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Boundary Harnack Inequalities for Operators of p-Laplace Type in Reifenberg Flat Domains

John L. Lewis, Niklas Lundström, and Kaj Nyström

Abstract. In this paper we highlight a set of techniques that recently have been used to establish boundary Harnack inequalities for p-harmonic functions vanishing on a portion of the boundary of a domain which is 'flat' in the sense that its boundary is well-approximated by hyperplanes. Moreover, we use these techniques to establish new results concerning boundary Harnack inequalities and the Martin boundary problem for operators of p-Laplace type with variable coefficients in Reifenberg flat domains.

1. Introduction and statement of main results

In [LN], [LN1], [LN2], see also [LN3] for a survey of these results, a number of results concerning the boundary behaviour of positive p-harmonic functions, $1 < p < \infty$, in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$ were proved. In particular, the boundary Harnack inequality, as well as Hölder continuity for ratios of positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of $\partial\Omega$ were established. Furthermore, the p-Martin boundary problem at $w \in \partial\Omega$ was resolved under the assumption that Ω is either convex, C^1-regular or a Lipschitz domain with small constant. Also, in [LN4] these questions were resolved for p-harmonic functions vanishing on a portion of certain Reifenberg flat and Ahlfors regular NTA-domains.

From a technological perspective the toolbox developed in [LN, LN1-LN4] can be divided into (i) techniques which can be used to establish boundary Harnack inequalities for p-harmonic functions vanishing on a portion of the boundary of a domain which is 'flat' in the sense that its boundary is well-approximated by hyperplanes and (ii) techniques which can be used to establish boundary Harnack inequalities for p-harmonic functions vanishing on a portion of the boundary of a Lipschitz domain or on a portion of the boundary of a domain which can be well approximated by Lipschitz graph domains. Domains in category (i) are called Reifenberg flat domains with small constant or just Reifenberg flat domains. They include domains with small Lipschitz constant, C^1-domains and certain quasi-balls. Domains in category (ii) include Lipschitz domains with large Lipschitz constant and certain Ahlfors regular NTA-domains, which can be well approximated by Lipschitz graph domains in the Hausdorff distance sense. The purpose of this paper is to highlight the techniques labeled as category (i) in the above discussion and to use these techniques to establish boundary Harnack inequalities as well as to

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resolve the Martin boundary problem for operators of p-Laplace type with variable coefficients in Reifenberg flat domains.

To state our results we need to introduce some notation. Points in Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \ldots, x_n)$ or (x', x_n) where $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $\bar{E}, \partial E$, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$ and let $d(y, E)$ equal the distance from $y \in \mathbf{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and $|x| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of x. Put $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ whenever $x \in \mathbf{R}^n$, $r > 0$, and let dx be Lebesgue n-measure on $\mathbf{R}^n$. We let

$$h(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$$

be the Hausdorff distance between the sets $E, F \subset \mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \ldots, f_{x_n})$, both of which are q th power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q-norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. By $\nabla \cdot$ we denote the divergence operator.

We first introduce the operators of p-Laplace type which we consider in this paper.

Definition 1.1. *Let $p, \beta, \alpha \in (1, \infty)$ and $\gamma \in (0, 1)$. Let $A = (A_1, ..., A_n) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, assume that $A = A(x, \eta)$ is continuous in $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ and that $A(x, \eta)$, for fixed $x \in \mathbf{R}^n$, is continuously differentiable in η_k, for every $k \in \{1, ..., n\}$, whenever $\eta \in \mathbf{R}^n \setminus \{0\}$. We say that the function A belongs to the class $M_p(\alpha, \beta, \gamma)$ if the following conditions are satisfied whenever x, y, $\xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^n \setminus \{0\}$:*

$$
\begin{aligned}
&(i) && \alpha^{-1} |\eta|^{p-2} |\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j, \\
&(ii) && \left| \frac{\partial A_i}{\partial \eta_j}(x, \eta) \right| \leq \alpha |\eta|^{p-2}, 1 \leq i, j \leq n, \\
&(iii) && |A(x, \eta) - A(y, \eta)| \leq \beta |x - y|^\gamma |\eta|^{p-1}, \\
&(iv) && A(x, \eta) = |\eta|^{p-1} A(x, \eta/|\eta|).
\end{aligned}
$$

For short, we write $M_p(\alpha)$ for the class $M_p(\alpha, 0, \gamma)$.

Definition 1.2. *Let $p \in (1, \infty)$ and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Given a bounded domain G we say that u is A-harmonic in G provided $u \in W^{1,p}(G)$ and*

$$\int \langle A(x, \nabla u(x)), \nabla \theta(x) \rangle \, dx = 0 \tag{1.3}$$

whenever $\theta \in W_0^{1,p}(G)$. If $A(x, \eta) = |\eta|^{p-2}(\eta_1, \ldots, \eta_n)$, then u is said to be p-harmonic in G. As a short notation for (1.3) we write $\nabla \cdot (A(x, \nabla u)) = 0$ in G.

The relevance and importance of the conditions imposed through the assumption $A \in M_p(\alpha, \beta, \gamma)$ will be discussed below. Initially we just note that the class $M_p(\alpha, \beta, \gamma)$ is, see Lemma 2.15, closed under translations, rotations and under dilations $x \to rx$, $r \in (0, 1]$. Moreover, we note that an important class of equations which is covered by Definition 1.1 and 1.2 is the class of equations of the type

$$\nabla \cdot \left[\langle A(x) \nabla u, \nabla u \rangle^{p/2-1} A(x) \nabla u \right] = 0 \text{ in } G \tag{1.4}$$

where $A = A(x) = \{a_{i,j}(x)\}$ is such that the conditions in Definition 1.1 (i) - (iv) are fulfilled.

Next we introduce the geometric notions used in this paper. We define,

Definition 1.5. *A bounded domain Ω is called non-tangentially accessible (NTA) if there exist $M \geq 2$ and $r_0 > 0$ such that the following are fulfilled:*

(i) *corkscrew condition: for any $w \in \partial\Omega, 0 < r < r_0$, there exists $a_r(w) \in \Omega \cap B(w, r/2)$, satisfying $M^{-1} r < d(a_r(w), \partial\Omega)$,*

(ii) *$\mathbf{R}^n \setminus \bar{\Omega}$ satisfies the corkscrew condition,*

(iii) *uniform condition: if $w \in \partial\Omega, 0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$, then there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and such that*

(a) $H^1(\gamma) \leq M\,|w_1 - w_2|$,

(b) $\min\{H^1(\gamma([0,t])),\, H^1(\gamma([t,1]))\} \leq M\, d(\gamma(t), \partial\Omega)$.

In Definition 1.5, H^1 denotes length or the one-dimensional Hausdorff measure. We note that (iii) is different but equivalent to the usual Harnack chain condition given in [JK] (see [BL], Lemma 2.5). M will be called the NTA-constant of Ω.

Definition 1.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, $w \in \partial\Omega$, and $0 < r < r_0$. Then $\partial\Omega$ is said to be uniformly (δ, r_0)-approximable by hyperplanes, provided there exists, whenever $w \in \partial\Omega$ and $0 < r < r_0$, a hyperplane Λ containing w such that*

$$h(\partial\Omega \cap B(w,r), \Lambda \cap B(w,r)) \leq \delta r.$$

We let $\mathcal{F}(\delta, r_0)$ denote the class of all domains Ω which satisfy Definition 1.6. Let $\Omega \in \mathcal{F}(\delta, r_0)$, $w \in \partial\Omega$, $0 < r < r_0$, and let Λ be as in Definition 1.6. We say that $\partial\Omega$ separates $B(w,r)$, if

$$\{x \in \Omega \cap B(w,r) : d(x, \partial\Omega) \geq 2\delta r\} \subset \text{ one component of } \mathbf{R}^n \setminus \Lambda. \tag{1.7}$$

Definition 1.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Then Ω and $\partial\Omega$ are said to be (δ, r_0)-Reifenberg flat provided $\Omega \in \mathcal{F}(\delta, r_0)$ and (1.7) hold whenever $0 < r < r_0, w \in \partial\Omega$.*

For short we say that Ω and $\partial\Omega$ are δ-Reifenberg flat whenever Ω and $\partial\Omega$ are (δ, r_0)-Reifenberg flat for some $r_0 > 0$. We note that an equivalent definition of a Reifenberg flat domain is given in [KT]. As in [KT] one can show that a δ-Reifenberg flat domain is an NTA-domain with constant $M = M(n)$, provided $0 < \delta < \hat{\delta}$ and $\hat{\delta}$ is small enough.

In this paper we first prove the following theorem.

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain. Let p, $1 < p < \infty$, be given and assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega, 0 < r < r_0$, and suppose that u, v are positive A-harmonic functions in $\Omega \cap B(w, 4r)$, continuous in $\bar{\Omega} \cap B(w, 4r)$, and $u = 0 = v$ on $\partial\Omega \cap B(w, 4r)$. There exists $\tilde{\delta} < \hat{\delta}, \sigma > 0$, and $c_1 \geq 1$, all depending only on $p, n, \alpha, \beta, \gamma$, such that if $0 < \delta < \tilde{\delta}$, then*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c_1 \left(\frac{|y_1 - y_2|}{r} \right)^{\sigma}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_1)$.

We note that in [LN] we obtained for p-harmonic functions u, v, in a bounded Lipschitz domain Ω,

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c$$

whenever $w \in \partial\Omega$, and $y_1, y_2 \in \Omega \cap B(w, r/c)$. Here c depends only on p, n, and the Lipschitz constant for Ω. Moreover, using this result, we showed, in [LN1], that the conclusion of Theorem 1 holds whenever u, v, are p-harmonic, and Ω is Lipschitz. Constants again depend only on p, n, and the Lipschitz constant for Ω.

In this paper we also prove the following theorem.

Theorem 2. *Let $\Omega \subset \mathbf{R}^n$, δ, r_0, p, α, β, γ, and A be as in the statement of Theorem 1. Then there exists $\delta^* = \delta^*(p, n, \alpha, \beta, \gamma) < \hat{\delta}$, such that the following is true. Let $w \in \partial\Omega$ and suppose that $\hat{u}, \hat{v}$ are positive A-harmonic functions in Ω with $\hat{u} = 0 = \hat{v}$ continuously on $\partial\Omega \setminus \{w\}$. If $0 < \delta < \delta^*$, then $\hat{u}(y) = \lambda \hat{v}(y)$ for all $y \in \Omega$ and for some constant λ.*

We remark, using terminology of the Martin boundary problem, that if $\hat{u}$ is as in Theorem 2, then $\hat{u}$ is called a minimal positive A-harmonic function in Ω, relative to $w \in \partial\Omega$. Moreover, the A-Martin boundary of Ω is the set of equivalence classes of positive minimal A-harmonic functions relative to all boundary points of Ω. Two minimal positive A-harmonic functions are in the same equivalence class if they correspond to the same boundary point and one is a constant multiple of the other. Note that the conclusion of Theorem 2 implies that $\hat{u}$ is unique up to constant multiples. Thus, since $w \in \partial\Omega$ is arbitrary, one can say that the A-Martin boundary of Ω is identifiable with $\partial\Omega$.

We remark that in [LN1] the Martin boundary problem for p-harmonic functions was resolved in domains which are either convex, C^1-regular or Lipschitz with sufficiently small constant. Also, in [LN4] the Martin boundary problem was resolved, again for p-harmonic functions, in Reifenberg flat domains and certain Ahlfors regular NTA-domains. Theorem 2 is new in the case of operators of p-Laplace type with variable coefficients.

Recall that Ω is said to be a bounded Lipschitz domain if there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned} \Omega \cap B(x_i, r_i) &= \{x = (x', x_n) \in \mathbf{R}^n : x_n > \phi_i(x')\} \cap B(x_i, r_i), \\ \partial\Omega \cap B(x_i, r_i) &= \{x = (x', x_n) \in \mathbf{R}^n : x_n = \phi_i(x')\} \cap B(x_i, r_i), \end{aligned} \tag{1.9}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constant of Ω is defined to be $M = \max_i \||\nabla\phi_i|\|_\infty$. If Ω is Lipschitz then Ω is NTA with $r_0 = \min r_i/c$, where $c = c(p,n,M) \geq 1$. Moreover, if each $\phi_i : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ can be chosen to be either C^1- or $C^{1,\sigma}$ -regular, then Ω is a bounded C^1- or $C^{1,\sigma}$-domain.

We say that Ω is a quasi-ball provided $\Omega = f(B(0,1))$, where $f = (f_1, f_2, ..., f_n) : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a $K > 1$ quasi-conformal mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$. That is, $f_i \in W^{1,n}(B(0,\rho)), 0 < \rho < \infty, 1 \leq i \leq n$, and for almost every $x \in \mathbf{R}^n$ with respect to Lebesgue n-measure the following hold,

$$\begin{aligned} &(i) \qquad |Df(x)|^n = \sup_{|h|=1} |Df(x)h|^n \leq K|J_f(x)|, \\ &(ii) \qquad J_f(x) \geq 0 \text{ or } J_f(x) \leq 0. \end{aligned}$$

In this display we have written $Df(x) = (\frac{\partial f_i}{\partial x_j})$ for the Jacobian matrix of f and $J_f(x)$ for the Jacobian determinant of f at x.

Remark 1.10. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. If M is small enough then Ω is (δ, r_0)-Reifenberg flat for some $\delta = \delta(M), r_0 > 0$ with $\delta(M) \rightarrow 0$ as $M \rightarrow 0$. Hence, Theorems 1-2 apply to any bounded Lipschitz domain with sufficiently small Lipschitz constant. Also, if $\Omega = f(B(0,1))$ where f is a K quasi-conformal mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$, then one can show that $\partial\Omega$ is δ-Reifenberg flat, with $r_0 = 1$, where $\delta \rightarrow 0$ as $K \rightarrow 1$ (see [R, Theorems 12.5 -12.7]). Thus Theorems 1, 2, apply when Ω is a quasi-ball and if $K = K(p,n)$ is close enough to 1.*

To state corollaries to Theorems 1-2 we next introduce the notion of Reifenberg flat domains with vanishing constant.

Definition 1.11. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain for some $0 < \delta < \hat{\delta}$, $r_0 > 0$, and let $w \in \partial\Omega$, $0 < r < r_0$. We say that $\partial\Omega \cap B(w,r)$ is Reifenberg flat with vanishing constant, if for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $x \in \partial\Omega \cap B(w,r)$ and $0 < \rho < \tilde{r}$, then there is a plane $P' = P'(x,\rho)$ containing x such that*

$$h(\partial\Omega \cap B(x,\rho), P' \cap B(x,\rho)) \leq \epsilon\rho.$$

The following corollaries are immediate consequences of Theorems 1-2.

Corollary 1. *Let $\Omega \subset \mathbf{R}^n$ be a domain which is Reifenberg flat with vanishing constant. Let p, $1 < p < \infty$, be given and assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega$, $0 < r < r_0$. Assume that u, v are positive A-harmonic functions in $\Omega \cap B(w, 4r)$, u, v are continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0 = v$ on $\partial\Omega \cap B(w, 4r)$. There exist $r_1^* = r_1^*(p, n, \alpha, \beta, \gamma) < r$ and $c_2 = c_2(p, n, \alpha, \beta, \gamma) \geq 1$ such that if $w' \in \partial\Omega \cap B(w, r)$ and $0 < r' < r_1^*$, then*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c_2 \left(\frac{|y_1 - y_2|}{r'} \right)^\sigma$$

whenever $y_1, y_2 \in \Omega \cap B(w', r')$.

Corollary 2. *Let $\Omega \subset \mathbf{R}^n$, p, α, β, γ and A be as in the statement of Corollary 1. Let $w \in \partial\Omega$ and suppose that $\hat{u}, \hat{v}$ are positive A-harmonic functions in Ω with*

$\hat u = 0 = \hat v$ continuously on $\partial\Omega \setminus \{w\}$. Then $\hat u(y) = \lambda \hat v(y)$ for all $y \in \Omega$ and for some constant λ.

Remark 1.12. *We note that if Ω is a bounded C^1-domain in the sense of (1.9) then Ω is also Reifenberg flat with vanishing constant. Hence Corollaries 1-2 apply to any bounded C^1-domain.*

Concerning proofs, we here outline the proof of Theorem 1.

Step 0. As a starting point we establish the conclusion of Theorem 1, see Lemma 2.8, when $A \in M_p(\alpha)$, Ω is equal to a truncated cylinder and w is the center on the bottom of $\bar\Omega$.

Step A. (Uniform non-degeneracy of $|\nabla u|$ - the 'fundamental inequality'). There exist $\delta_1 = \delta_1(p,n,\alpha,\beta,\gamma)$, $\hat c_1 = \hat c_1(p,n,\alpha,\beta,\gamma)$ and $\bar\lambda = \bar\lambda(p,n,\alpha,\beta,\gamma)$, such that if $0 < \delta < \delta_1$, then

$$\bar\lambda^{-1}\frac{u(y)}{d(y,\partial\Omega)} \le |\nabla u(y)| \le \bar\lambda \frac{u(y)}{d(y,\partial\Omega)} \text{ whenever } y \in \Omega \cap B(w, r/\hat c_1). \tag{1.13}$$

If (1.13) holds then we say that $|\nabla u|$ satisfies the 'fundamental inequality' in $\Omega \cap B(w, r/\hat c_1)$.

Step B. (Extension of $|\nabla u|^{p-2}$ to an A_2-weight). There exist $\delta_2 = \delta_2(p,n,\alpha,\beta,\gamma)$ and $\hat c_2 = \hat c_2(p,n,\alpha,\beta,\gamma)$ such that if $0 < \delta < \delta_2$, then $|\nabla u|^{p-2}$ extends to an $A_2(B(w, r/(\hat c_1 \hat c_2))$-weight with constant depending only on p,n,α,β,γ.

For the definition of an A_2-weight, see section 4. The 'fundamental inequality' established in Step A is crucial to our arguments and section 3 is devoted to its proof. Armed with the results established in Step A and Step B we introduce certain deformations of A-harmonic functions. In particular, to describe the constructions we let $\Omega \subset \mathbf{R}^n$, δ, r_0, p, α, β, γ, A, w, r, u and v be as in the statement of Theorem 1. Let $\tilde\delta = \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 are given in Step A and Step B respectively. We extend u and v to $B(w,4r)$ by defining $u \equiv 0 \equiv v$ on $B(w,4r)\setminus\bar\Omega$.

Step C. (Deformation of A-harmonic functions). Let $r^* = r/c$ and assume that

$$\begin{aligned}
&(a) \quad 0 \le u \le v/2 \text{ in } \bar\Omega \cap \bar B(w, 4r^*),\\
&(b) \quad c^{-1} \le u(a_{r^*}(w)), v(a_{r^*}(w)) \le c,\\
(1.14)&(c) \quad c^{-1} h(a_{r^*}(w)) \le \max_{\bar\Omega \cap \bar B(w,4r^*)} h \le c h(a_{r^*}(w)) \text{ whenever } h = u \text{ or } v.
\end{aligned}$$

Here $c \ge 1$ depends only on p,n,α,β,γ. At the end of section 4 we then show that the assumptions in (1.14) can be easily removed. Hence, to prove Theorem 1 we can without loss of generality assume that (1.14) holds. We let $\tilde u(\cdot,\tau)$, $0 \le \tau \le 1$, be the A-harmonic function in $\Omega \cap B(w,4r^*)$ with continuous boundary values,

$$\tilde u(y,\tau) = \tau v(y) + (1-\tau)u(y) \text{ whenever } y \in \partial(\Omega \cap B(w,4r^*)) \text{ and } \tau \in [0,1]. \tag{1.15}$$

Using (1.14), (1.15), we see that if $t, \tau \in [0,1]$, then

$$0 \le \frac{\tilde u(\cdot,t) - \tilde u(\cdot,\tau)}{t-\tau} = v - u \le c(p,n,\alpha,\beta,\gamma) \tag{1.16}$$

on $\partial(\Omega\cap B(w,4r^*))$. From the maximum principle for A-harmonic functions it then follows that the inequality in (1.16) also holds in $\Omega\cap B(w,4r^*)$. Therefore, using (1.16) we see that $\tau\to\tilde u(y,\tau), \tau\in[0,1]$, for fixed $y\in\Omega\cap B(w,4r^*)$, is Lipschitz continuous with Lipschitz norm $\leq c$. Thus $\tilde u_\tau(y,\cdot)$ exists, for fixed $y\in\Omega\cap B(w,4r^*)$, almost everywhere in [0,1]. Let $\{y_\nu\}$ be a dense sequence of $\Omega\cap B(w,4r^*)$ and let W be the set of all $\tau\in[0,1]$ for which $u_\tau(y_m,\cdot)$ exists, in the sense of difference quotients, whenever $y_m\in\{y_\nu\}$. We note that $H^1([0,1]\setminus W)=0$ where H^1 is one-dimensional Hausdorff measure. Next, applying the 'fundamental inequality', established in Step A, to $\tilde u(\cdot,\tau)$, $\tau\in[0,1]$, we see that there exist constants $\hat c$ and $\bar\lambda$, which depend on p,n,α,β,γ, but are independent of τ, $\tau\in[0,1]$, such that if $y\in\Omega\cap B(w,16r')$, $r'=r^*/\hat c$ and $\tau\in[0,1]$, then

$$\bar\lambda^{-1}\frac{\tilde u(y,\tau)}{d(y,\partial\Omega)}\leq|\nabla\tilde u(y,\tau)|\leq\bar\lambda\frac{\tilde u(y,\tau)}{d(y,\partial\Omega)}. \tag{1.17}$$

One can then deduce, using the fundamental theorem of calculus and arguing as in [LN4, displays (1.15)-(1.23)], that

$$\log\left(\frac{v(y_m)}{u(y_m)}\right)=\log\left(\frac{\tilde u(y_m,1)}{\tilde u(y_m,0)}\right)=\int_0^1\frac{f(y_m,\tau)}{\tilde u(y_m,\tau)}\,d\tau \tag{1.18}$$

whenever $y_m\in\{y_\nu\}$, $y_m\in\Omega\cap B(w,r')$, and for a function f which has the following important properties,

$$\begin{array}{lll}(i) & & f\geq 0 \text{ is continuous in } \bar B(w,r') \text{ with } f\equiv 0 \text{ on } \bar B(w,r')\setminus\Omega,\\ (ii) & & f(y_m,\tau)=\tilde u_\tau(y_m,\tau)\end{array} \tag{1.19}$$

whenever $y_m\in\{y_\nu\}$, $y_m\in\Omega\cap B(w,r'),\tau\in W$. Moreover, f is locally a weak solution in $\Omega\cap B(w,r')$ to the equation

$$\tilde L\zeta\;=\;\sum_{i,j=1}^n\frac{\partial}{\partial y_i}\left(\,\tilde b_{ij}(y,\tau)\zeta_{y_j}(y)\,\right)=0 \tag{1.20}$$

where

$$\tilde b_{ij}(y,\tau)=\frac{\partial A_i}{\partial\eta_j}(y,\nabla\tilde u(y,\tau)) \tag{1.21}$$

whenever $y\in\Omega\cap B(w,r')$ and $1\leq i,j\leq n$. Also, using Definition 1.1 (i) and (ii) we see that

$$\alpha^{-1}\tilde\lambda(y,\tau)|\xi|^2\leq\sum_{i,j}\tilde b_{ij}(y,\tau)\xi_i\xi_j\leq\alpha\tilde\lambda(y,\tau)|\xi|^2 \tag{1.22}$$

whenever $y\in\Omega\cap B(w,r')$ and where $\tilde\lambda(y,\tau)=|\nabla\tilde u(y,\tau)|^{p-2}$. Finally, a key observation in this step is that $\zeta=\tilde u(\cdot,\tau)$ is also a weak solution to $\tilde L$ in $\Omega\cap B(w,r')$. Indeed, using the homogeneity in Definition 1.1 (iv) we see that

$$\begin{aligned}\sum_j\tilde b_{ij}(y,\tau)\tilde u_{y_j}(y,\tau)\quad&=\sum_j\frac{\partial A_i}{\partial\eta_j}(y,\nabla\tilde u(y,\tau))\tilde u_{y_j}(y,\tau)\\ &=(p-1)A_i(y,\nabla\tilde u(y,\tau)).\end{aligned} \tag{1.23}$$

We conclude from (1.23) that $\zeta=\tilde u(\cdot,\tau)$ is also a weak solution to $\tilde L$.

Step D. (Boundary Harnack inequalities for degenerate elliptic equations). Using the deformations introduced in Step C the proof of Theorem 1 therefore boils down to proving boundary Harnack inequalities for the operator $\tilde{L}$. The idea here is to make use of Step B to conclude that $\tilde{\lambda}(\cdot,\tau)$, $\tau\in[0,1]$, can be extended to A_2-weights in $B(w,4r'')$, $r''=r'/(4\hat{c}_2)$. Then the operator $\tilde{L}$ can be considered as a degenerate elliptic operator in the sense of [FKS], [FJK], [FJK1], and we can apply results of these authors. In particular, to do this we first observe that the sequence $\{y_\nu\}$, introduced below (1.16), is a dense sequence in $\Omega\cap B(w,r')$, and $v_1(\cdot)=f(\cdot,\tau)$, $v_2(\cdot)=u(\cdot,\tau)$, are positive solutions to $\tilde{L}$, see (1.20)-(1.22), vanishing continuously on $\Omega\cap B(w,r')$. Second, we observe from Step B that $\tilde{\lambda}(y,\tau)=|\nabla\tilde{u}(y,\tau)|^{p-2}$ can be extended to an $A_2(B(w,4r''))$-weight. Hence, from [FKS], [FJK] and [FJK1], we can conclude that there exist a constant $c=c(p,n,\alpha,\beta,\gamma)$, $1\leq c<\infty$, and $\sigma=\sigma(p,n,\alpha,\beta,\gamma)$, $\sigma\in(0,1)$, such that if $r'''=r''/c$, then

$$\left|\frac{v_1(y_1)}{v_2(y_1)}-\frac{v_1(y_2)}{v_2(y_2)}\right|\leq c\,\frac{v_1(a_{r'''}(w))}{v_2(a_{r'''}(w))}\left(\frac{|y_1-y_2|}{r''}\right)^{\sigma} \tag{1.24}$$

whenever $y_1,y_2\in\Omega\cap B(w,r''')$. Hence, assuming (1.14) we see that Theorem 1 now follows from (1.18), (1.24), as

$$0\leq f(a_{r'''}(w),\tau)\leq c,\ u(a_{r'''}(w),\tau)\geq c^{-1},\ \text{whenever } \tau\in(0,1]. \tag{1.25}$$

(1.25) is a consequence of (1.16) and (1.14) (b).

The proof of Theorem 2 can also be decomposed into steps similar to steps A-D stated above. Still in this case details are more involved and we refer to section 5 for details.

The rest of the paper is organized as follows. In section 2 we state a number of basic estimates for A-harmonic functions in NTA-domains and we obtain the conclusion of Theorem 1 when $A\in M_p(\alpha)$, Ω is equal to a truncated cylinder (see (2.7) and Lemma 2.8), and w is the center of the bottom of $\bar{\Omega}$ (Step 0). In section 3 we establish the 'fundamental inequality' for A-harmonic functions, u, vanishing on a portion of a Reifenberg flat domain (Step A). In section 4 we first state a number of results for degenerate elliptic equations tailored to our situation and we then extend $|\nabla u|^{p-2}$ to an A_2-weight (Step B). In this section we also complete the proof of Theorem 1 by showing that the technical assumption in (1.14) can be removed. In section 5 we prove Theorem 2. Finally in an Appendix to this paper (section 6), we point out an alternative argument to Step C based on an idea in [W].

2. Basic estimates for A-harmonic functions and boundary Harnack inequalities in a prototype case

In this section we first state and prove some basic estimates for non-negative A-harmonic functions in a bounded NTA domain $\Omega\subset\mathbf{R}^n$. We then prove the boundary Harnack inequality for non-negative A-harmonic functions, $A\in M_p(\alpha)$, vanishing on a portion of a hyperplane. Throughout this section we will assume that $A\in M_p(\alpha,\beta,\gamma)$ or $A\in M_p(\alpha)$ for some (α,β,γ) and $1<p<\infty$. Also in this paper, unless otherwise stated, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending only on $p,n,M,\alpha,\beta,\gamma$ where M denotes the NTA-constant for $\Omega\subset\mathbf{R}^n$. In general, $c(a_1,\ldots,a_m)$ denotes a positive constant ≥ 1, which may depend only on $p, n, M, \alpha, \beta, \gamma$ and $a_1, \dots, a_m$, not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on $p, n, M, \alpha, \beta, \gamma$. Moreover, we let $\max\limits_{B(z,s)} u$, $\min\limits_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z,s)$ whenever $B(z,s) \subset \mathbf{R}^n$ and whenever u is defined on $B(z,s)$. We put $\Delta(w,r) = \partial\Omega \cap B(w,r)$ whenever $w \in \partial\Omega$, $0 < r$. Finally, $e_i, 1 \leq i \leq n$, denotes the point in $\mathbf{R}^n$ with one in the i th coordinate position and zeroes elsewhere.

Lemma 2.1. *Given $p, 1 < p < \infty$, assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let u be a positive A-harmonic function in $B(w, 2r)$. Then*

$$(i) \qquad r^{p-n} \int\limits_{B(w,r/2)} |\nabla u|^p \, dx \;\leq\; c\,(\max_{B(w,r)} u)^p,$$

$$(ii) \qquad \max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Furthermore, there exists $\tilde\sigma = \tilde\sigma(p, n, \alpha, \beta, \gamma) \in (0,1)$ such that if $x, y \in B(w,r)$, then

$$(iii) \qquad |u(x) - u(y)| \leq c \left(\tfrac{|x-y|}{r} \right)^{\tilde\sigma} \max_{B(w,2r)} u.$$

Proof: Lemma 2.1 $(i), (ii)$ are standard Caccioppoli and Harnack inequalities while (iii) is a standard Hölder estimate (see [S]). $\Box$

Lemma 2.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain, suppose that p, $1 < p < \infty$, is given and that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous A-harmonic function in $\bar\Omega \cap B(w, 2r)$ and that $u = 0$ on $\Delta(w, 2r)$. Then*

$$(i) \qquad r^{p-n} \int\limits_{\Omega \cap B(w,r/2)} |\nabla u|^p \, dx \;\leq\; c\,(\max_{\Omega \cap B(w,r)} u)^p.$$

Furthermore, there exists $\tilde\sigma = \tilde\sigma(p, n, M, \alpha, \beta, \gamma) \in (0,1)$ such that if $x, y \in \Omega \cap B(w,r)$, then

$$(ii) \qquad |u(x) - u(y)| \leq c \left(\tfrac{|x-y|}{r} \right)^{\tilde\sigma} \max_{\Omega \cap B(w,2r)} u.$$

Proof: Lemma 2.2 (i) is a standard subsolution inequality while (ii) follows from a Wiener criteria first proved in [M] and later generalized in [GZ]. $\Box$

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain, suppose that p, $1 < p < \infty$, is given and that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous A-harmonic function in $\bar\Omega \cap B(w, 2r)$ and that $u = 0$ on $\Delta(w, 2r)$. There exists $c = c(p, n, M, \alpha, \beta, \gamma)$, $1 \leq c < \infty$, such that if $\tilde r = r/c$, then*

$$\max_{\Omega \cap B(w,\tilde r)} u \;\leq\; c\, u(a_{\tilde r}(w)).$$

Proof: A proof of Lemma 2.3 for linear elliptic PDE can be found in [CFMS]. The proof uses only analogues of Lemmas 2.1, 2.2 for linear PDE and Definition 1.5. In particular, the proof also applies in our situation. □

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain, suppose that p, $1<p<\infty$, is given and that $A \in M_p(\alpha,\beta,\gamma)$ for some (α,β,γ). Let $w \in \partial\Omega$, $0<r<r_0$, and suppose that u is a non-negative continuous A-harmonic function in $\bar\Omega \cap \bar B(w,4r)$ and that $u=0$ on $\Delta(w,4r)$. Extend u to $B(w,4r)$ by defining $u\equiv 0$ on $B(w,4r)\setminus\Omega$. Then u has a representative in $W^{1,p}(B(w,4r))$ with Hölder continuous partial derivatives of first order in $\Omega \cap B(w,4r)$. In particular, there exists $\hat\sigma \in (0,1]$, depending only on p,n,α,β,γ such that if $x,y \in B(\hat w,\hat r/2)$, $B(\hat w,4\hat r) \subset \Omega \cap B(w,4r)$, then*

$$c^{-1}\,|\nabla u(x)-\nabla u(y)| \,\leq\, (|x-y|/\hat r)^{\hat\sigma} \max_{B(\hat w,\hat r)} |\nabla u| \,\leq\, c\,\hat r^{-1}\,(|x-y|/\hat r)^{\hat\sigma} \max_{B(\hat w,2\hat r)} u.$$

Proof: Given $\epsilon>0$ and small, let

$$A(y,\eta,\epsilon)=\int\limits_{\mathbf{R}^n} A(y,\eta-x)\theta_\epsilon(x)dx \text{ whenever } (y,\eta)\in\mathbf{R}^n\times\mathbf{R}^n, \tag{2.5}$$

where $\theta \in C_0^\infty(B(0,1))$ with $\int_{\mathbf{R}^n}\theta dx=1$ and $\theta_\epsilon(x)=\epsilon^{-n}\theta(x/\epsilon)$ whenever $x\in\mathbf{R}^n$. From Definition 1.1 and standard properties of approximations to the identity, we deduce for some $c=c(p,n)\geq 1$ that

$$\begin{aligned}
&(i) \qquad (c\alpha)^{-1}(\epsilon+|\eta|)^{p-2}|\xi|^2 \leq \sum_{i,j=1}^n \frac{\partial A_i}{\partial\eta_j}(y,\eta,\epsilon)\xi_i\xi_j,\\
&(ii) \qquad \left|\frac{\partial A_i}{\partial\eta_j}(y,\eta,\epsilon)\right| \leq c\alpha(\epsilon+|\eta|)^{p-2}, 1\leq i,j\leq n,\\
&(iii) \qquad |A(x,\eta,\epsilon)-A(y,\eta,\epsilon)| \,\leq\, c\beta|x-y|^\gamma(\epsilon+|\eta|)^{p-1}
\end{aligned} \tag{2.6}$$

whenever $x,y,\eta\in\mathbf{R}^n$. Moreover, $A(y,\cdot,\epsilon)$ is, for fixed (y,ϵ), infinitely differentiable.

To prove Lemma 2.4 we choose $u(\cdot,\epsilon)$, a weak solution to the PDE with structure as in (2.6), in such a way that $u(\cdot,\epsilon)$ is continuous in $\bar\Omega\cap\bar B(w,3r)$ and $u(\cdot,\epsilon)=u$ on $\partial[\Omega\cap B(w,3r)]$. Existence of $u(\cdot,\epsilon)$ follows from the Wiener criteria in [GZ] mentioned in the proof of Lemma 2.2, the maximum principle for A-harmonic functions, and the fact that the $W^{1,p}$-Dirichlet problem for these functions, in $\Omega\cap B(w,3r)$, always has a unique solution (see [HKM, Appendix I]). Moreover, from [T], [T1], it follows that $u(\cdot,\epsilon)$ is in $C^{1,\hat\sigma}(\Omega\cap B(w,2r))$ for some $\hat\sigma>0$ with constants independent of ϵ. Letting $\epsilon\to 0$ one can show, using Definition 1.1, that subsequences of $\{u(\cdot,\epsilon)\},\{\nabla u(\cdot,\epsilon)\}$, converge pointwise to $u,\nabla u$. In view of Lemma 2.1 and the result in [T] it follows that this convergence is uniform on compact subsets of $\Omega\cap B(w,3r)$. Using this fact we get the last display in Lemma 2.4.

Finally we note that in [T] a stronger assumption, compared to (2.6) *(iii)*, is imposed. However, other authors later obtained the results in [T] under assumption (2.6) (see [Li] for references). □

Next we show that the conclusion of Theorem 1 holds in the case of a truncated cylinder with w the center on the bottom of the cylinder (Step 0). To this end we introduce, for $a,b\in\mathbf{R}^+$ and $w=(w_1,...,w_n)\in\mathbf{R}^n$, the truncated cylinders,

$$
\begin{aligned}
Q_{a,b}(w) &= \{y=(y',y_n): |y'-w'|<a, |y_n-w_n|<b\},\\
Q^+_{a,b}(w) &= \{y=(y',y_n): |y'-w'|<a,\ 0<y_n-w_n<b\},\\
Q^-_{a,b}(w) &= \{y=(y',y_n): |y'-w'|<a,\ -b<y_n-w_n<0\}.
\end{aligned}
\tag{2.7}
$$

Furthermore, if $a=b$ then we let $Q_a(w)=Q_{a,a}(w)$, $Q^+_a(w)=Q^+_{a,a}(w)$, $Q^-_a(w)=Q^-_{a,a}(w)$.

Lemma 2.8. *Suppose that p, $1<p<\infty$, is given and that $A\in M_p(\alpha)$ for some α. Assume also, that u, v are non-negative A-harmonic functions in $Q^+_1(0)$, continuous on the closure of $Q^+_1(0)$, and with $u=0=v$ on $\partial Q^+_1(0)\cap\{y_n=0\}$. Then there exist $c=c(p,n,\alpha), 1\leq c<\infty$, and $\sigma=\sigma(p,n,\alpha)\in(0,1]$ such that*

$$\left|\log\left(\frac{u(y_1)}{v(y_1)}\right)-\log\left(\frac{u(y_2)}{v(y_2)}\right)\right|\leq c|y_1-y_2|^\sigma$$

whenever y_1, $y_2\in Q^+_{1/4}(0)$.

Proof. Let $A=A(\eta)$ be as in Lemma 2.8 and let p be fixed, $1<p<\infty$. Note that y_n is A-harmonic and that it suffices to prove Lemma 2.8 when $v=y_n$. Define $A(\eta,\epsilon)$ as in (2.5) relative to A and let $u(\cdot,\epsilon)$ be the solution to $\nabla\cdot(A(\nabla u(y,\epsilon),\epsilon))=0$ with continuous boundary values equal to u on $\partial Q^+_1(0)$. Let

$$A^*_{ij}(y,\epsilon)=\tfrac{1}{2}(\epsilon+|\nabla u(y,\epsilon)|)^{2-p}\left[\frac{\partial A_i}{\partial\eta_j}(\nabla u(y,\epsilon),\epsilon)+\frac{\partial A_j}{\partial\eta_i}(\nabla u(y,\epsilon),\epsilon)\right]$$

whenever $y\in Q^+_{1/2}(0)$ and $1\leq i,j\leq n$. From (2.6) (ii) and Schauder type estimates we see that $u(\cdot,\epsilon), y_n$, are classical solutions to the non-divergence form uniformly elliptic equation,

$$L^*\zeta=\sum_{i,j=1}^n A^*_{ij}(y,\epsilon)\zeta_{y_iy_j}=0, \tag{2.9}$$

for $y\in Q^+_{1/2}(0)$. Note also from (2.6) that the ellipticity constant for $(A^*_{ij}(y,\epsilon))$ and the L^∞-norm for $A^*_{ij}(y,\epsilon), 1\leq i,j\leq n$, in $Q^+_{1/2}(0)$, depend only on α,p,n. From this note we see that if $z=(z',z_n)\in Q^+_{1/2}(0)$ and $10^{-3}<\rho_1<\rho_2<10^3$, then

$$\psi(y)=\frac{e^{-N|y-z|^2}-e^{-N\rho_2^2}}{e^{-N\rho_1^2}-e^{-N\rho_2^2}} \tag{2.10}$$

is a subsolution to L^* in $Q^+_1(0)\cap[B(z,\rho_2)\setminus B(z,\rho_1)]$, if $N=N(\alpha,p,n)$ is sufficiently large, and $\psi\equiv1$ on $\partial B(z,\rho_1)$ while $\psi\equiv0$ on $\partial B(z,\rho_2)$. Using this fact, with $z=(z',1/16)$, $|z'|<1/2$, $\rho_1=1/64,\rho_2=1/16$ and Harnack's inequality for L^* (see [GT, Corollary 9.25]) we get

$$c^{-1}y_n\,u(e_n/4,\epsilon)\ \leq\ u(y,\epsilon) \tag{2.11}$$

whenever $y\in Q^+_{1/4}(0)$. Moreover, using $1-\psi, z=(z',-e_n/64)$, $|z'|<1/2$, $\rho_1=1/64,\rho_2=1/16$, in a similar argument it follows that

$$u(y,\epsilon)\leq cy_n\max_{Q^+_{1/4}(0)}u(\cdot,\epsilon)\ \leq c^2\,y_n\,u(e_n/4,\epsilon) \tag{2.12}$$

in $Q^+_{1/4}(0)$. In particular, the right-hand inequality in (2.12) follows from the analogue of Lemma 2.3 for L^*.

Fix $x \in \partial Q^+_{1/2}(0) \cap \{y : y_n = 0\}$. From (2.11), (2.12), and linearity of L^* one can deduce (see for example [LN, Lemma 3.27]) that there exists $\theta, 0 < \theta < 1$, such that

$$\text{osc}\ (\rho/4) \leq \theta\ \text{osc}\ (\rho) \tag{2.13}$$

when $0 < \rho \leq 1/4$, where $\text{osc}\ (t) = M(t) - m(t)$ and we have put

$$M(t) = \max_{Q^+_t(x)} \tfrac{u(y,\epsilon)}{y_n},\ m(t) = \min_{Q^+_t(x)} \tfrac{u(y,\epsilon)}{y_n}.$$

To get (2.13) one can simply apply the same argument as in (2.11), (2.12) to $u - m(\rho)y_n, y_n$ and $M(\rho)y_n - u, y_n$ in $Q^+_\rho(x)$. Iterating (2.13), we obtain for some $\lambda > 0, c > 1$, depending on α, p, n, that

$$\text{osc}\ (s) \leq c\,(s/t)^\lambda\ \text{osc}\ (t), 0 < s < t \leq 1/4. \tag{2.14}$$

Letting $\epsilon \to 0$ it follows as in the proof of Lemma 2.4 that $u(\cdot, \epsilon)$ converges uniformly to u on compact subsets of $Q^+_{1/2}(0)$. Thus (2.11), (2.12) and (2.14) also hold for u. Moreover, (2.11), (2.12), (2.14), arbitrariness of x, and interior Harnack - Hölder continuity of u are easily shown to be equivalent to the conclusion of Lemma 2.8 when $v(y) = y_n$. $\square$

We note that boundary Harnack inequalities for non-divergence form linear symmetric operators in Lipschitz domains can be found in either [B] or [FGMS].

We end this section by proving the following lemma.

Lemma 2.15. *Let $G \subset \mathbf{R}^n$ be an open set, suppose that p, $1 < p < \infty$, is given and let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be the composition of a translation, a rotation and a dilation $z \to rz$, $r \in (0, 1]$. Suppose that u is A-harmonic in G and define $\hat{u}(z) = u(F(z))$ whenever $F(z) \in G$. Then $\hat{u}$ is $\hat{A}$-harmonic in $F^{-1}(G)$ and $\hat{A} \in M_p(\alpha, \beta, \gamma)$.*

Proof. Suppose that $F(z) = z + w$ for some $w \in \mathbf{R}^n$, i.e., F is a translation. In this case the conclusion follows immediately with $\hat{A}(z, \eta) = A(z + w, \eta)$ and $\hat{A} \in M_p(\alpha, \beta, \gamma)$. Suppose that $F(z) = \Gamma z$, where Γ is an orthogonal matrix with $\det \Gamma = 1$. In this case the conclusion follows with $\hat{A}(z, \eta) = A(\Gamma z, \Gamma \eta)$ and $\hat{A} \in M_p(\alpha, \beta, \gamma)$. Finally, suppose that $F(z) = rz$ for some $r \in (0, 1]$. Then $\hat{u}$ is $\hat{A}$-harmonic in $F^{-1}(G)$ with $\hat{A}(z, \eta) = r^{p-1} A(rz, r^{-1}\eta)$. Moreover, property (i), (ii) and (iv) in Definition 1.1 follow readily. To prove (iii) in Definition 1.1 we see that

$$|\hat{A}(z, \eta) - \hat{A}(y, \eta)| \leq \beta r^\gamma |z - y|^\gamma |\eta|^{p-1} \leq \beta |z - y|^\gamma |\eta|^{p-1}$$

whenever $r \in (0, 1]$. This completes the proof of Lemma 2.15. $\square$

3. Non-degeneracy of $|\nabla u|$

In this section we establish the 'fundamental inequality' referred to as Step A in the introduction. To do this we first prove a few technical results.

Lemma 3.1. *Let $1 < p < \infty$, and assume that $A_1, A_2 \in M_p(\alpha,\beta,\gamma)$ with*

$$|A_1(y,\eta) - A_2(y,\eta)| \leq \epsilon|\eta|^{p-1} \text{ whenever } y \in Q_1^+(0)$$

for some $0 < \epsilon < 1/2$. Let u_2 be a non-negative A_2-harmonic function in $Q_1^+(0)$, continuous on the closure $Q_1^+(0)$, and with $u_2 = 0$ on $\partial Q_1^+(0) \cap \{y_n = 0\}$. Moreover, let u_1 be the A_1-harmonic function in $Q_{1/2}^+(0)$ which is continuous on the closure of $Q_{1/2}^+(0)$ and which coincides with u_2 on $\partial Q_{1/2}^+(0)$. Then there exist, given $\rho \in (0, 1/16)$, c, $\tilde{c}$, θ, and τ, all depending only on $p, n, \alpha, \beta, \gamma$, such that

$$|u_2(y) - u_1(y)| \leq c\epsilon^\theta u_2(e_n/2) \leq \tilde{c}\epsilon^\theta \rho^{-\tau} u_2(y) \text{ whenever } y \in Q_{1/4}^+(0) \setminus Q_{1/4,\rho}^+(0)\,.$$

Proof. To begin the proof of Lemma 3.1 we note that the existence of u_1 in Lemma 3.1 follows from the Wiener criteria in [GZ], see the discussion after Lemma 2.2, the maximum principle for A-harmonic functions, and the fact that the $W^{1,p}$-Dirichlet problem for these functions in $Q_{1/2}^+(0)$ always has a unique solution (see [HKM, Appendix I]). Observe for $x \in \mathbf{R}^n, \lambda \in \mathbf{R}^n, \xi \in \mathbf{R}^n \setminus \{0\}$, and $A \in M_p(\alpha,\beta,\gamma)$, that

$$A_i(x,\lambda) - A_i(x,\xi) = \sum_{j=1}^n (\lambda_j - \xi_j) \int_0^1 \frac{\partial A_i}{\partial \eta_j}(x, t\lambda + (1-t)\xi)dt \tag{3.2}$$

for $i \in \{1,..,n\}$. Using (3.2) and Definition 1.1 $(i), (ii)$, we see that

$$c^{-1}\,(|\lambda|+|\xi|)^{p-2}\,|\lambda-\xi|^2 \;\leq\; \langle A(x,\lambda)-A(x,\xi), \lambda-\xi\rangle \;\leq c\,(|\lambda|+|\xi|)^{p-2}|\lambda-\xi|^2\,. \tag{3.3}$$

Moreover, from (3.3) we deduce that if

$$I = \int_{Q_{1/2}^+(0)} |\nabla u_2 - \nabla u_1|^p dy,$$

then,

$$I \leq cJ,\; J := \int_{Q_{1/2}^+(0)} \langle A_1(y, \nabla u_1(y)) - A_1(y, \nabla u_2(y)), \nabla u_2(y) - \nabla u_1(y)\rangle dy, \tag{3.4}$$

whenever $p \geq 2$. Also, if $1 < p < 2$, we see from (3.3) and Hölder's inequality that

$$I \leq cJ^{p/2} \left(\int_{Q_{1/2}^+(0)} |\nabla u_1|^p + |\nabla u_2|^p dx \right)^{1-p/2} \tag{3.5}$$

where J is as defined in (3.4). As $\nabla \cdot (A_1(y, \nabla u_1(y))) = 0 = \nabla \cdot (A_2(y, \nabla u_2(y)))$ whenever $y \in Q_{1/2}^+(0)$ and as $\theta = u_2 - u_1 \in W_0^{1,p}(Q_{1/2}^+(0))$, we see from the definition of J in (3.4) that

$$J = \int_{Q_{1/2}^+(0)} \langle A_2(y, \nabla u_2(y)) - A_1(y, \nabla u_2(y)), \nabla u_2(y) - \nabla u_1(y)\rangle dy. \tag{3.6}$$

Hence, using (3.4), (3.6), the assumption on the difference $|A_1(y,\eta) - A_2(y,\eta)|$ stated in the lemma and Hölder's inequality we can conclude, for $p \geq 2$, that

$$I \leq c\epsilon \int\limits_{Q^+_{1/2}(0)} (|\nabla u_1|^p + |\nabla u_2|^p) dx. \tag{3.7}$$

Also, for $1 < p < 2$, we can use (3.5) to find that

$$I \leq c\epsilon^{p/2} \int\limits_{Q^+_{1/2}(0)} (|\nabla u_1|^p + |\nabla u_2|^p) dx. \tag{3.8}$$

Now from the observation above (3.6), (3.3) with $\xi = 0$, and Hölder's inequality we see that

$$\begin{aligned} \int\limits_{Q^+_{1/2}(0)} |\nabla u_1|^p dx \quad &\leq c \int\limits_{Q^+_{1/2}(0)} \langle A_1(x, \nabla u_1(x)), \, \nabla u_2(x)\rangle dx \\ &\leq (1/2) \int\limits_{Q^+_{1/2}(0)} |\nabla u_1|^p dx \, + c \int\limits_{Q^+_{1/2}(0)} |\nabla u_2|^p dx. \end{aligned}$$

Thus,

$$\int\limits_{Q^+_{1/2}(0)} |\nabla u_1|^p dx \, \leq \, c \int\limits_{Q^+_{1/2}(0)} |\nabla u_2|^p dx. \tag{3.9}$$

Let $a = \min\{1, p/2\}$. Using (3.9) in (3.8), (3.7), and Lemmas 2.1 - 2.3 for u_2 we obtain

$$I \leq c\epsilon^a (u_2(e_n/2))^p. \tag{3.10}$$

Next using the Poincáre inequality for functions in $W^{1,p}_0(Q^+_{1/2}(0))$ we deduce from (3.10) that

$$\int\limits_{Q^+_{1/2}(0)} |u_2 - u_1|^p \, dx \, \leq \, c \int\limits_{Q^+_{1/2}(0)} |\nabla u_2 - \nabla u_1|^p \, dx \, \leq \, c\epsilon^a (u_2(e_n/2))^p. \tag{3.11}$$

In the following we let $\eta = a/(p+2)$ and we introduce the sets

$$E = \{y \in Q^+_{1/2}(0) : \; |u_2(y) - u_1(y)| \leq \epsilon^\eta u_2(e_n/2)\}, \; F = Q^+_{1/2}(0) \setminus E. \tag{3.12}$$

Moreover, for a measurable function f defined on $Q^+_{1/2}(0)$ we introduce, whenever $y \in Q^+_{1/2}(0)$, the Hardy-Littlewood maximal function

$$M(f)(y) := \sup_{\{r>0, \; Q_r(y) \subset Q^+_{1/2}(0)\}} \frac{1}{|Q_r(y)|} \int\limits_{Q_r(y)} |f(z)| dz. \tag{3.13}$$

Let

$$G = \{y \in Q^+_{1/2}(0) : \; M(\chi_F)(y) \leq \epsilon^\eta\} \tag{3.14}$$

where χ_F is the indicator function for the set F. Then using weak (1,1)-estimates for the Hardy-Littlewood maximal function, (3.11) and (3.12) we see that

$$|Q^+_{1/2}(0) \setminus G| \leq c\epsilon^{-\eta}|F| \leq c\epsilon^{-\eta}\epsilon^{-p\eta}\epsilon^a = c\epsilon^\eta \tag{3.15}$$

by our choice for η. Also, using continuity of $u_2(y) - u_1(y)$ we find for $y \in G$ that

$$|u_2(y) - u_1(y)| = \lim_{r \to 0} \frac{1}{|B(y,r)|} \int\limits_{B(y,r)} |u_2(z) - u_1(z)| dz \leq c\epsilon^{\eta} u_2(e_n/2). \tag{3.16}$$

If $y \in Q^+_{1/4}(0) \setminus G$, then from (3.15) we see there exists $\hat{y} \in G$ such that $|y - \hat{y}| \leq c(n)\epsilon^{\eta/n}$. Using Lemmas 2.1, 2.2, we hence get that

$$\begin{aligned} |u_2(y) - u_1(y)| &\leq |u_2(\hat{y}) - u_1(\hat{y})| + |u_2(y) - u_2(\hat{y})| + |u_1(y) - u_1(\hat{y})| \\ &\leq c(\epsilon^{\eta} + \epsilon^{\tilde{\sigma}\eta/n}) u_2(e_n/2). \end{aligned} \tag{3.17}$$

This completes the proof of the first inequality stated in Lemma 3.1. Finally, using the Harnack inequality we see that there exists $\tau = \tau(p, n, \alpha, \beta, \gamma) \geq 1$ such that $u_2(e_n/2) \leq c\rho^{-\tau} u_2(y)$ whenever $y \in Q^+_{1/4}(0) \setminus Q^+_{1/4,\rho}(0)$. $\Box$

We continue by proving the following important technical lemma.

Lemma 3.18. *Let $O \subset \mathbf{R}^n$ be an open set, suppose $1 < p < \infty$, and that $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$. Also, suppose that $\hat{u}_1, \hat{u}_2$ are non-negative functions in O, that $\hat{u}_1$ is A_1-harmonic in O and that $\hat{u}_2$ is A_2-harmonic in O. Let $\tilde{a} \geq 1, y \in O$ and assume that*

$$\frac{1}{\tilde{a}} \frac{\hat{u}_1(y)}{d(y, \partial O)} \leq |\nabla \hat{u}_1(y)| \leq \tilde{a} \frac{\hat{u}_1(y)}{d(y, \partial O)}.$$

Let $\tilde{\epsilon}^{-1} = (c\tilde{a})^{(1+\hat{\sigma})/\hat{\sigma}}$, where $\hat{\sigma}$ is as in Lemma 2.4. If

$$(1 - \tilde{\epsilon})\hat{L} \leq \frac{\hat{u}_2}{\hat{u}_1} \leq (1 + \tilde{\epsilon})\hat{L} \ \text{ in } B(y, \tfrac{1}{100} d(y, \partial O))$$

for some $\hat{L}, 0 < \hat{L} < \infty$, then for $c = c(p, n, \alpha, \beta, \gamma)$ suitably large,

$$\frac{1}{c\tilde{a}} \frac{\hat{u}_2(y)}{d(y, \partial O)} \leq |\nabla \hat{u}_2(y)| \leq c\tilde{a} \frac{\hat{u}_2(y)}{d(y, \partial O)}.$$

Proof. Let $\tilde{a} \geq 1, y \in O$ be as in the statement of the lemma. Using Lemma 2.4 and the Harnack inequality in Lemma 2.1 (ii) we see that,

$$|\nabla \hat{u}_2(z_1) - \nabla \hat{u}_2(z_2)| \leq ct^{\hat{\sigma}} \max_{B(y, td(y, \partial O))} |\nabla \hat{u}_2(\cdot)| \leq c^2 t^{\hat{\sigma}}\, \hat{u}_2(y)/d(y, \partial O) \tag{3.19}$$

whenever $z_1, z_2 \in \bar{B}(y, td(y, \partial O))$ and $0 < t \leq 10^{-3}$. Here c depends only on $p, n, \alpha, \beta, \gamma$. Using (3.19) we see that we only have to prove bounds from below for the gradient of $\hat{u}_2$ at y. To achieve this we suppose that,

$$|\nabla \hat{u}_2(y)| \leq \zeta\, \hat{u}_2(y)/d(y, \partial O), \tag{3.20}$$

for some small $\zeta > 0$ to be chosen. From (3.19) with $z = z_1, y = z_2$ and (3.20) we then deduce that

$$|\nabla \hat{u}_2(z)| \leq [\zeta + c^2 t^{\hat{\sigma}}]\ \hat{u}_2(y)/d(y, \partial O) \tag{3.21}$$

whenever $z \in B(y, td(y, \partial O))$. Integrating, it follows that if $\hat{y} \in \partial B(y, td(y, \partial O))$, $|y - \hat{y}| = td(y, \partial O)$, $t = \zeta^{1/\hat{\sigma}}$, then

$$|\hat{u}_2(\hat{y}) - \hat{u}_2(y)| \leq c' \zeta^{1+1/\hat{\sigma}}\, \hat{u}_2(y). \tag{3.22}$$

The constants in (3.21),(3.22) depend only on $p, n, \alpha, \beta, \gamma$.

Next we note that (3.19) also holds with $\hat u_2$ replaced by $\hat u_1$. Let $\lambda = \frac{\nabla \hat u_1(y)}{|\nabla \hat u_1(y)|}$. Then from (3.19) for $\hat u_1$ and the non-degeneracy assumption on $|\nabla \hat u_1|$ in Lemma 3.18, we find that

$$\langle \nabla \hat u_1(z), \lambda\rangle \geq (1 - c\,\tilde a \zeta)|\nabla \hat u_1(y)| \text{ whenever } z \in \bar B(y, \zeta^{1/\hat\sigma} d(y, \partial O)),$$

for some $c = c(p, n, \alpha, \beta, \gamma)$. If $\zeta \leq (2c\tilde a)^{-1}$, where c is the constant in the last display, then we get from integration that

$$c^*(\hat u_1(\hat y) - \hat u_1(y)) \geq \tilde a^{-1} \zeta^{1/\hat\sigma} \hat u_1(y) \tag{3.23}$$

with $\hat y = y + \zeta^{1/\hat\sigma} d(y, \partial O)\lambda$ and where the constant c^* depends only on $p, n, \alpha, \beta, \gamma$. From (3.23), (3.22), we see that if $\tilde\epsilon$ is as in Lemma 3.18, then

$$\begin{aligned}
(1-\tilde\epsilon)\hat L \quad &\leq \quad \frac{\hat u_2(\hat y)}{\hat u_1(\hat y)} \leq \left(\frac{1 + c'\zeta^{1+1/\hat\sigma}}{1 + \zeta^{1/\hat\sigma}/(\tilde a c^*)}\right)\frac{\hat u_2(y)}{\hat u_1(y)} \\
&\leq \quad (1+\tilde\epsilon)\left(\frac{1 + c'\zeta^{1+1/\hat\sigma}}{1 + \zeta^{1/\hat\sigma}/(\tilde a c^*)}\right)\hat L < (1-\tilde\epsilon)\hat L
\end{aligned} \tag{3.24}$$

provided $1/(\tilde a \tilde c)^{1/\hat\sigma} \geq \zeta^{1/\hat\sigma} \geq \tilde a \tilde c\, \tilde\epsilon$ for some large $\tilde c = \tilde c(p, n, \alpha, \beta, \gamma)$. This inequality and (3.23) are satisfied if $\tilde\epsilon^{-1} = (\tilde c \tilde a)^{(1+\hat\sigma)/\hat\sigma}$ and $\zeta^{-1} = \tilde c \tilde a$. Moreover, if the hypotheses of Lemma 3.18 hold for this $\tilde\epsilon$, then in order to avoid the contradiction in (3.24) it must be true that (3.20) is false for this choice of ζ. Hence Lemma 3.18 is true. $\Box$

Armed with Lemma 3.1 and Lemma 3.18 we prove the 'fundamental inequality' for A-harmonic functions, $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ), vanishing on a portion of $\{y : y_n = 0\}$.

Lemma 3.25. *Let $1 < p < \infty$, and $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Suppose that u is a positive A-harmonic function in $Q_1^+(0)$, continuous on the closure of $Q_1^+(0)$, and that $u = 0$ on $\partial Q_1^+(0) \cap \{y_n = 0\}$. Then there exist $\hat c = \hat c(p, n, \alpha, \beta, \gamma)$ and $\bar\lambda = \bar\lambda(p, n, \alpha, \beta, \gamma)$, such that*

$$\bar\lambda^{-1}\frac{u(y)}{y_n} \leq |\nabla u(y)| \leq \bar\lambda \frac{u(y)}{y_n} \text{ whenever } y \in Q^+_{1/\hat c}(0).$$

Proof. Let $A \in M_p(\alpha, \beta, \gamma)$, $A = A(y, \eta)$, be given. Put $A_2(y, \eta) = A(y, \eta)$, $A_1(\eta) = A(0, \eta)$. Clearly, $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$. We decompose the proof into the following steps.

Step 1. Lemma 3.25 holds for the operator A_1. To see this we note once again that $\hat u_1(y) = y_n$ is A_1-harmonic and $\hat u_1 = 0$ on $\partial Q_1^+(0) \cap \{y_n = 0\}$. Let $\hat u_2 = u$. Applying Lemma 2.8 to the pair $\hat u_1, \hat u_2$ we see that

$$\left|\log\left(\frac{\hat u_1(y_1)}{\hat u_2(y_1)}\right) - \log\left(\frac{\hat u_1(y_2)}{\hat u_2(y_2)}\right)\right| \leq c|y_1 - y_2|^\sigma \tag{3.26}$$

whenever $y_1,\, y_2 \in Q^+_{1/4}(0)$. Exponentiation of this inequality yields the equivalent inequality

$$\left|\frac{\hat u_1(y_1)}{\hat u_2(y_1)} - \frac{\hat u_1(y_2)}{\hat u_2(y_2)}\right| \leq c' \frac{\hat u_1(y_2)}{\hat u_2(y_2)}|y_1 - y_2|^\sigma \tag{3.27}$$

whenever $y_1,\ y_2 \in Q^+_{1/4}(0)$. Let $O = Q^+_{1/4}(0)$ and note that if $y_2 \in Q^+_{1/8}(0)$ then obviously

$$\frac{1}{\tilde a}\frac{\hat u_1(y_2)}{d(y_2,\partial O)} \leq |\nabla \hat u_1(y_2)| \leq \tilde a \frac{\hat u_1(y_2)}{d(y_2,\partial O)} \tag{3.28}$$

for some $\tilde a = \tilde a(n)$. Let r be defined through the relation $c' r^\sigma = \frac{1}{2}\tilde\epsilon$ where $\tilde\epsilon$ is as in Lemma 3.18. Using (3.27) we then see that

$$(1-\tilde\epsilon/2)\frac{\hat u_1(y_2)}{\hat u_2(y_2)} \leq \frac{\hat u_1(y_1)}{\hat u_2(y_1)} \leq (1+\tilde\epsilon/2)\frac{\hat u_1(y_2)}{\hat u_2(y_2)} \tag{3.29}$$

whenever $y_1 \in B(y_2, r)$. From (3.28), (3.29), and Lemma 3.18 we conclude that Lemma 3.25 holds for the operator A_1.

Step 2. Lemma 3.25 is valid for the operator A_2. We let $\rho \in (0, 1/16)$ and $\bar\delta \in (0, 1/8)$ be degrees of freedom to be chosen below. Let $\hat u_1$ be the A_1-harmonic function in $Q^+_{\bar\delta/2}(0)$ which is continuous on the closure of $Q^+_{\bar\delta/2}(0)$ and which satisfies $\hat u_1 = u$ on $\partial Q^+_{\bar\delta/2}(0)$. Using Step 1 we see there exist $\lambda_1 = \lambda_1(p, n, \alpha)$, $\hat c_1 = \hat c_1(p, n, \alpha) \geq 1$, such that

$$\lambda_1^{-1}\frac{\hat u_1(y)}{y_n} \leq |\nabla \hat u_1(y)| \leq \lambda_1 \frac{\hat u_1(y)}{y_n} \text{ whenever } y \in Q^+_{\bar\delta/\hat c_1}(0). \tag{3.30}$$

Moreover, using Definition 1.1 (iii) we have

$$|A_2(y,\eta) - A_1(y,\eta)| \leq \epsilon|\eta|^{p-2} \text{ with } \epsilon = 2\beta\bar\delta^\gamma \text{ whenever } y \in Q^+_{\bar\delta}(0). \tag{3.31}$$

Let $\hat u_2 = u$. From Lemma 2.15 and Lemma 3.1 we see there exist c', θ, τ, each depending only on $p, n, \alpha, \beta, \gamma$, such that

$$|\hat u_2(y) - \hat u_1(y)| \leq c'\epsilon^\theta\rho^{-\tau}\,\hat u_2(y) \text{ whenever } y \in Q^+_{\bar\delta/4}(0) \setminus Q^+_{\bar\delta/4,\rho\bar\delta}(0). \tag{3.32}$$

Let $\tilde\epsilon$ be as in the statement of Lemma 3.18 with $\tilde a$ replaced by λ_1 and put $\rho = 1/(32\hat c_1)$. Fix $\bar\delta$ subject to $c'\epsilon^\theta\rho^{-\tau} = c'\,(2\beta\bar\delta^\gamma)^\theta\rho^{-\tau} = \min\{\tilde\epsilon/2, 10^{-8}\}$. In particular, we note that $\bar\delta = \bar\delta(p, n, \alpha, \beta, \gamma)$. Then from (3.32) we see that

$$1-\tilde\epsilon \leq \frac{\hat u_2(y)}{\hat u_1(y)} \leq 1+\tilde\epsilon \text{ whenever } y \in Q^+_{\bar\delta/4}(0) \setminus Q^+_{\bar\delta/4,\rho\bar\delta}(0). \tag{3.33}$$

Using (3.30), (3.33), and Lemma 3.18 we therefore conclude that

$$\lambda_2^{-1}\frac{\hat u_2(y)}{y_n} \leq |\nabla \hat u_2(y)| \leq \lambda_2 \frac{\hat u_2(y)}{y_n} \text{ whenever } y \in Q^+_{\bar\delta/\hat c_1}(0) \setminus Q^+_{\bar\delta/\hat c_1, 2\rho\bar\delta}(0), \tag{3.34}$$

for some $\lambda_2 = \lambda_2(p, n, \alpha, \beta, \gamma)$. Moreover, if $y \in Q^+_{\bar\delta/\hat c_1, 2\rho\bar\delta}(0)$, then we can also prove that (3.34) is valid at y by iterating the previous argument and by making use of the invariance of the class $M_p(\alpha, \beta, \gamma)$ with respect to translations and dilations, see Lemma 2.15. This completes the proof of Lemma 3.25. □

Finally we use Lemma 3.25 to establish the main result of this section.

Lemma 3.35. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain, $w \in \partial\Omega$, and $0 < r < \min\{r_0, 1\}$. Let $p, 1 < p < \infty$, be given and assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Suppose that u is a positive A-harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\bar\Omega \cap B(w, 4r)$, and that $u = 0$ on $\Delta(w, 4r)$. There exist $\hat\delta =$*

$\hat\delta(p,n,\alpha,\beta,\gamma)$, $\hat c=\hat c(p,n,\alpha,\beta,\gamma)$ and $\bar\lambda=\bar\lambda(p,n,\alpha,\beta,\gamma)$, such that if $0<\delta\leq\hat\delta$, then

$$\bar\lambda^{-1}\frac{u(y)}{d(y,\partial\Omega)}\leq|\nabla u(y)|\leq\bar\lambda\frac{u(y)}{d(y,\partial\Omega)}\ \textit{whenever}\ y\in\Omega\cap B(w,r/\hat c).$$

Proof. Let $A\in M_p(\alpha,\beta,\gamma)$, $A=A(y,\eta)$ be given. Let $w\in\partial\Omega$, $0<r<r_0$, suppose that u is a positive A-harmonic function in $\Omega\cap B(w,4r)$, that u is continuous in $\bar\Omega\cap B(w,4r)$, and that $u=0$ on $\Delta(w,4r)$. We intend to use Lemma 3.25 and Lemma 3.1 to prove Lemma 3.35. Let $u\equiv0$ in $B(w,4r)\setminus\Omega$. Then $u\in W^{1,p}(B(w,2r))$ and u is continuous in $B(w,4r)$. Let $c_1=\hat c$ be as in Lemma 3.25 and choose $c'\geq100c_1$ so that if $\hat y\in\Omega\cap B(w,r/c'), s=4c_1d(\hat y,\partial\Omega)$, and $z\in\partial\Omega$ with $|\hat y-z|=d(\hat y,\partial\Omega)$, then

$$\max_{B(z,4s)}u\leq cu(\hat y) \tag{3.36}$$

for some $c=c(p,n,\alpha,\beta,\gamma)$. Using Definition 1.6 with w,r replaced by $z,4s$, we see that there exists a hyperplane Λ such that

$$h(\partial\Omega\cap B(z,4s),\Lambda\cap B(z,4s))\leq4\delta s. \tag{3.37}$$

For the moment we allow $\hat\delta$ in Lemma 3.35 to vary but shall later fix it as a number satisfying several conditions. Using (1.7) we deduce that

$$\{y\in\Omega\cap B(z,4s):d(y,\partial\Omega)\geq8\delta s\}\subset\text{ one component of }\mathbf{R}^n\setminus\Lambda.$$

Moreover, using Lemma 2.15 we see that we may without loss of generality assume that $\Lambda=\{(y',y_n):y'\in\mathbf{R}^{n-1},y_n=0\}$ and

$$\{y\in\Omega\cap B(z,4s):d(y,\partial\Omega)\geq8\delta s\}\subset\{y\in\mathbf{R}^n:y_n>0\}. \tag{3.38}$$

From (3.38) we find that if we define

$$\Lambda'\ =\ \{(y',0)+20\delta se_n,\ y'\in\mathbf{R}^{n-1}\},\ \Omega'=\{y\in\mathbf{R}^n:y_n>20\delta s\},$$

then

$$\Omega'\cap B(z,2s)\subset\Omega\cap B(z,2s). \tag{3.39}$$

Let v be a A-harmonic function in $\Omega'\cap B(z,2s)$ with continuous boundary values on $\partial(\Omega'\cap B(z,2s))$ and such that $v\leq u$ on $\partial(\Omega'\cap B(z,2s))$. Moreover, we choose v so that

$$\begin{aligned}v(y)&=u(y)\text{ whenever }y\in\partial[\Omega'\cap B(z,2s)]\text{ and }y_n>40\delta s,\\ v(y)&=0\quad\text{ whenever }y\in\partial[\Omega'\cap B(z,2s)]\text{ and }y_n<30\delta s.\end{aligned}$$

Existence of v follows once again from the Wiener criteria of [GZ], the maximum principle for A-harmonic functions, and the fact that the $W^{1,p}$-Dirichlet problem for these functions in $\Omega'\cap B(z,2s)$ always has a solution. By construction and the maximum principle for A-harmonic functions we have $v\leq u$ in $\Omega'\cap B(z,2s)$. Also, since each point of $\partial[\Omega'\cap B(z,2s)]$ where $u\neq v$ lies within $80\delta s$ of a point where u is zero, it follows from (3.36) and Lemmas 2.2, 2.3 that $u\leq v+c\delta^{\tilde\sigma}u(\hat y)$ on $\partial[\Omega'\cap B(z,2s)]$. In particular, again using the maximum principle for p-harmonic functions we conclude that

$$v\leq u\leq v+c\delta^{\tilde\sigma}u(\hat y)\text{ in }\Omega'\cap B(z,2s).$$

Thus, using the last inequality and (3.36) we see that

$$1 \leq \frac{u(y)}{v(y)} \leq (1 - c\delta^{\tilde{\sigma}})^{-1} \text{ whenever } y \in \Omega' \cap B(\hat{y}, \tfrac{1}{2}d(\hat{y}, \partial\Omega')) \tag{3.40}$$

provided $\hat{\delta}$ is small enough. Using Lemma 3.25 and the construction we also have

$$\hat{\lambda}^{-1}\frac{v(\hat{y})}{d(\hat{y}, \partial\Omega)} \leq |\nabla v(\hat{y})| \leq \hat{\lambda}\frac{v(\hat{y})}{d(\hat{y}, \partial\Omega)}. \tag{3.41}$$

for some $\hat{\lambda} = \hat{\lambda}(p, n)$. In particular, from (3.40), (3.41) we see for $0 < \delta < \hat{\delta}$, and $\hat{\delta} = \hat{\delta}(p, n, \alpha, \beta, \gamma)$ suitably small, that the hypotheses of Lemma 3.18 are satisfied with $O = \Omega' \cap B(z, 2s)$ and $\tilde{a} = \hat{\lambda}$. We now fix $\hat{\delta}$ and from Lemma 3.18 we conclude that

$$\bar{\lambda}_1^{-1}\frac{u(\hat{y})}{d(\hat{y}, \partial\Omega)} \leq |\nabla u(\hat{y})| \leq \bar{\lambda}_1\frac{u(\hat{y})}{d(\hat{y}, \partial\Omega)}$$

for some $\bar{\lambda}_1 = \bar{\lambda}_1(p, n, \alpha, \beta, \gamma)$. Since $\hat{y} \in \Omega \cap B(w, r/c')$ is arbitrary, the proof of Lemma 3.35 is complete. $\Box$

4. Degenerate elliptic equations and extension of $|\nabla u|^{p-2}$ to an A_2-weight

Let $w \in \mathbf{R}^n$, $0 < r$ and let $\lambda(x)$ be a real valued Lebesgue measurable function defined almost everywhere on $B(w, 2r)$. $\lambda(x)$ is said to belong to the class $A_2(B(w, r))$ if there exists a constant Γ such that

$$\tilde{r}^{-2n} \int\limits_{B(\tilde{w},\tilde{r})} \lambda \, dx \cdot \int\limits_{B(\tilde{w},\tilde{r})} \lambda^{-1} dx \leq \Gamma \tag{4.1}$$

whenever $\tilde{w} \in B(w, r)$ and $0 < \tilde{r} \leq r$. If $\lambda(x)$ belongs to the class $A_2(B(w, r))$ then λ is referred to as an $A_2(B(w, r))$-weight. The smallest Γ such that (4.1) holds is referred to as the A_2-constant of λ.

In the following we let $\Omega \subset \mathbf{R}^n$ be a bounded (δ, r_0)-Reifenberg flat domain with NTA-constant M. We let $w \in \partial\Omega$, $0 < r < r_0$, and we consider the operator

$$\hat{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\hat{b}_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{4.2}$$

in $\Omega \cap B(w, 2r)$. We assume that the coefficients $\{\hat{b}_{ij}(x)\}$ are bounded, Lebesgue measurable functions defined almost everywhere on $B(w, 2r)$. Moreover,

$$c^{-1}\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} \hat{b}_{ij}(x)\xi_i\xi_j \leq c|\xi|^2\lambda(x) \tag{4.3}$$

for almost every $x \in B(w, 2r)$, where $\lambda \in A_2(B(w, r))$. By definition $\hat{L}$ is a degenerate elliptic operator (in divergence form) in $B(w, 2r)$ with ellipticity measured by the function λ. If $O \subset B(w, 2r)$ is open then we let $\tilde{W}^{1,2}(O)$ be the weighted Sobolev space of equivalence classes of functions v with distributional gradient ∇v and norm

$$\|v\tilde{\|}_{1,2}^2 = \int\limits_O v^2\lambda dx + \int\limits_O |\nabla v|^2\lambda dx < \infty.$$

Let $\tilde W_0^{1,2}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $\tilde W^{1,2}(O)$. We say that v is a weak solution to $\hat L v = 0$ in O provided $v \in \tilde W^{1,2}(O)$ and

$$\int_O \sum_{i,j} \hat b_{ij} v_{x_i} \phi_{x_j} dx = 0 \tag{4.4}$$

whenever $\phi \in C_0^\infty(O)$.

The following three lemmas, Lemmas 4.5-4.7, are tailored to our situation and based on the results in [FKS], [FJK] and [FJK1]. We note that these authors assumed $\hat L$ was symmetric, i.e., $\hat b_{ij} = \hat b_{ji}, 1 \leq i, j \leq n$, but this assumption was not needed in the proof of these lemmas. Essentially one can say 'ditto' to the discussion in [KKPT, section 1] for nonsymmetric uniformly elliptic divergence form PDE.

Lemma 4.5. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant Γ. Suppose that v is a positive weak solution to $\hat L v = 0$ in $\Omega \cap B(w, 2r)$. Then there exists a constant c, $1 \leq c < \infty$, depending only on n, M and Γ, such that if $\tilde w \in \Omega$, $0 < \tilde r$, $B(\tilde w, 2\tilde r) \subset \Omega \cap B(w, r)$, then*

$$(i) \qquad c^{-1}\tilde r^2 \int_{B(\tilde w,\tilde r/2)} |\nabla v|^2 \lambda dx \leq c\biggl(\int_{B(\tilde w,\tilde r)} \lambda dx\biggr)(\max_{B(\tilde w,\tilde r)} v)^2 \leq c \int_{B(\tilde w,2\tilde r)} |v|^2 \lambda dx,$$

$$(ii) \qquad \max_{B(\tilde w,\tilde r)} v \leq c \min_{B(\tilde w,\tilde r)} v.$$

Furthermore, there exists $\tilde\alpha = \tilde\alpha(n, M, \Gamma) \in (0, 1)$ such that if $x, y \in B(\tilde w, \tilde r)$ then

$$(iii) \qquad |v(x) - v(y)| \leq c\biggl(\tfrac{|x-y|}{\tilde r}\biggr)^{\tilde\alpha} \max_{B(\tilde w,2\tilde r)} v.$$

Lemma 4.6. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant Γ. Suppose that v is a positive weak solution to $\hat L v = 0$ in $\Omega \cap B(w, 2r)$ and that $v = 0$ on $\Delta(w, 2r)$ in the weighted Sobolev sense. Then there exists $\tilde c = \tilde c(n, M, \Gamma)$, $1 \leq \tilde c < \infty$, such that the following holds with $\tilde r = r/\tilde c$.*

$$(i) \qquad r^2 \int_{\Omega\cap B(w,r/2)} |\nabla v|^2 \lambda dx \leq \tilde c \int_{\Omega\cap B(w,r)} |v|^2 \lambda dx,$$

$$(ii) \qquad \max_{\Omega\cap B(w,\tilde r)} v \leq \tilde c v(a_{\tilde r}(w)).$$

Furthermore, there exists $\tilde\alpha = \alpha(n, M, \Gamma) \in (0, 1)$ such that if $x, y \in \Omega \cap B(w, \tilde r)$, then

$$(iii) \qquad |v(x) - v(y)| \leq c\biggl(\tfrac{|x-y|}{r}\biggr)^{\tilde\alpha} \max_{\Omega\cap B(w,2\tilde r)} v.$$

Lemma 4.7. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant Γ. Suppose that v_1 and v_2 are two positive weak solutions to $\hat L v = 0$ in $\Omega \cap B(w, 2r)$ and $v_1 = 0 = v_2$ on $\Delta(w, 2r)$ in the weighted Sobolev sense. Then there exist $c = c(n, M, \Gamma)$, $1 \leq c < \infty$, and*

$\sigma = \sigma(n, M, \Gamma) \in (0, 1)$ such that if $\tilde r = r/c$, $v_1(a_{\tilde r}(w)) = v_2(a_{\tilde r}(w))$, then $v_1/v_2 \leq c$ in $\Omega \cap B(w, r/c)$ and if $y_1, y_2 \in \Omega \cap B(w, r/c)$, then

$$\left|\frac{v_1(y_1)}{v_2(y_1)} - \frac{v_1(y_2)}{v_2(y_2)}\right| \leq c\left(\frac{|y_1 - y_2|}{r}\right)^{\sigma}.$$

To continue the proof of Theorem 1, we have the following lemmas.

Lemma 4.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded (δ, r_0)-Reifenberg flat domain. Let $p, 1 < p < \infty$, be given and assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a positive A-harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar\Omega \cap \bar B(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Then there exist, for $\epsilon^* > 0$ given, $\hat\delta = \hat\delta(p, n, \alpha, \beta, \gamma, \epsilon^*) > 0$ and $c = c(p, n, \alpha, \beta, \gamma, \epsilon^*)$, $1 \leq c < \infty$, such that*

$$c^{-1}\left(\frac{\hat r}{r}\right)^{1+\epsilon^*} \leq \frac{u(a_{\hat r}(w))}{u(a_r(w))} \leq c\left(\frac{\hat r}{r}\right)^{1-\epsilon^*}$$

whenever $0 < \delta \leq \hat\delta$ and $0 < \hat r < r/4$.

Lemma 4.9. *Let $\Omega \subset \mathbf{R}^n$ be a bounded (δ, r_0)-Reifenberg flat domain. Let $p, 1 < p < \infty$, be given and assume that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \partial\Omega$, $0 < r < \min\{r_0, 1\}$, and suppose that u is a positive A-harmonic function in $\Omega \cap B(w, 2r)$, u is continuous in $\bar\Omega \cap B(w, 2r)$, and $u = 0$ on $\Delta(w, 2r)$. There exist $\delta' = \delta'(p, n, \alpha, \beta, \gamma)$, and $c = c(p, n, \alpha, \beta, \gamma) \geq 1$ such that if $0 < \delta < \delta'$, and $\hat r = r/c$, then $|\nabla u|^{p-2}$ extends to an $A_2(B(w, \hat r))$-weight with constant depending only on $p, n, \alpha, \beta, \gamma$.*

Proof of Lemma 4.8: Let $A \in M_p(\alpha, \beta, \gamma)$, $A = A(y, \eta)$ be given and set $A_2(y, \eta) = A(y, \eta)$, $A_1(\eta) = A(w, \eta)$. Then $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$. Let u be a A_2-harmonic function as in the statement of the lemma. We extend u to $B(w, 4r) \setminus \Omega$ by putting $u \equiv 0$ in this set and then note that u is continuous in $B(w, 4r)$. We also observe from Definition 1.8 that it suffices to prove Lemma 4.9 for $\delta = \hat\delta$. Also, as discussed after Definition 1.8 we may assume that Ω is a NTA-domain. Moreover, using Lemma 2.15 and Definition 1.1 (iv), we can without loss of generality assume that $r = 4$, $w = 0$ and $u(a_1(0)) = 1$.

In the following we let ξ be a small constant to be chosen below. In particular, ξ will be fixed to depend only on $p, n, \alpha, \beta, \gamma$. For ξ fixed we can, again using Lemma 2.15, without loss of generality also assume that

$$h(P \cap B(0, 4\xi), \partial\Omega \cap B(0, 4\xi)) \leq 4\hat\delta\xi,$$

where $P = \{y \in \mathbf{R}^n : y_n = 0\}$. Furthermore, if $\bar\delta = 4\hat\delta$ is small enough, then we may assume, as in (3.38), that

$$\begin{aligned} B(0,4) \cap \{(y', y_n) : y_n \geq 2\bar\delta\xi\} &\subset \Omega \\ B(0,4) \cap \{(y', y_n) : y_n \leq -2\bar\delta\xi\} &\subset \mathbf{R}^n \setminus \Omega. \end{aligned} \tag{4.10}$$

Moreover, we see that to prove Lemma 4.8 it suffices to show that

$$c^{-1}\hat r^{1+\epsilon^*} \leq u(a_{\hat r}(0)) \leq c\hat r^{1-\epsilon^*} \text{ whenever } 0 < \hat r < \xi. \tag{4.11}$$

In the following we will use the notation introduced in (2.7).

To begin the proof of (4.11) we introduce two auxiliary functions u^+ and u^-. In particular, we define u^+ to be A_2-harmonic in $Q^+_{\xi,(1-8\bar\delta)\xi}(8\bar\delta\xi e_n)$ with continuous boundary values on $\partial Q^+_{\xi,(1-8\bar\delta)\xi}(8\bar\delta\xi e_n)$ defined as follows,

$$\begin{aligned} u^+(y) &= u(y) && \text{if } y\in\partial Q^+_{\xi,(1-8\bar\delta)\xi}(8\bar\delta\xi e_n)\cap\{y:16\bar\delta\xi\leq y_n\},\\ u^+(y) &= \frac{(y_n-8\bar\delta\xi)}{8\bar\delta\xi}u(y) && \text{if } y\in\partial Q^+_{\xi,(1-8\bar\delta)\xi}(8\bar\delta\xi e_n)\cap\{y:8\bar\delta\xi<y_n<16\bar\delta\xi\},\\ u^+(y) &= 0 && \text{if } y\in\partial Q^+_{\xi,(1-8\bar\delta)\xi}(8\bar\delta\xi e_n)\cap\{y:y_n=8\bar\delta\xi\}. \end{aligned}$$

Similarly, we define u^- to be the A_2-harmonic function in $Q^+_{\xi,(1+8\bar\delta)\xi}(-8\bar\delta\xi e_n)$ which satisfies $u^-=u$ on $\partial Q^+_{\xi,(1+8\bar\delta)\xi}(-8\bar\delta\xi e_n)$. From the maximum principle for A-harmonic functions and (4.10) we see, by construction, that

$$u^+(y)\leq u(y)\leq u^-(y) \text{ whenever } y\in Q^+_{\xi,(1-8\bar\delta)\xi}(8\bar\delta\xi e_n). \tag{4.12}$$

Using Definition 1.1 (iii) we next note that

$$|A_2(y,\eta)-A_1(y,\eta)|\leq\epsilon|\eta|^{p-1} \text{ whenever } y\in Q^+_{\xi,(1+8\bar\delta)\xi}(-8\bar\delta\xi e_n),\ \epsilon=2\beta\xi^\gamma. \tag{4.13}$$

To proceed we let $\bar u^+$ be the A_1-harmonic function in $Q^+_{\xi/2,(1/2-8\bar\delta)\xi}(8\bar\delta\xi e_n)$ which is continuous on the closure of $Q^+_{\xi/2,(1/2-8\bar\delta)\xi}(8\bar\delta\xi e_n)$ and which coincides with u^+ on $\partial Q^+_{\xi/2,(1/2-8\bar\delta)\xi}(8\bar\delta\xi e_n)$. Similarly, we let $\bar u^-$ be the A_1-harmonic function in $Q^+_{\xi/2,(1/2+8\bar\delta)\xi}(-8\bar\delta\xi e_n)$ which is continuous on the closure of $Q^+_{\xi/2,(1/2+8\bar\delta)\xi}(-8\bar\delta\xi e_n)$ and coincides with u^- on $\partial Q^+_{\xi/2,(1/2+8\bar\delta)\xi}(-8\bar\delta\xi e_n)$. Finally, we define $v^+(y):=y_n-8\bar\delta\xi$, $v^-(y):=y_n+8\bar\delta\xi$ whenever $y\in\mathbf{R}^n$. Hence v^+ and v^- are A_1-harmonic functions and grow linearly in the e_n-direction.

We first focus on the right hand inequality in (4.11). Using (4.13), Lemma 2.15, and Lemma 3.1 we see that

$$u^-(y)\leq(1-\tilde c\epsilon^\theta\bar\delta^{-\tau})^{-1}\bar u^-(y) \tag{4.14}$$

for $y\in Q^+_{\xi/4,(1/4+8\bar\delta)\xi}(-8\bar\delta\xi e_n)\cap\{-4\bar\delta\xi<y_n<\xi/2\}$ and for a constant $\tilde c=\tilde c(p,n,\alpha,\beta,\gamma)$. Moreover, using (4.12), the maximum principle and the Harnack inequality for A-harmonic functions, (4.14), as well as Lemma 2.8 applied to the functions $\bar u^-$, v^- we see that there exists a constant $\bar c=\bar c(p,n,\alpha)$, $1\leq\bar c<\infty$, such that

$$u(y)\leq u^-(y)\leq(1-\tilde c\epsilon^\theta\bar\delta^{-\tau})^{-1}\bar u^-(y)\leq c(1-\tilde c\epsilon^\theta\bar\delta^{-\tau})^{-1}\,\bar u^-(a_{\xi/8}(0))\frac{v^-(y)}{\xi} \tag{4.15}$$

whenever $y\in\Omega\cap B(0,\xi/\bar c)$. From (4.15) we conclude that

$$u(y)\leq c(1-\tilde c\epsilon^\theta\bar\delta^{-\tau})^{-1}\bar u^-(a_{\xi/8}(0))\ (y_n/\xi) \tag{4.16}$$

whenever $y\in\Omega\cap B(0,\xi/\bar c)$. Let $\bar\delta<1/(16\bar c)$ and let ξ be defined though the relation

$$1/2=\tilde c\epsilon^\theta\bar\delta^{-\tau}=\tilde c(2\beta\xi^\gamma)^\theta\ \bar\delta^{-\tau}.$$

Then $\xi=\xi(p,n,\alpha,\beta,\gamma,\bar\delta)$ and from Lemmas 3.1, 2.2, 2.3, as well as the maximum principle for A harmonic functions, we observe that

$$u(a_{\xi/8}(0))\approx u^-(a_{\xi/8}(0))\approx\bar u^-(a_{\xi/8}(0))\approx\max_{Q^+_{\xi/2,(1/2+8\bar\delta)\xi}(-8\bar\delta\xi e_n)}u$$

where proportionality constants depend only on $p, n, \alpha, \beta, \gamma$. Using these displays in (4.16), we get $u(a_{\bar\delta\xi}(0)) \leq \hat c\bar\delta\, u(a_{\xi/8}(0))$. Moreover, suppose by way of induction that we have shown, for some $k \in \{1, 2, ...\}$,

$$u(a_{\bar\delta^k\xi}(0)) \leq (\hat c\bar\delta)^k\, u(a_{\xi/8}(0)) \tag{4.17}$$

where $\hat c$ depends only on $p, n, \alpha, \beta, \gamma$. Then, from Reifenberg flatness we see there exists a plane P' containing 0 such that

$$h(P' \cap B(0, 4\bar\delta^k\xi), \partial\Omega \cap B(0, 4\bar\delta^k\xi)) \leq 4\hat\delta\bar\delta^k\xi.$$

We can now repeat the above argument with P replaced by P' and 4 replaced by $4\bar\delta^k\xi$. Here however we use a cylinder with radius and height $\approx \bar\delta^k\xi$, since we can already apply Lemma 3.1. We get

$$u(a_{\bar\delta^{k+1}\xi}(0)) \leq \hat c\,\bar\delta\, u(a_{\bar\delta^k\xi}(0)) \leq (\hat c\bar\delta)^{k+1} u(a_{\xi/8}(0)).$$

Thus by induction the inequality in (4.17) is true for all positive integers k. Next we fix $\bar\delta$ so that $\bar\delta^{-\epsilon^*} = \hat c$ where $\hat c$ is the constant in the above display. Then $\bar\delta$ and ξ both depend only on $p, n, \alpha, \beta, \gamma$ and ϵ^*. Given $0 < \hat r < \xi$, let k be the smallest integer such that $\bar\delta^k\xi \leq \hat r$. Then from (4.17) and our choice of $\bar\delta$ we see that $u(a_{\hat r}(0)) \leq c\hat r^{1-\epsilon^*}$, for some $c = c(p, n, \alpha, \beta, \gamma, \epsilon^*)$. Here we have also used the fact that $u(a_{\xi/8}(0)) \leq c_* = c_*(p, n, \alpha, \beta, \gamma)$, which follows from Lemmas 2.2, 2.3, and fact that $u(a_1(0)) = 1$. This completes the proof of the right-hand side inequality in (4.11).

Second we focus on the left-hand inequality in (4.11). In this case we first apply Lemma 2.8 to the functions $\bar u^+$, v^+ in $Q^+_{\xi/2,(1/2-8\bar\delta)\xi}(8\bar\delta\xi e_n)$. Indeed, using Lemma 2.8 and the Harnack inequality we see, provided $\bar\delta$ is small enough, that

$$\frac{\bar u^+(a_{32\bar\delta\xi}(0))}{v^+(a_{32\bar\delta\xi}(0))} \approx \frac{\bar u^+(a_{\xi/8}(0))}{v^+(a_{\xi/4}(0))} \approx \frac{\bar u^+(a_{\xi/8}(0))}{\xi}. \tag{4.18}$$

Here $A \approx B$ means that A/B is bounded from above and below by constants which only depend on $p, n, \alpha, \beta, \gamma$. From (4.18) we get

$$\bar u^+(a_{32\bar\delta\xi}(0)) \geq \bar c^{-1}\bar\delta\bar u^+(a_{\xi/8}(0)) \tag{4.19}$$

for some $\bar c = c(p, n, \alpha, \xi)$, $1 \leq \bar c < \infty$. Moreover, using Lemma 3.1 we also see that

$$\bar u^+(y) \leq (1 - \tilde c\,\epsilon^\theta\bar\delta^{-\tau})^{-1} u^+(y) \tag{4.20}$$

for $y \in Q^+_{\xi/2,(1/2-8\bar\delta)\xi}(8\bar\delta\xi\delta e_n) \cap \{16\bar\delta\xi < y_n < \xi/2\}$ and for a constant $\tilde c = \tilde c(p, n, \alpha, \beta, \gamma)$. Using (4.19), (4.20), the fact that the class $M_p(\alpha, \beta, \gamma)$ is closed under translations, rotations, suitable dilations, and multiplication by constants (see Lemma 2.15 and Definition 1.1 (iv)) , we can argue as in the proof of the right-hand inequality in (4.11). Thus by induction we obtain

$$u(a_{(32\bar\delta)^k\xi}(0)) \geq (\bar c^{-1}\bar\delta)^k u(a_{\xi/8}(0)) \text{ for } k = 1, 2, ... \tag{4.21}$$

To complete the proof we let $\bar\delta$ be so small that $\bar c^{-1}\bar\delta \geq (32\bar\delta)^{1+\epsilon^*}$ and assume that $\hat r \in [(32\bar\delta)^{k+1}\xi, (32\bar\delta)^k\xi]$. With $\bar\delta(p, n, \alpha, \beta, \gamma, \epsilon^*)$ now fixed, it follows from Harnack's inequality for A-harmonic functions that

$$u(a_{\hat r}(0)) \geq c^{-1}u(a_{(32\bar\delta)^k\xi}(0)) \geq c^{-1}(32\bar\delta)^{k(1+\epsilon^*)}u(a_{\xi/8}(0)) \geq c^{-1}\hat r^{(1+\epsilon^*)}. \tag{4.22}$$

for some $c = c(p, n, \alpha, \beta, \gamma, \epsilon^*)$. In (4.22) we have also used the fact that $u(a_{\xi/8}(0)) \geq 1/c_+(p, n, \alpha, \beta, \gamma, \epsilon^*)$, for some $c_+ \geq 1$, which follows from the definition of $\bar\delta$ in terms of ϵ^*, Harnack's inequality, and the fact that $u(a_1(0)) = 1$. (4.22) completes the proof of (4.11) and hence the proof of Lemma 4.8. $\Box$

Proof of Lemma 4.9. Lemma 4.9 follows from Lemma 4.8, in exactly the same way as Lemma 3.30 in [LN4] followed from Lemma 3.15 in [LN4]. For the readers convenience we include the details of the proof. Let $Q_j = Q(x_j, r_j)$, $j = 1, 2, ...$ be a Whitney decomposition of $\mathbf{R}^n \setminus \bar\Omega$ into open cubes with center at x_j and sidelength r_j. Then $\cup_j \bar Q(x_j, r_j) = \mathbf{R}^n \setminus \bar\Omega$ and $Q(x_j, r_j) \cap Q(x_i, r_i) = \emptyset$ when $i \neq j$. We furthermore construct the Whitney cubes in such a way that $10^{-4n} d(Q_j, \partial\Omega) \leq r_j \leq 10^{-2n} d(Q_j, \partial\Omega)$. Let $\hat r = r/\tilde c^2$, where $\tilde c = \tilde c(p, n, \alpha, \beta, \gamma)$, $1 \leq \tilde c < \infty$, is so large that the 'fundamental inequality' in Lemma 3.35 holds in $\Omega \cap B(w, r/\tilde c)$. From the NTA property of Ω we may also suppose $\tilde c$ is so large that if $Q_j \cap B(w, 50\hat r) \neq \emptyset$, then there is a $w_j \in \Omega \cap B(w, \tilde c \hat r)$ for which $d(w_j, \partial\Omega) \sim d(w_j, x_j) \sim d(x_j, \partial\Omega)$. Here $A \sim B$ means that A/B is bounded from above and below by constants which only depend on n.

Next we define $\lambda(x) = |\nabla u(x)|^{p-2}$ whenever $x \in \Omega \cap B(w, 50\hat r)$ and we let Γ be the set of all j such that if $j \in \Gamma$ then $Q_j \cap B(w, 50\hat r) \neq \emptyset$. Moreover, if $j \in \Gamma$ then we choose $w_j \in \Omega \cap B(w, \tilde c \hat r)$ as above and define $\lambda(x) = \lambda(w_j)$ when $x \in Q_j$. This defines λ almost everywhere on $B(w, 50\hat r)$ with respect to Lebesgue n measure, since it follows from (4.27) that for δ small enough, $\partial\Omega \cap B(w, r)$ has Lebesgue n measure zero. From the definition of λ, Lemma 3.35, and the Harnack inequality for A-harmonic functions we see that

$$\lambda(x) = \lambda(w_j) \approx \lambda(z) \text{ whenever } x \in Q_j \text{ and } z \in B(w_j, d(w_j, \partial\Omega)/2). \tag{4.23}$$

Let $\hat\lambda = \lambda$ if $p \geq 2$ and $\hat\lambda = 1/\lambda$ if $1 < p \leq 2$. If $\tilde w \in B(w, r)$ and $d(\tilde w, \partial\Omega)/2 < \tilde r \leq \hat r$, then from Lemmas 2.1 - 2.3, (4.23), and Hölder's inequality it follows that

$$\int_{B(\tilde w, \tilde r)} \hat\lambda dx \leq c u(a_{\tilde r}(\hat w))^{|p-2|} \, \tilde r^{n-|p-2|}. \tag{4.24}$$

Here $\hat w \in \partial\Omega$ with $|\tilde w - \hat w| = d(\tilde w, \partial\Omega)$. Also, from Lemma 4.8 we get for $\hat\delta$ small enough and $y \in \Omega \cap B(\hat w, c\tilde r)$, that

$$c u(y) \geq u(a_{\tilde r}(\hat w)) \left(\frac{d(y, \partial\Omega)}{\tilde r} \right)^{1+\epsilon^*}. \tag{4.25}$$

Here $\epsilon^* > 0$ is a small positive number which will be fixed after the display following (4.27). From (4.25) and Lemma 3.35, we see that if $d(\tilde w, \partial\Omega)/2 < \tilde r \leq \hat r$, then

$$\int_{B(\tilde w, \tilde r)} \hat\lambda^{-1} dx \leq c \tilde r^{(1+\epsilon^*)|p-2|} u(a_{\tilde r}(\hat w))^{-|p-2|} \int_{\Omega \cap B(\hat w, c\tilde r)} d(y, \partial\Omega)^{-\epsilon^*|p-2|} dy. \tag{4.26}$$

To complete the estimate in (4.26) we need to estimate the integral involving the distance function. To do this we define

$$I(z, s) = \int_{\Omega \cap B(z, s)} d(y, \partial\Omega)^{-\epsilon^*|p-2|} dy$$

whenever $z \in \partial\Omega \cap B(w, r)$, $0 < s < r$. Let

$$E_k = \Omega \cap B(z, s) \cap \{y : d(y, \partial\Omega) \leq \delta^k s\} \text{ for } k = 0, 1, 2, \ldots$$

Then since $\partial\Omega$ is δ-Reifenberg flat we deduce that

$$\int_{E_k} dy \leq c_+^{k+1}\delta^k s^n \tag{4.27}$$

where $c_+ = c_+(p,n)$. Indeed, from δ-Reifenberg flatness it is easily seen that this statement holds for E_0, E_1. Moreover, E_1 can be covered by at most c/δ^{n-1} balls of radius $100\delta s$ with centers in $\partial\Omega \cap B(z,s)$. We can then repeat the argument in each ball to get that (4.27) holds for E_2. Continuing in this way we get (4.27) for all positive integers k. Using (4.27) and writing $I(z,s)$ as a sum over $E_k \setminus E_{k+1}, k = 0, 1, 2, \dots$ we get

$$I(z,s) \leq c s^{n-\epsilon^*|p-2|}\left[1 + \delta^{-\epsilon^*|p-2|}\sum_{k=1}^{\infty} c_+^k\, \delta^{k(1-\epsilon^*|p-2|)}\right] < c_-\, s^{n-\epsilon^*|p-2|},$$

for some $c_- = c_-(p,n) \geq 1$, provided $4\epsilon^*|p-2| \leq 1$ and $\delta' > 0$ is small enough. Using this estimate with $z = \hat{w}, s = c\tilde{r}$, we can continue our calculation in (4.26) and conclude that

$$\int\limits_{B(\tilde{w},\tilde{r})} \hat{\lambda}^{-1} dx \leq c u(a_{\tilde{r}}(\hat{w}))^{-|p-2|}\ \tilde{r}^{n+|p-2|}. \tag{4.28}$$

Combining (4.24), (4.28), we get

$$\int\limits_{B(\tilde{w},\tilde{r})} \hat{\lambda}^{-1} dx \ \cdot \int\limits_{B(\tilde{w},\tilde{r})} \hat{\lambda} dx\ \leq c\tilde{r}^{2n}$$

when $d(\tilde{w},\partial\Omega)/2 \leq \tilde{r} \leq \hat{r}$. This inequality is also valid if $\tilde{r} \leq d(\tilde{w},\partial\Omega)/2$, as follows easily from Lemma 3.35. We conclude from this inequality and arbitrariness of $\tilde{w}, \tilde{r}$, that Lemma 4.9 is true. □

4.1. Proof of Theorem 1. From the results proved or stated in section 2, 3, 4, we see that Steps 0, A, B, C and D outlined in the introduction are now completed. Hence, to prove Theorem 1 it only remains to remove assumption (1.14). To do this we first note from Definition 1.1 (iv) that Theorem 1 is invariant under multiplication of u, v by constants. Using this note and Lemma 2.3 we see that if $r^* = r/c$, for $c = c(p,n,\alpha,\beta,\gamma)$ large enough, then we may assume that

$$\max_{\Omega\cap B(w,4r^*)} h \approx h(a_{r^*}(w)) = 1 \text{ whenever } h = u \text{ or } v. \tag{4.29}$$

Let $\tilde{u}, \tilde{v}$ be the A-harmonic functions in $\Omega \cap B(w,4r^*)$ with boundary values $\tilde{u} = \min(u,v)$ and $\tilde{v} = 2\max(u,v)$ respectively on $\partial(\Omega\cap B(w,4r^*))$. From the maximum principle for A-harmonic functions we then see that $\tilde{u} \leq u, v \leq \tilde{v}/2$ in $\Omega \cap B(w,4r^*)$. Using this inequality and applying Theorem 1 to $\tilde{u}, \tilde{v}$ with r replaced by r^*, we get

$$\max(u/v,\ v/u)\ \leq\ \tilde{v}/\tilde{u}\ \leq c \text{ in } \Omega\cap B(w,\tilde{r}).$$

Finally we note from boundedness of u/v that (1.14) (a) can be achieved in $\Omega \cap B(w,4r^*)$ by multiplying v by a suitably large constant which can be chosen to depend depend only on $p, n, \alpha, \beta, \gamma$. Thus Theorem 1 is true. □

5. The Martin boundary problem: preliminary reductions

Let $\Omega \subset \mathbf{R}^n$, δ, r_0, p, α, β, γ, and A be as in the statement of Theorem 2. Moreover, let $w \in \partial\Omega$ and let $0 < r' \ll \tilde r_0$, where $\tilde r_0 = \min\{r_0, 1\}$. Assume that $\hat u$ is an A-harmonic function in $\Omega \setminus B(w, r')$ and that $\hat u = 0$ continuously on $\partial\Omega \setminus B(w, r')$. We can apply Lemma 3.35 to conclude that there exist $\delta^*, 0 < \delta^* < 1, \bar c, \bar\lambda \geq 1$, depending only on $p, n, \alpha, \beta, \gamma$, such that if $0 < \delta \leq \delta^*$, then, for each $\hat y \in \partial\Omega \setminus B(w, \bar c r')$, the 'fundamental inequality',

$$\bar\lambda^{-1} \frac{\hat u(y)}{d(y, \partial\Omega)} \leq |\nabla \hat u(y)| \leq \bar\lambda \frac{\hat u(y)}{d(y, \partial\Omega)} \tag{5.1}$$

holds whenever $y \in \partial\Omega \cap B(\hat y, |\hat y - w|/\bar c) \cap B(w, \tilde r_0)$. Using this fact we see that if $0 < \delta \leq \delta^*$ then there exists $\tilde\eta$, depending only on $p, n, \alpha, \beta, \gamma$, such that if we define a non-tangential approach region at $w \in \partial\Omega$, denoted $\tilde\Omega(w, \tilde\eta)$, by $\tilde\Omega(w, \tilde\eta) = \{y \in \Omega : \ d(y, \partial\Omega) \geq \tilde\eta |y - w|\}$, then

$$\hat u \text{ satisfies (5.1) in } [\Omega \setminus \tilde\Omega(w, \tilde\eta)] \cap (B(w, \tilde r_0) \setminus B(w, \bar c r')). \tag{5.2}$$

We observe that the above argument applies for any small $r' > 0$ if $\hat u$ is a minimal positive A-harmonic function with respect to w. We note, in analogy with the proof of Theorem 1, that if we apriori knew that (5.1) held in $\Omega \cap B(w, \tilde r)$ for some $\tilde r > 0$, then we could apply the argument in Steps C, D of the introduction to get an analogue of Theorem 1 in $\Omega \cap B(w, \tilde r) \setminus B(w, cr')$. Letting $r' \to 0$ we would then get Theorem 2. Unfortunately though we do not know this apriori and we do not see how to 'deduce' this inequality from simpler functions as in the proof of Lemma 3.35. Still, if (5.1) holds in $\Omega \cap B(w, \tilde r)$, whenever $A \in M_p(\alpha)$, then we can make use of appropriate versions of Lemmas 3.1 and 3.18, as well as Definition 1.1 (*iii*), to conclude that (5.1) holds in $\Omega \cap B(w, \tilde s)$, for some $\tilde s < \tilde r$, whenever $A \in M_p(\alpha, \beta, \gamma)$. Thus to prove Theorem 2 we first prove Theorem 2 under the assumption that

$$A \in M_p(\alpha). \tag{5.3}$$

In particular, we start by showing that if one such A-harmonic function satisfies the 'fundamental inequality' then all such functions, relative to the given A, have this property. More specifically we prove,

Lemma 5.4. *Let Ω be a bounded (δ, r_0)-Reifenberg flat domain and let $w \in \partial\Omega$. Let $A \in M_p(\alpha)$ for some α and $1 < p < \infty$. Let $\hat u, \hat v > 0$ be A-harmonic in $\Omega \setminus B(w, r')$, continuous in $\mathbf{R}^n \setminus B(w, r')$, with $\hat u \equiv \hat v \equiv 0$ on $\mathbf{R}^n \setminus [\Omega \cup B(w, r')]$. Suppose for some $r_1, r' < r_1 < \tilde r_0$, and $b \geq 1$, that*

$$b^{-1} \frac{\hat u(y)}{d(y, \partial\Omega)} \leq |\nabla \hat u(y)| \leq b \frac{\hat u(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap [B(w, r_1) \setminus B(w, r')]$. There exists $\tilde\delta^* > 0, \lambda, c \geq 1$, depending on p, n, α, b, such that if $0 < \delta < \tilde\delta^* < \tilde\delta$ ($\tilde\delta$ as in Theorem 1), then

$$\lambda^{-1} \frac{\hat v(y)}{d(y, \partial\Omega)} \leq |\nabla \hat v(y)| \leq \lambda \frac{\hat v(y)}{d(y, \partial\Omega)}$$

whenever $y \in \Omega \cap [B(w, r_1/c) \setminus B(w, cr')]$. Moreover,

$$\left| \log\left(\frac{\hat u(z)}{\hat v(z)}\right) - \log\left(\frac{\hat u(y)}{\hat v(y)}\right) \right| \leq c \left(\frac{r'}{\min(r_1, |z - w|, |y - w|)} \right)^{\sigma}$$

whenever $z, y \in \Omega \setminus B(w, cr')$.

Proof: We note that to prove the last statement of Lemma 5.4 we can assume that $r'/r_1 << 1$, since otherwise there is nothing to prove. Let $\tilde r = \hat c r'$. If $\hat c = \hat c(p, n, \alpha)$ is large enough, we may assume

$$\hat u \leq \hat v/2 \leq \hat c \hat u \text{ in } \Omega \setminus B(w, \tilde r), \tag{5.5}$$

as we see from Theorem 1, Harnack's inequality, and the maximum principle for A-harmonic functions. As in (1.15) we let $u(\cdot, t), t \in [0, 1]$, be A-harmonic in $\Omega \setminus \bar B(w, \tilde r)$, with continuous boundary values,

$$u(\cdot, t) = (1 - t)\hat u(\cdot) + t\hat v(\cdot) \text{ on } \partial[\Omega \setminus \bar B(w, \tilde r)]. \tag{5.6}$$

Extend $u(\cdot, t), t \in [0, 1]$, to be continuous on $\mathbf{R}^n \setminus [\Omega \cup \bar B(w, \tilde r)]$ by setting $u(\cdot, t) \equiv 0$ on this set. Next we note from Lemma 3.18 that there exists $\epsilon_0 = \epsilon_0(p, n, \alpha, b)$ such that if s_1 and ρ_1 satisfy $\tilde r \leq s_1 < \rho_1/4 \leq r_1/16$, $t \in [0, 1]$, and

$$(1 - \epsilon_0)\tilde L \leq u(\cdot, t)/\hat u(\cdot) \leq (1 + \epsilon_0)\tilde L, \tag{5.7}$$

in $\Omega \cap [B(w, 2\rho_1) \setminus B(w, s_1)]$ for some $\tilde L$, then

$$\hat\lambda^{-1} \frac{u(y, t)}{d(y, \partial\Omega)} \leq |\nabla u(y, t)| \leq \hat\lambda \frac{u(y, t)}{d(y, \partial\Omega)} \tag{5.8}$$

whenever $y \in \Omega \cap [B(w, \rho_1) \setminus B(w, 2s_1)]$ where $\hat\lambda = \hat\lambda(p, n, \alpha, b)$. Observe from (5.5), (5.6), that if $t_1, t_2 \in [0, 1]$, then

$$\begin{aligned} c^{-1} u(\cdot, t_1) &\leq U(\cdot, t_1, t_2) = \frac{u(\cdot, t_2) - u(\cdot, t_1)}{t_2 - t_1} \\ &= \hat v(\cdot) - \hat u(\cdot) \leq c\, u(\cdot, t_1) \end{aligned} \tag{5.9}$$

on $\partial[\Omega \setminus B(w, \tilde r)]$. Moreover, from the maximum principle we see that this inequality also holds in $\Omega \setminus \bar B(w, \tilde r)$. Thus for ϵ_0 as in (5.7), there exists $\epsilon_0', 0 < \epsilon_0' \leq \epsilon_0$, with the same dependence as ϵ_0, such that if $|t_2 - t_1| \leq \epsilon_0'$, then

$$1 - \epsilon_0/2 \leq \frac{u(\cdot, t_2)}{u(\cdot, t_1)} \leq 1 + \epsilon_0/2 \text{ in } \Omega \setminus B(w, \tilde r). \tag{5.10}$$

Divide [0,1] into closed intervals, disjoint except for endpoints, of length $\epsilon_0'/2$ except possibly for the interval containing 1 which is of length $\leq \epsilon_0'/2$. Let $\xi_1 = 0 < \xi_2 < ... < \xi_m = 1$ be the endpoints of these intervals. Thus [0,1] is divided into $\{[\xi_k, \xi_{k+1}]\}_1^m$. Next suppose for some $l, 1 \leq l \leq m - 1$, that (5.8) is valid whenever $t \in [\xi_l, \xi_{l+1}]$ and $y \in \Omega \cap [B(w, \rho_1) \setminus B(w, 2s_1)]$. Under this assumption we claim for some $\hat c_1, \hat c_2, \sigma$, depending only on p, n, α, b, that

$$\left| \log \frac{u(z, \xi_{l+1})}{u(z, \xi_l)} - \log \frac{u(y, \xi_{l+1})}{u(y, \xi_l)} \right| \leq \hat c_1 \left(\frac{s_1}{\min(|z - w|, |y - w|)} \right)^\sigma \tag{5.11}$$

whenever $z, y \in \Omega \cap [B(w, \rho_1/\hat c_2) \setminus B(w, \hat c_2 s_1)]$. Indeed we can retrace the argument in Step C of the introduction to get, for $z, y \in \Omega \cap [B(w, \rho_1/c) \setminus B(w, cs_1)]$, that there exists f as in (1.19) and $\sigma > 0$ as in (1.24) such that

$$\begin{aligned}\left|\log \frac{u(z,\xi_{l+1})}{u(z,\xi_l)} - \log \frac{u(y,\xi_{l+1})}{u(y,\xi_l)}\right| &\leq \int_{\xi_l}^{\xi_{l+1}} \left|\frac{f(z,t)}{u(z,t)} - \frac{f(y,t)}{u(y,t)}\right| dt \\ &\leq c\left(\frac{s_1}{\min(|z-w|,|y-w|)}\right)^{\sigma}\end{aligned} \tag{5.12}$$

To get the last inequality in (5.12) we have used a slightly more general version of Lemma 4.7.

We now proceed by induction. Observe from (5.10) and $u(\cdot,\xi_1) = \hat{u}$, that (5.7) (5.8) hold whenever $t \in [\xi_1, \xi_2]$. Thus (5.11) is true for $l = 1$ with $s_1 = \tilde{r}, \rho_1 = r_1/4$. Let $s_2 = \hat{c}_2 s_1$, $\rho_2 = \rho_1/\hat{c}_2$. By induction, suppose for some $2 \leq k < m$,

$$\left|\log \frac{u(z,\xi_k)}{\hat{u}(z)} - \log \frac{u(y,\xi_k)}{\hat{u}(y)}\right| \leq (k-1)\hat{c}_1 \left(\frac{s_k}{\min(|z-w|,|y-w|)}\right)^{\sigma} \tag{5.13}$$

whenever $z, y \in \Omega \cap [B(w,\rho_k) \setminus B(w,s_k)]$, where $\sigma, \hat{c}_1$ are the constants in (5.11). For $\eta > 0$ given and small we choose $s_k' \geq 2s_k$, so that

$$\left|\frac{u(z,\xi_k)}{\hat{u}(z)} - \frac{u(y,\xi_k)}{\hat{u}(y)}\right| \leq \eta \frac{u(z,\xi_k)}{\hat{u}(z)}$$

whenever $z, y \in \Omega \cap [B(w,\rho_k) \setminus B(w,s_k')]$. Moreover, fix z as in the last display and choose $\eta > 0$ so small that

$$(1-\epsilon_0)\frac{u(z,\xi_k)}{\hat{u}(z)} \leq \frac{u(y,t)}{\hat{u}(y)} \leq (1+\epsilon_0)\frac{u(z,\xi_k)}{\hat{u}(z)}\,. \tag{5.14}$$

whenever $y \in \Omega \cap [B(w,\rho_k) \setminus B(w,s_k')]$ and $t \in [\xi_k, \xi_{k+1}]$. To estimate the size of η observe, for $t \in [\xi_k, \xi_{k+1}]$, that

$$\frac{u(y,t)}{\hat{u}(y)} = \frac{u(y,t)}{u(y,\xi_k)} \cdot \frac{u(y,\xi_k)}{\hat{u}(y)} \leq (1+\epsilon_0/2)(1+\eta)\frac{u(z,\xi_k)}{\hat{u}(z)}.$$

Thus if $\eta = \epsilon_0/4$ (ϵ_0 small), then the right hand inequality in (5.14) is valid. A similar argument gives the left hand inequality in (5.14) when $\eta = \epsilon_0/4$. Also since $k \leq 2/\epsilon_0'$, and ϵ_0', σ depend only on p, n, α, b, we deduce from (5.13) that one can take $s_k' = \hat{c}_3 s_k$ for $\hat{c}_3 = \hat{c}(p,n,\alpha,b)$ large enough. From (5.14) we first find that (5.7) holds with $\tilde{L} = \frac{u(z,\xi_k)}{\hat{u}(z)}$ in $\Omega \cap [B(w,\rho_k) \setminus B(w,s_k')]$ and thereupon that (5.8) also holds. From (5.8) we now get, as in (5.12), that (5.11) is valid for $l = k$ in $\Omega \cap [B(w, \frac{\rho_k}{2\hat{c}_2}) \setminus B(w, 2\hat{c}_2 s_k')]$. Let $s_{k+1} = 2\hat{c}_3\hat{c}_2 s_k$ and $\rho_{k+1} = \frac{\rho_k}{2\hat{c}_2}$. Using (5.11) and the induction hypothesis we have

$$\begin{aligned}\left|\log \frac{u(z,\xi_{k+1})}{\hat{u}(z)} - \log \frac{u(y,\xi_{k+1})}{\hat{u}(y)}\right| &\leq \left|\log \frac{u(z,\xi_{k+1})}{u(z,\xi_k)} - \log \frac{u(y,\xi_{k+1})}{u(y,\xi_k)}\right| \\ &\quad + \left|\log \frac{u(z,\xi_k)}{\hat{u}(z)} - \log \frac{u(y,\xi_k)}{\hat{u}(y)}\right| \\ &\leq k\hat{c}_1\left(\frac{s_{k+1}}{\min(|z-w|,|y-w|)}\right)^{\sigma}\end{aligned} \tag{5.15}$$

whenever $z, y \in \Omega \cap [B(w, \rho_{k+1}) \setminus B(w, s_{k+1})]$. From (5.15) and induction we get (5.13) with $k = m$. Since $u(\cdot, \xi_m) = \hat{v}$ and $s_m \leq cr'$, $\rho_m \geq r_1/c$, for some large $c = c(p, n, \alpha)$, we can now argue as in (5.14) to first get (5.7) with $u(\cdot, t)$ replaced by $\hat{v}$ and then (5.8) for $\hat{v}$. We conclude that Lemma 5.4 is valid for $z, y \in \Omega \cap [B(w, r_1/c) \setminus B(w, cr')]$ provided c is large enough. Using the maximum principle for A-harmonic functions it follows that the last display in Lemma 5.4 is also valid for $z, y \in \Omega \setminus B(w, r_1/c)$. $\Box$

5.1. Proof of Theorem 2 when $A \in M_p(\alpha)$. Let $\Omega \subset \mathbf{R}^n, w \in \partial\Omega, \delta, p, r_0, \alpha, \beta, \gamma$, be as in Theorem 2. Let $A \in M_p(\alpha)$, and suppose that u, v, are minimal positive A-harmonic functions relative to $w \in \partial\Omega$. If (5.1) holds for u in $\Omega \cap B(w, r_1)$, then we can apply Lemma 5.4 to u, v and let $r' \to 0$. We then get that u/v equals a constant, which is the conclusion of Theorem 2. Thus to complete the proof of Theorem 2 for $A \in M_p(\alpha)$, it suffices to show the existence of a minimal positive A-harmonic function u relative to $w \in \partial\Omega$ and $0 < r_1 < \tilde{r}_0$ for which the 'fundamental inequality' in (5.1) holds in $\Omega \cap B(w, r_1)$. Moreover, it suffices to show that (5.1) holds for some $r_1 = r_1(p, n, \alpha), 0 < r_1 < \tilde{r}_0, \bar{\lambda} = \bar{\lambda}(p, n, \alpha) \geq 1$, in $\tilde{\Omega}(w, \tilde{\eta}) \cap B(w, r_1)$ where $\tilde{\eta} = \tilde{\eta}(p, n, \alpha)$ is as in (5.2). To this end we show there exists $c = c(p, n, \alpha) \geq 1$ such that if $c^2 r' < r < \tilde{r}_0/n$, and $\rho = r/c$, then (5.1) holds for $\hat{u}$ on $\tilde{\Omega}(w, \tilde{\eta}) \cap \partial B(w, \rho)$. Here $\hat{u} > 0$ is A-harmonic in $\Omega \setminus \bar{B}(w, r')$ with continuous boundary values and $\hat{u} \equiv 0$ on $\partial\Omega \setminus \bar{B}(w, r')$. It then follows from arbitrariness of r, r', the above discussion, and Lemma 5.4 that Theorem 2 is valid whenever $A \in M_p(\alpha)$ and u is a minimal positive A-harmonic function relative to $w \in \partial\Omega$. With this game plan in mind, observe from Lemma 2.15 and (1.7), that we may assume $r = 1, w = 0$, and

(5.16) $B(0, 4n) \cap \{y : y_n \geq \mu\} \subset \Omega,\ B(0, 4n) \cap \{y : y_n \leq -\mu\} \subset \mathbf{R}^n \setminus \Omega,$

where $\mu = 500n\delta^*, 0 < \mu < 10^{-100}$ and $r' < (\delta^*)^2$. Here δ^* is temporarily allowed to vary but will be fixed after the proof of Lemma 5.19. Extend $\hat{u}$ to be continuous on $\mathbf{R}^n \setminus B(0, r')$, by putting $\hat{u} \equiv 0$ on $\mathbf{R}^n \setminus (\Omega \cup B(0, r'))$. Using the notation in (2.7), let $Q = Q^+_{1,1-\mu}(\mu e_n) \setminus \bar{B}(0, \sqrt{\mu})$ and let v_1 be the A-harmonic function in Q with the following continuous boundary values,

$$
\begin{aligned}
v_1(y) &= \hat{u}(y), \qquad\qquad y \in \partial Q \cap \{y : 2\mu \leq y_n\}, \\
v_1(y) &= \frac{(y_n - \mu)}{\mu}\hat{u}(y),\ y \in \partial Q \cap \{y : \mu \leq y_n < 2\mu\}.
\end{aligned}
$$

Comparing boundary values and using the maximum principle for A-harmonic functions, it follows that

$$v_1 \leq \hat{u} \text{ in } Q\,. \tag{5.17}$$

We now set $\mu = \mu(\epsilon) = \exp(-1/\epsilon)$. We shall prove,

Lemma 5.18. *Let $0 < \epsilon \leq \hat{\epsilon}$, $\mu = \mu(\epsilon)$ be as above and let $\tilde{\eta}$ be as in (5.2). If $\hat{\epsilon}$ is small enough, then there exists $\hat{\theta} = \hat{\theta}(p, n, \alpha), 0 < \hat{\theta} \leq 1/2$, such that if $\hat{\rho} = \mu^{1/2 - \hat{\theta}}$, then*

$$1 \,\leq\, \hat{u}(y)/v_1(y) \,\leq\, 1 + \epsilon$$

whenever $y \in \tilde{\Omega}(0, \tilde{\eta}/4) \cap [B(0, \hat{\rho}) \setminus B(0, 2\sqrt{\mu})]$.

Lemma 5.19. *Let $v_1, \epsilon, \hat\epsilon, \hat\theta, \mu$ be as in Lemma 5.18 and let $\tilde\eta$ be as in (5.2). If $\hat\epsilon$ is small enough, there exist $\theta = \theta(p,n,\alpha), 0 < \theta < \hat\theta/4, \tilde\lambda = \tilde\lambda(p,n,\alpha) > 1$, such that if $\rho = \mu^{1/2-2\theta}, a = \mu^{-\theta}$, then*

$$\tilde\lambda^{-1}\frac{v_1(y)}{d(y,\partial\Omega)} \leq |\nabla v_1(y)| \leq \tilde\lambda\frac{v_1(y)}{d(y,\partial\Omega)}$$

whenever $y \in \tilde\Omega(0,\tilde\eta/2) \cap [B(0,a\rho) \setminus B(0,\rho/a)]$.

Assuming Lemmas 5.18, 5.19, are true we complete the proof of Theorem 2 when $A \in M_p(\alpha)$ as follows. From these lemmas and Lemma 3.18 we deduce, for sufficiently small $\hat\epsilon = \hat\epsilon(p,n,\alpha) > 0$, that (5.1) is valid for $\hat u$ and for some $\bar\lambda = \bar\lambda(p,n,\alpha) \geq 1$ in $\Omega(w,\tilde\eta) \cap \partial B(0,\rho)$. With $\hat\epsilon$ now fixed we put $\delta^* = \mu(\hat\epsilon)/(500n)$ and conclude from (5.2), Lemma 2.15, arbitrariness of r, that (5.1) holds in $\Omega \cap [B(w,r_1) \setminus B(w,r')]$ with $r_1 = \tilde r_0/c, r' \leq r_0/c'$, provided c, c' are large enough, depending only on p,n,α. Thus we can apply Lemma 5.4 and proceed as in the discussion after that lemma to get Theorem 2 under the assumption $A \in M_p(\alpha)$.

Proof of Lemma 5.18. To begin the proof of Lemma 5.18 observe from (5.17) that it suffices to prove the righthand inequality in this display. We note that if $y \in \partial Q$ and $\hat u(y) \neq v_1(y)$, then y lies within 4μ of a point in ∂Q. Also $\max_{\partial B(0,t)} \hat u$ is non-increasing as a function of $t \geq r'$, as we see from the maximum principle for A-harmonic functions. Using these facts and Lemmas 2.1- 2.3 we find that

$$\hat u \leq v_1 + c\mu^{\tilde\sigma/2}\,\hat u(\sqrt{\mu}e_n), \tag{5.20}$$

on ∂Q. By the maximum principle this inequality also holds in Q. Here $\tilde\sigma$ is the exponent of Hölder continuity in Lemma 2.2. Using Harnack's inequality, we also find that there exist $\tau = \tau(p,n,\alpha) \geq 1$ and $c = c(p,n,\alpha) > 1$ such that

$$\max\{\psi(z),\psi(y)\} \leq c(d(z,\partial Q)/d(y,\partial Q))^{\tau}\min\{\psi(z),\psi(y)\} \tag{5.21}$$

whenever $z \in Q$, $y \in Q \cap B(z, 4d(z,\partial Q))$ and $\psi = \hat u$ or v_1. Also from Lemmas 2.1-2.3 applied to v_1, we get

$$v_1(2\sqrt{\mu}e_n) \geq c^{-1}\,\hat u(\sqrt{\mu}e_n). \tag{5.22}$$

Let $\hat\rho, \hat\theta$ be as in Lemma 5.18. Using (5.20) - (5.22), we see that if $y \in \tilde\Omega(0,\tilde\eta/4) \cap [B(0,\hat\rho) \setminus B(0,2\sqrt{\mu})]$, then

$$\hat u(y) \leq v_1(y) + c\mu^{\tilde\sigma/2}\,\hat u(\sqrt{\mu}e_n) \leq \left(1 + c^2\mu^{\tilde\sigma/2-\hat\theta\tau}\right) v_1(y) \leq (1+\epsilon)v_1(y) \tag{5.23}$$

provided $\hat\epsilon$ is small enough and $\hat\theta\tau = \tilde\sigma/4$. The proof of Lemma 5.18 is complete. □

Proof of Lemma 5.19. To prove Lemma 5.19 we let $v_1, \epsilon, \hat\epsilon, \hat\theta, \mu$ be as in Lemma 5.18. Using Lemmas 2.2 - 2.3 and Harnack's inequality we see that there exists $\phi = \phi(p,n,\alpha) > 0, 0 < \phi \leq 1/2$, and $c = c(p,n,\alpha) > 1$ with

$$\hat u(y) \leq c(s/t)^{\phi}\hat u(se_n) \tag{5.24}$$

provided $y \in \mathbf{R}^n \setminus B(0,t), t \geq s \geq 2r'$. Using (5.24) with $t = 1, s = 2\sqrt{\mu}$, and Lemmas 2.1 - 2.3 we see that

$$v_1 \leq c\mu^{\phi/2}\hat u(\sqrt{\mu}e_n) \text{ on } \partial Q \setminus \bar B(0,\sqrt{\mu}), \tag{5.25}$$

where c depends only on p, n, α. Let $\tilde v$ be the A-harmonic function in Q with continuous boundary values $\tilde v = 0$ on $\partial Q \setminus \bar B(0, \sqrt{\mu})$, and $\tilde v = v_1$ on $\partial B(0, \sqrt{\mu})$. Then from (5.25) and the maximum principle, we see that

$$\tilde v \leq v_1 \leq \tilde v + c\mu^{\phi/2}\hat u(\sqrt{\mu} e_n) \text{ in } Q. \tag{5.26}$$

Let $\rho = \mu^{1/2-2\theta}, \theta$ small, and $a = \mu^{-\theta}$ be as in Lemma 5.19. Using (5.21) applied to $\psi = \tilde v$ we find

$$\tilde v \geq c^{-1}(\mu^{1/2}/a\rho)^{\tau}\hat u(\sqrt{\mu} e_n) = c^{-1}\mu^{3\theta\tau}\hat u(\sqrt{\mu} e_n) \tag{5.27}$$

on $\tilde\Omega(0, \tilde\eta/8) \cap [B(0, 2a\rho) \setminus B(0, \rho/(2a))]$, where τ is as in (5.21) and the nontangential approach region $\tilde\Omega$ was defined above (5.2) relative to $w, \tilde\eta$. Also, since $\tilde\eta$ depends only on p, n, α, it follows that $c = c(p, n, \alpha)$ in (5.27). If we define θ by $\theta = \min\{\phi/(12\tau), \hat\theta/4\}$, then from (5.26), (5.27) we get

$$1 \leq \frac{v_1}{\tilde v} \leq 1 + \epsilon \tag{5.28}$$

in $\tilde\Omega(0, \tilde\eta/8) \cap B(0, 2a\rho) \setminus B(0, \rho/(2a))$, whenever $0 \leq \epsilon \leq \hat\epsilon$, provided $\hat\epsilon$ is sufficiently small.

Next let v be the A-harmonic function in

$$Q' = Q^{+}_{1,1-\mu}(\mu e_n) \setminus \bar B(2\sqrt{\mu} e_n, \sqrt{\mu})$$

with continuous boundary values $v = 0$ on $\partial Q' \setminus \bar B(2\sqrt{\mu} e_n, \sqrt{\mu})$ and $v = 1$ on $\partial B(2\sqrt{\mu} e_n, \sqrt{\mu})$. We claim that

$$v(y) \leq c\langle 2\sqrt{\mu} e_n - y, \nabla v(y)\rangle \tag{5.29}$$

when $y \in Q'$. Assuming claim (5.29) we can complete the proof of Lemma 5.19 in the following manner. First observe that (5.29) implies there exists $c = c(p, n, \eta) \geq 1$, for given $\eta, 0 < \eta \leq 1/2$, with

$$c^{-1}\frac{v(y)}{d(y, \partial Q')} \leq |\nabla v(y)| \leq c\frac{v(y)}{d(y, \partial Q')} \tag{5.30}$$

in $\tilde Q(0, \eta) \setminus \bar B(0, 10\sqrt{\mu})$, where $\tilde Q(0, \eta)$ is the non-tangential approach region defined relative to $0, \eta, Q$, as above (5.2). From the observation in (5.2) with $\hat u, \Omega$, replaced by v, Q and (5.30) for suitable $\eta = \eta(p, n, \alpha)$ we deduce that (5.30) in fact holds in $Q \setminus \bar B(0, 10\sqrt{\mu})$. We can now use Lemma 5.4 in $Q \setminus \bar B(0, 10\sqrt{\mu})$ with $v, \tilde v$ playing the role of $\hat u, \hat v$, respectively. In particular, we get for some large $c = c(p, n, \alpha)$ that

$$c^{-1}\frac{\tilde v(y)}{d(y, \partial\Omega)} \leq |\nabla \tilde v(y)| \leq c\frac{\tilde v(y)}{d(y, \partial\Omega)} \tag{5.31}$$

in $Q \cap B(0, 1/c^*) \setminus B(0, c^*\sqrt{\mu})$ for some $c^* = c^*(p, n, \alpha)$. Finally, note that if $0 \leq \epsilon \leq \hat\epsilon$ and if $\hat\epsilon$ is sufficiently small, then $1/c^* > 2a\rho > \rho/(2a) > c^*\sqrt{\mu}$. From this fact, (5.31), (5.28), and Lemma 3.18 applied to $\tilde v, v_1$, we deduce that Lemma 5.19 is valid subject to claim (5.29).

To prove claim (5.29) we first observe from Lemmas 2.2, 2.3 that $v(z) \leq 1/2$ in $Q' \cap B(0, 10\sqrt{\mu})$ for some z whose distance from ∂Q is at least $c^{-1}\sqrt{\mu}$ where $c = c(p, n, \alpha)$. Using Harnack's inequality it follows for some $c' > 1$ that $v \leq 1 - 1/c'$ on $\partial B(2\sqrt{\mu} e_n, 3\sqrt{\mu}/2)$. If $y \in \bar B(2\sqrt{\mu} e_n, 3\sqrt{\mu}/2) \setminus B(2\sqrt{\mu} e_n, \sqrt{\mu})$, set

$$\zeta(y) = \frac{e^{N|y-z|^2/\mu} - e^N}{e^{9N/4} - e^N} \tag{5.32}$$

where $z = 2\sqrt{\mu}e_n$. Then $\zeta \equiv 0$ on $\partial B(2\sqrt{\mu}e_n, \sqrt{\mu})$, and $\zeta \equiv 1$ on $\partial B(2\sqrt{\mu}e_n, 3\sqrt{\mu}/2)$. Also, if $N = N(p, n, \alpha)$ is large enough in (5.32), then from direct calculation and Definition 1.1, we find $\nabla \cdot A(\nabla\zeta) \geq 0$ in $B(2\sqrt{\mu}e_n, 3\sqrt{\mu}/2) \setminus \bar{B}(2\sqrt{\mu}e_n, \sqrt{\mu})$. Moreover, using these facts and the maximum principle we deduce

$$1 - v(y) \geq (c_+\sqrt{\mu})^{-1} d(y, \partial B(2\sqrt{\mu}e_n, \sqrt{\mu})) \tag{5.33}$$

in $B(2\sqrt{\mu}e_n, 3\sqrt{\mu}/2) \setminus B(2\sqrt{\mu}e_n, \sqrt{\mu})$ provided $c_+ = c_+(p, n, \alpha)$ is large enough. Next for fixed $t > 1$ put

$$\begin{aligned} O &= \{y \in Q' : 2\sqrt{\mu}e_n + t(y - 2\sqrt{\mu}e_n) \in Q'\}, \\ F(y) &= F(y,t) = \frac{v(y) - v(2\sqrt{\mu}e_n + t(y - 2\sqrt{\mu}e_n))}{t-1} \text{ whenever } y \in O. \end{aligned}$$

From (5.33) for $t > 1$ fixed, t near 1, and basic geometry it follows that

$$F \geq c^{-1}\, v \text{ on } \partial O. \tag{5.34}$$

We note that (iv) of Definition 1.1 and $A \in M_p(\alpha)$ imply that an A-harmonic function remains A-harmonic under scaling, translation, and multiplication by a constant. From this fact we see that F is the difference of two A-harmonic functions in O and one of them is a constant multiple of v. Using this fact, (5.34), and the maximum principle for A-harmonic functions, it follows that $F \geq c^{-1}v$ in O. Letting $t \rightarrow 1$, using Lemma 2.4 and the chain rule, we get claim (5.29). The proof of Lemma 5.19 is finished. $\Box$

As mentioned earlier, Lemmas 5.18, 5.19 together with Lemma 5.4 imply Theorem 2 when $A \in M_p(\alpha)$.

5.2. Proof of Theorem 2. We are now ready to prove Theorem 2 in the general case.

Lemma 5.35. *Let Ω be a bounded (δ, r_0)-Reifenberg flat domain and let $w \in \partial\Omega$. Let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) and $1 < p < \infty$. Let $\hat{u}, \hat{v} > 0$ be A-harmonic in $\Omega \setminus B(w, r')$, continuous in $\mathbf{R}^n \setminus B(w, r')$, with $\hat{u} \equiv \hat{v} \equiv 0$ on $\mathbf{R}^n \setminus [\Omega \cup B(w, r')]$. Then there exists $\delta_*, \sigma > 0, c_+ \geq 1$, depending on $p, n, \alpha, \beta, \gamma$, such that if $0 < \delta < \delta_* < \tilde{\delta}$ ($\tilde{\delta}$ as in Theorem 1) and $r_1 = \tilde{r}_0/c_+$, then*

$$\left| \log\left(\frac{\hat{u}(z)}{\hat{v}(z)}\right) - \log\left(\frac{\hat{u}(y)}{\hat{v}(y)}\right) \right| \leq c_+ \left(\frac{r'}{\min(r_1, |z-w|, |y-w|)}\right)^{\sigma}$$

whenever $z, y \in \Omega \setminus B(w, c_+r')$.

Proof: Once again we assume that $r'/r_1 << 1$, since otherwise there is nothing to prove. As in (5.5) we may assume for some $c = c(p, n, \alpha, \beta, \gamma)$ that

$$\hat{u} \leq \hat{v}/2 \leq c\hat{u} \text{ in } \Omega \setminus B(w, 2r'). \tag{5.36}$$

Let $u(\cdot, t), t \in [0, 1]$, be A-harmonic in $\Omega \setminus \bar{B}(w, 2r')$, with continuous boundary values,

$$u(\cdot, t) = (1-t)\hat{u}(\cdot) + t\hat{v}(\cdot) \text{ on } \partial[\Omega \setminus \bar{B}(w, 2r')]. \tag{5.37}$$

We claim there exists $c, \bar\lambda \geq 1$ depending only on $p, n, \alpha, \beta, \gamma$ such that if $t \in [0,1]$, and $y \in \Omega \cap [B(w, \tilde r_0/c) \setminus \bar B(w, cr')]$, then

$$\bar\lambda^{-1} \frac{u(y,t)}{d(y,\partial\Omega)} \leq |\nabla u(y,t)| \leq \bar\lambda \frac{u(y,t)}{d(y,\partial\Omega)}. \tag{5.38}$$

Indeed let $A_1(y,\eta) = A(w,\eta)$ whenever $y \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^n \setminus \{0\}$. Let $1 < a < b$ and suppose that ρ is such that $2r' \leq \rho/a < b\rho \leq \tilde r_0/2$. Let $v(\cdot,t)$, for $t \in [0,1]$, be A_1-harmonic in $\Omega \setminus \bar B(w, \rho/a)$ with continuous boundary values equal to $u(\cdot,t)$. Then, from Lemmas 5.4, 5.18, 5.19 we see that if $a = a(p,n,\alpha)$ is large enough, then

$$|\nabla v(\cdot,t)| \approx v(\cdot,t)/d(\cdot,\partial\Omega) \tag{5.39}$$

on $\Omega \cap \partial B(w,\rho)$. Here $\approx$ means with constants depending only on p, n, α. Let $h(\cdot,t)$ be the A_1-harmonic function in $\Omega_1 = \Omega \cap [B(w, b\rho) \setminus \bar B(w, \rho/a)]$ with continuous boundary values equal to $u(\cdot,t)$. We claim that if $b = b(p,n,\alpha,\beta,\gamma)$ is large enough then (5.39) is also valid for h. In fact (5.24) holds with u replaced by $u(\cdot,t)$ for $t \in [0,1]$, where now $\phi = \phi(p,n,\alpha,\beta,\gamma)$ and $s \geq 2r'$. Using (5.24) for $u(\cdot,t)$ we get

$$v(\cdot,t) \leq h(\cdot,t) \leq v(\cdot,t) + c(ab)^{-\phi} u(\rho e_n/a, t) \text{ on } \partial\Omega_1.$$

From the maximum principle this inequality also holds in Ω_1. Moreover, for τ as in (5.21) we deduce,

$$v(\cdot,t) \geq c^{-1} a^{-\tau} u(\rho e_n/a, t)$$

on $\tilde\Omega(w, \tilde\eta/2) \cap (B(w, 2\rho) \setminus \bar B(w, \rho/2))$. Thus, for some $c' = c'(p,n,\alpha,\beta,\gamma) \geq 1$,

$$v(\cdot,t) \leq h(\cdot,t) \leq (1 + c' a^{\tau-\phi} b^{-\phi}) v(\cdot,t) \tag{5.40}$$

on $\tilde\Omega(w, \tilde\eta/2) \cap (B(w, 2\rho) \setminus \bar B(w, \rho/2))$. Choosing $b = b(p,n,\alpha,\beta,\gamma)$ large enough in (5.40), using (5.39), Lemma 3.18, it follows that

$$\lambda_+^{-1} h(y,t)/d(y,\partial\Omega) \leq |\nabla h(y,t)| \leq \lambda_+ h(y,t)/d(y,\partial\Omega) \tag{5.41}$$

whenever $y \in \tilde\Omega(w,\tilde\eta) \cap \partial B(w,\rho)$ for some $\lambda_+ = \lambda_+(p,n,\alpha,\beta,\gamma) \geq 1$.

From (5.2) we see that (5.41) holds on $\Omega \cap \partial B(w,\rho)$ provided $\lambda_+(p,n,\alpha,\beta,\gamma)$ is large enough. With a, b, now fixed, depending only on $p, n, \alpha, \beta, \gamma$, we can use Lemma 2.15 and argue as in Lemma 3.1 to conclude for given $\epsilon > 0$, the existence of $r_1 = r_1(p,n,\alpha,\beta,\gamma,\epsilon)$ so small that if $b\rho \leq r_1 < \tilde r_0$, then

$$1 - \epsilon \leq u(\cdot,t)/h(\cdot,t) \leq 1 + \epsilon$$

on $\tilde\Omega(w, \tilde\eta/2) \cap (B(w, 2\rho) \setminus \bar B(w, \rho/2))$. In view of this inequality, (5.41), and Lemma 3.18, we see that if $\epsilon = \epsilon(p,n,\alpha,\beta,\gamma)$ is small enough, then

$$|\nabla u(\cdot,t)| \approx u(\cdot,t)/d(\cdot,\partial\Omega) \tag{5.42}$$

on $\tilde\Omega(w,\tilde\eta) \cap \partial B(w,\rho)$, where proportionality constants depend only on $p, n, \alpha, \beta, \gamma$. In view of (5.2), this inequality holds on $\Omega \cap \partial B(w,\rho)$. With r_1, a, b fixed we see from arbitrariness of ρ that (5.38) is true. We can now argue as in Lemma 5.4 or just repeat the argument in (1.18) - (1.25) to conclude Lemma 5.35. $\Box$

As pointed out earlier in this section, if u, v are minimal A-harmonic functions relative to $w \in \partial\Omega$, then we can apply Lemma 5.35 and let $r' \to 0$ to get Theorem 2. The proof of Theorem 2 is now complete. $\Box$

6. Appendix : an alternative approach to deformations

In this section we show that Step C in Theorem 1 can be replaced by a somewhat different argument based on ideas in [W]. The first author would like to thank Mikhail Feldman for making him aware of the ideas in [W]. In the following all constants will depend only on $p, n, \alpha, \beta, \gamma$ and we suppose that u, v are A-harmonic in $\Omega \cap B(w, 4r)$ and continuous in $B(w, 4r)$ with $u = v = 0$ on $B(w, 4r) \setminus \Omega$. From Lemma 3.35 we see that if δ is small enough, $\hat r = r/c$, and c is large enough, then for some $\mu \geq 1$,

$$\mu^{-1} \frac{h(y)}{d(y, \partial\Omega)} \leq |\nabla h(y)| \leq \mu \frac{h(y)}{d(y, \partial\Omega)} \tag{6.1}$$

whenever $y \in \Omega \cap B(w, 4\hat r)$, $h \in \{u, v\}$. Also from Lemma 4.8 we see that there exists $\mu_* \geq 1$, for $\epsilon^* > 0$ fixed, such that

$$\mu_*^{-1} \left(\frac{s}{\hat r}\right)^{1+\epsilon^*} \leq \frac{h(a_s(w))}{h(a_{\hat r}(w))} \leq \mu_* \left(\frac{s}{\hat r}\right)^{1-\epsilon^*} \tag{6.2}$$

whenever $y \in \Omega \cap B(w, \hat r)$, $h \in \{u, v\}$, where $0 < s \leq 4\hat r$. Observe again, for $x, \lambda \in \mathbf{R}^n, \xi \in \mathbf{R}^n \setminus \{0\}$, that

$$\begin{aligned} A_i(x, \lambda) - A_i(x, \xi) &= \int_0^1 \frac{d}{dt} A_i(x, t\lambda + (1-t)\xi) dt \\ &= \sum_{j=1}^n (\lambda_j - \xi_j) \int_0^1 \frac{\partial A_i}{\partial \eta_j}(x, t\lambda + (1-t)\xi) dt \end{aligned} \tag{6.3}$$

for $i \in \{1, ..., n\}$. In view of (6.3), (6.1), and A-harmonicity of u, v, we deduce that $u - v$ is a weak solution to $\bar L \zeta = 0$ in $\Omega \cap B(w, \hat r)$, where

$$\begin{aligned} \bar L \zeta(x) &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x) \zeta_{x_j}) \\ \text{and } a_{ij}(x) &= \int_0^1 \frac{\partial A_i}{\partial \eta_j}(t \nabla u(x) + (1-t) \nabla v(x)) dt, \end{aligned} \tag{6.4}$$

for $1 \leq i, j \leq n$. Moreover, from the structure assumptions on A, see Definition 1.1, we find that

$$\begin{aligned} c_+^{-1} (|\nabla u(x)| + |\nabla v(x)|)^{p-2} |\xi|^2 &\leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \\ &\leq c_+ (|\nabla u(x)| + |\nabla v(x)|)^{p-2} |\xi|^2 \end{aligned} \tag{6.5}$$

whenever $x \in \Omega \cap B(w, \hat r)$. Next we prove the following lemma.

Lemma 6.6. *There exists $c \geq 1, \delta_0 > 0$, such that if $r^* = \hat r / c$, and $0 < \delta < \delta_0$, then $(|\nabla u| + |\nabla v|)^{p-2}$ extends to an A_2-weight in $B(w, r^*)$ with A_2-constant depending only on $p, n, \alpha, \beta, \gamma$.*

Proof: The proof is essentially the same as the proof of Lemma 4.9. That is, we use a Whitney cube decomposition of $\mathbf{R}^n \setminus \Omega$ to extend $(|\nabla u| + |\nabla v|)^{p-2}$ to a function λ on $B(w, 4r^*)$. Let $\tilde w \in B(w, r^*)$ and $0 < \tilde r < r^*$. Let $\hat w \in \partial\Omega$ with $|\hat w - \tilde w| = d(\tilde w, \partial\Omega)$ and suppose that $|\hat w - \tilde w|/2 < \tilde r < r^*$. We assume, as we may, that

$$\max\{u(a_{\tilde r}(\hat w)), v(a_{\tilde r}(\hat w))\} = u(a_{\tilde r}(\hat w)). \tag{6.7}$$

Let $\hat\lambda = \lambda$ when $p \geq 2$ and $\hat\lambda = 1/\lambda$ for $1 < p < 2$. As in (4.25) - (4.28), it follows, for $\epsilon^* > 0$, small enough, that

$$\int_{B(\tilde w, \tilde r)} \hat\lambda dx \ \leq\ c u(a_{\tilde r}(\hat w))^{|p-2|}\, \tilde r^{n-|p-2|} \tag{6.8}$$

and

$$\begin{aligned} \int_{B(\tilde w,\tilde r)} \hat\lambda^{-1} dx \quad &\leq \quad c\tilde r^{(1+\epsilon^*)(|p-2|)} u(a_{\tilde r}(\hat w))^{-|p-2|} \int_{\Omega\cap B(\tilde w, 50\tilde r)} d(y,\partial\Omega)^{-\epsilon^*(|p-2|)} dy \\ &\leq \quad c u(a_{\tilde r}(\hat w))^{-|p-2|}\, \tilde r^{n+|p-2|}. \end{aligned} \tag{6.9}$$

These inequalities remain true if $\tilde r \leq |\tilde w - \hat w|/2$, as follows easily from (6.1). Combining (6.8), (6.9), and using arbitrariness of $\tilde w, \tilde r$, we get Lemma 6.6. $\Box$

Using the ideas in [W] we continue by proving the following.

Lemma 6.10. *Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that $\hat u$ and $\hat v$ are non-negative A-harmonic functions in $\Omega \cap B(w, 2r)$ with $\hat v \leq \hat u$. Assume also that $\hat u, \hat v$, are continuous in $B(w, 2r)$ with $\hat u \equiv 0 \equiv \hat v$ on $B(w, 2r) \setminus \Omega$. Let r^* be as in Lemma 6.6. There exists $c \geq 1$ such that if $\tilde r = r^*/c$, then*

$$c^{-1}\frac{\hat u(a_{\tilde r}(w)) - \hat v(a_{\tilde r}(w))}{\hat v(a_{\tilde r}(w))} \leq \frac{\hat u(y) - \hat v(y)}{\hat v(y)} \leq c\frac{\hat u(a_{\tilde r}(w)) - \hat v(a_{\tilde r}(w))}{\hat v(a_{\tilde r}(w))}$$

whenever $y \in \Omega \cap B(w, \tilde r)$.

Proof: We first prove the lefthand inequality in Lemma 6.10. To do so we show the existence of Λ, $1 \leq \Lambda < \infty$, and $\hat c \geq 1$, such that if $r' = r^*/\hat c$ and if

$$e(y) = \Lambda\left(\frac{\hat u(y) - \hat v(y)}{\hat u(a_{r^*}(w)) - \hat v(a_{r^*}(w))}\right) - \frac{\hat v(y)}{\hat v(a_{r^*}(w))} \tag{6.11}$$

for $y \in \Omega \cap B(w, r^*)$, then

$$e(y) \geq 0 \text{ whenever } y \in \Omega \cap B(w, 2r'). \tag{6.12}$$

To do this, we initially allow $\Lambda, \hat c \geq 1$ to vary in (6.11). $\Lambda, \hat c$, are then fixed near the end of the argument. Put

$$\begin{aligned} u'(y) &= \frac{\Lambda\, \hat u(y)}{\hat u(a_{r^*}(w)) - \hat v(a_{r^*}(w))}, \\ v'(y) &= \frac{\Lambda\, \hat v(y)}{\hat u(a_{r^*}(w)) - \hat v(a_{r^*}(w))} + \frac{\hat v(y)}{\hat v(a_{r^*}(w))}. \end{aligned}$$

Observe from (6.11) that $e = u' - v'$. Using Definition 1.1 (iv) we see that u', v' are A-harmonic functions. Let $\bar L$ be defined as in (6.4) using u', v', instead of u, v, and let e_1, e_2 be the solutions to $\bar{L}e_i = 0, i = 1, 2$, in $\Omega \cap B(w, r^*)$, with continuous boundary values:

$$e_1(y) = \frac{\hat{u}(y) - \hat{v}(y)}{\hat{u}(a_{r^*}(w)) - \hat{v}(a_{r^*}(w))}, \; e_2(y) = \frac{\hat{v}(y)}{\hat{v}(a_{r^*}(w))}, \tag{6.13}$$

whenever $y \in \partial(\Omega \cap B(w, r^*))$. From Lemma 6.6 we see that Lemma 4.7 can be applied and we get, for some $c_+ \geq 1$ and $r_+ = r^*/c_+$, that

$$c_+^{-1} \frac{e_1(a_{r_+}(w))}{e_2(a_{r_+}(w))} \leq \frac{e_1(y)}{e_2(y)} \leq c_+ \frac{e_1(a_{r_+}(w))}{e_2(a_{r_+}(w))} \tag{6.14}$$

whenever $y \in \Omega \cap B(w, 2r_+)$. We now put

$$\hat{c} = c_+, \; r' = r_+, \; \Lambda = \hat{c}\, \frac{e_2(a_{r'}(w))}{e_1(a_{r'}(w))},$$

and observe from (6.14) that

$$\Lambda e_1(y) - e_2(y) \geq 0 \text{ whenever } y \in \Omega \cap B(w, 2r'). \tag{6.15}$$

Let $\hat{e} = \Lambda\, e_1 - e_2$ and note from linearity of $\bar{L}$ that $\hat{e}, e$, both satisfy the same linear locally uniformly elliptic pde in $\Omega \cap B(w, r^*)$ and also that these functions have the same continuous boundary values on $\partial(\Omega \cap B(w, r^*))$. Hence, using the maximum principle for the operator $\bar{L}$ it follows that $e = \hat{e}$ and then by (6.15) that $e(y) \geq 0$ in $\Omega \cap B(w, 2r')$. To complete the proof of the left-hand inequality in Lemma 6.10 with $\tilde{r} = 2r'$, we observe from Lemmas 4.5, 4.6, that $\Lambda \leq c$. The proof of the right-hand inequality in Lemma 6.11 is similar. We omit the details. □

We note that in [LN5] Lemma 6.10 was proved under the assumptions that $\hat{u}$ and $\hat{v}$ are non-negative p-harmonic functions in $\Omega \cap B(w, 2r)$ and that $\Omega \subset \mathbf{R}^n$ is a Lipschitz domain. In this case the constants in Lemma 6.10 depend only on p, n and the Lipschitz constant of Ω. Moreover, in [LN5] this result is used to prove regularity of a Lipschitz free boundary in a general two-phase free boundary problem for the p-Laplace operator.

Proof of Theorem 1. Let u, v, A, Ω, w, r be as in Theorem 1 and let $\hat{u}, \hat{v}$ be the A-harmonic functions in $\Omega \cap B(w, 2r)$ with

$$\hat{u} = \max\{u, v\} \text{ and } \hat{v} = \min\{u, v\} \text{ on } \partial[\Omega \cap B(w, 2r)].$$

From the maximum principle for A-harmonic functions we have $\hat{u} \geq \hat{v}$ and hence we can apply Lemma 6.10 to conclude that

$$c^{-1} \frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))} \leq \frac{\hat{u}(y)}{\hat{v}(y)} \leq c \frac{\hat{u}(a_{\tilde{r}}(w))}{\hat{v}(a_{\tilde{r}}(w))}$$

whenever $y \in \Omega \cap B(w, \tilde{r})$. Moreover, using the definition of $\hat{u}$, $\hat{v}$, and the inequalities in the last display we can conclude that

$$\frac{u(y)}{v(y)} \leq c \frac{u(z)}{v(z)} \text{ whenever } y, z \in \Omega \cap B(w, \tilde{r}). \tag{6.16}$$

Next if $x \in \partial\Omega \cap B(w, \tilde{r}/8)$, then we let

$$M(\rho) = \sup_{B(x,\rho)} \frac{u}{v} \text{ and } m(\rho) = \inf_{B(x,\rho)} \frac{u}{v}$$

when $0 < \rho < \tilde{r}$. If ρ is fixed we can apply Lemma 6.10 with $\hat{u} = u, \hat{v} = m(\rho)v$, and $2r$ replaced by ρ to conclude the existence of c^*, c_*, such that if $\tilde{\rho} = \rho/c^*$, then

$$M(\tilde{\rho}) - m(\rho) \leq c_*(m(\tilde{\rho}) - m(\rho)). \tag{6.17}$$

Likewise, we can apply Lemma 6.10 with $\hat{u} = M(\rho)\, v$ and $\hat{v} = u$ to conclude

$$(M(\rho)\, v - u)/u \approx \text{ constant on } \Omega \cap B(w, \tilde{\rho}).$$

Using this inequality together with (6.16) it follows that

$$(M(\rho)v - u)/v \approx \text{ constant on } \Omega \cap B(w, \tilde{\rho}).$$

Here we have used heavily the fact that A-harmonic functions remain A-harmonic after multiplication by a constant as follows from Definition 1.1 (iv). Thus if c_* is large enough, then

$$M(\rho) - m(\tilde{\rho}) \leq c_*(M(\rho) - M(\tilde{\rho})). \tag{6.18}$$

If osc $(t) = M(t) - m(t)$, then we can add (6.17), (6.18) and we get, after some arithmetic, that

$$\text{osc } (\tilde{\rho}) \leq \frac{c_* - 1}{c_* + 1} \text{ osc } (\rho). \tag{6.19}$$

We can now use (6.19), since c^* is independent of $\rho.$ in an iterative argument. Doing this we can conclude that

$$\text{osc } (s) \leq c(s/t)^\theta \text{ osc } (t) \text{ whenever } 0 < s < t \leq r/2 \tag{6.20}$$

for some $\theta > 0, c \geq 1$. (6.20), (6.16), along with arbitrariness of $x \in \partial\Omega \cap B(w, \tilde{r}/8)$ and interior Hölder continuity - Harnack inequalities for u, v, are easily seen to imply Theorem 1. □

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Current address: Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA

E-mail address: john@ms.uky.edu

Current address: Department of Mathematics, Umeå University, S-90187 Umeå, Sweden

E-mail address: email: niklas.lundstrom@math.umu.se

Current address: Department of Mathematics, Umeå University, S-90187 Umeå, Sweden

E-mail address: kaj.nystrom@math.umu.se