfplane. This statement contradicts (5.51). The proof of (1.14) and Theorem 3 is now complete. \square

6 Proof of Theorem 4 and Closing Remarks

The purpose of this section is to prove Theorem 4 and we note that to prove Theorem 4 we again have to prove (i) and (ii) in (1.14). However in this case we have to prove these statements in the setting of Ahlfors regular NTA-domains and without any initial flatness assumption. Thus if (1.14) (i) or (ii) is false we can repeat the corresponding blow-up argument in Theorem 3 to get p -harmonic functions u_∞^i in domains Ω_∞^i , $i = 1, 2$, which are continuous in \mathbf{R}^n with $u_\infty^i \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\infty^i$. Moreover $\mathbf{R}^n \setminus \bar{\Omega}_\infty^1 = \Omega_\infty^2$. Let μ_∞^i be the measures corresponding to u_∞^i , $i = 1, 2$. Applying Lemma 5.2-Lemma 5.4 we deduce that if σ_∞ denotes the $(n-1)$ -dimensional Hausdorff measure on $\partial\Omega_\infty^1 = \partial\Omega_\infty^2$, then

$$\mu_\infty^i \equiv \sigma_\infty \text{ for } i = 1, 2. \quad (6.1)$$

Put $\hat{u} = u_\infty^1$ in Ω_∞^1 and $\hat{u} = -u_\infty^2$ in $\bar{\Omega}_\infty^2$. Then from Lemma 2.5 we see that \hat{u} is p -harmonic in \mathbf{R}^n . Also from Lemma 5.4 we have $|\nabla \hat{u}| \leq 1$. Using this fact and Hölder continuity of $\nabla \hat{u}$ in Lemma 2.4 we see, for given $x \in \mathbf{R}^n$, that if $R > 0$ is large enough then

$$|\nabla \hat{u}(x) - \nabla \hat{u}(0)| \leq c(|x|/R)^\sigma, \quad (6.2)$$

where $c = c(p, n)$. Letting $R \rightarrow \infty$ we conclude that $\nabla \hat{u}(x) = \nabla \hat{u}(0)$. Thus \hat{u} is linear and since $\partial\Omega_\infty^1 = \{\hat{u} = 0\}$, we conclude that Ω_∞^1 and Ω_∞^2 are half spaces. This conclusion leads to a contradiction exactly as in the proof of Theorem 3. The proof of Theorem 4 is therefore complete. \square

6.1 Closing Remarks

We note that a problem related to Theorem 4 was considered in [KPT] and [B1]. Indeed, suppose Ω_+ is a NTA-domain with parameters $M, r_0 = \infty$, and that $\Omega_- = \mathbf{R}^n \setminus \Omega_+$ is also a NTA-domain with parameters $M, r_0 = \infty$. For fixed $p, 1 < p < \infty$, suppose u_+, u_- , are positive p -harmonic functions in Ω_+, Ω_- with continuous boundary value 0 on $\partial\Omega_+ = \partial\Omega_-$. Extend u_{\pm} to Ω_{\mp} by putting $u_{\pm} \equiv 0$. Let μ_{\pm} be the measures corresponding to u_{\pm} as in Lemma 2.5. Next let

$$A = \{y \in \partial\Omega_+ : \lim_{t \rightarrow 0} \frac{\mu_-(B(y, t))}{\mu_+(B(y, t))} = f(y), 0 < f(y) < \infty\}.$$

From differentiation theory it follows that

$$\partial\Omega_+ = \partial\Omega_- = A \cup B \cup C$$

where μ_+, μ_- , are mutually absolutely continuous on A , and $\mu_+(B) = 0 = \mu_-(C)$. Let H^α denote α -dimensional Hausdorff measure on \mathbf{R}^n and let $\text{H-dim } A$ denote the Hausdorff dimension of A defined by

$$\text{H-dim } A = \inf\{\alpha : H^\alpha(A) = 0\}.$$

We state

Proposition 6.1 *H-dim* $A \leq n - 1$.

The case $p = 2$ of Proposition 6.1 is due to [KPT]. Moreover using Lemma 2.5, as well as the blow-up argument in Theorem 4, we believe that one can essentially copy the proof of Theorem 4.1 and Corollary 4.1 for harmonic functions in [KPT] with slight adjustments. For example in [KPT] the authors quote a result of Hardt and Simon in order to show, for a harmonic function v in \mathbf{R}^n , that $|\nabla v| \neq 0$ somewhere on $\{x : v(x) = 0\}$. If v is p -harmonic in \mathbf{R}^n , for some $1 < p < \infty$, then this statement follows easily from Lemma 2.4 and a barrier type argument. For $p \geq n$, Proposition 6.1 is not surprising in view of the results in [LNV].

In [KPT] it is also shown, for $p = 2$, that if $\mu_{\pm}(A) > 0$, then $\text{H-dim } A = n - 1$. The proof uses Proposition 6.1 and a well known monotonicity formula of Alt, Caffarelli, and Friedman. The proof fails for $p \neq 2$, since it is not known whether an analogue of this monotonicity formula holds when $p \neq 2$. Once again the results in [LNV] provide interesting examples of Reifenberg domains where $\text{H-dim } A < n - 1$ when $p \geq n$.

Finally we note that the arguments in [KPT] are further enhanced by Badger in [B1]. In order to state these results let $\text{VMO}(\mu_+)$ denote the space of functions on $\partial\Omega_+$ that are of vanishing mean oscillation with respect to μ_+ . This space is defined in a way similar to the VMO space defined after (1.5) but with σ replaced by μ_+ . Badger proves the following.

Proposition 6.2 *Suppose $p = 2$ and $\log(d\mu_-/d\mu_+) \in \text{VMO}(\mu_+)$. Then for some positive integer d , $\partial\Omega_+ = \Gamma_1 \cup \Gamma_2 \cdots \cup \Gamma_d$, where $\Gamma_k, 1 \leq k \leq d$, are pairwise disjoint. Also each nontrivial blow-up of u_+ at points in Γ_k produces a homogeneous polynomial of degree k .*

We conjecture that an analogue of Proposition 6.2 holds for fixed $p, 1 < p < \infty$, with homogeneous polynomial replaced by homogeneous p -harmonic function. Finally, if Ω_+ is (δ, ∞) -Reifenberg flat and $\delta = \delta(p) > 0$ is small enough, then Proposition 6.2 with $d = 1$ is proved in [LuN].

7 References

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