On Very Weak Solutions of Certain Elliptic Systems

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0 Introduction

In many elliptic problems, a *weak solution* u is required to satisfy a certain integral identity - obtained from integrating the pde by parts with respect to a class of smooth test functions - and also to lie in a certain Sobolev space. The Sobolev space is chosen so that powers of u times a smooth cutoff function can be used as a test function, to derive properties of u ; higher integrability, uniqueness, Hölder continuity, etc. On the other hand the integral identity often makes sense under weaker assumptions on the Sobolev space. Thus the question arises as to whether the Sobolev space assumption is in fact necessary to guarantee properties of u. This question was partially answered by Serrin[S] who constructed the following examples. Given $\epsilon \in (0,1)$ let

$$u(x) \,=\, x_1\,|x|^{1-n-\epsilon},\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

$$a_{ij}(x) \,=\, \delta_{ij} \,+\, (a-1)\tfrac{x_i x_j}{|x|^2},\ 1 \leq\, i, j\, \leq n,$$

where $\{\delta_{ij}\}$ is the Kronecker delta, $a \,=\, \frac{n-1}{\epsilon(\epsilon+n-2)}$, and $|x|$ denotes the norm of x. We note that

$$\begin{aligned} &(a) \qquad \lambda\,|\xi|^2 \leq \sum_{i,j} a_{ij}\,\xi_i\xi_j \,\leq\, \Lambda\,|\xi|^2, \\ &(b) \qquad u \in W_{1,r}(\mathbb{R}^n) \quad \text{locally for} \quad r < \tfrac{n}{n+\epsilon-1}, \\ &(c) \qquad \sum_{i,j} \int a_{ij}\,\tfrac{\partial u}{\partial x_j}\,\tfrac{\partial \phi}{\partial x_j}\,dx \,=\, 0,\ \phi \in C_0^\infty(\mathbb{R}^n), \end{aligned} \tag{0.1}$$

where λ, Λ, depend only on ϵ. Clearly, $u(x) \to \infty$ as $x \to 0$. We remark that if $(0.1)(a),(c)$ hold for some measurable (a_{ij}) defined on an open set O with $\phi \in C_0^\infty(O)$ and $u \in W_{1,2}(O)$ locally, then classical pde theory shows that u is Hölder continuous.

After Serrin's example, related results were obtained by [EM], [HR], [GM], and [GS] on problems of the above type. We note that Elcrat and Meyers[EM] were the first to show for a broad range of elliptic problems that a classical weak solution actually lies in a higher Sobolev space. Using this result and duality they obtained for $O, (a_{ij}), u, \phi$, satisfying $(0.1)(a),(c)$ as above, that there exists $\delta > 0$ depending only on λ, Λ, n, with the property that if $u \in W_{1,2-\delta}(O)$ locally, then $u \in W_{1,2+\delta}$ locally, so in fact u is a classical weak solution to the pde. However their method was unable to handle nonlinear pde's. Recently, Iwaniec and Sbordone[IS] have

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shown that the usual Sobolev assumption for weak solutions of p Laplacian type ($u \in W_{1.p}$ locally) can be relaxed to $u \in W_{1,p-\delta}$ for some $\delta > 0$, where δ depends only on p, n, and the structure constants, in order to conclude that $u \in W_{1,p+\delta}$, locally. Hence in this case it still turns out that u is a classical weak solution. Essentially they use the Hodge decomposition to construct a suitable test function.

In this paper we introduce another method for constructing suitable test functions, based on the Whitney extension theorem, Gehring's reverse Hölder inequality, and the theory of A_p weights. We show that our method can be used to relax the Sobolev assumptions for the full range of elliptic problems considered by Elcrat-Meyers and still one can conclude that a solution to the integral identity associated with the pde, is in fact a classical weak solution. Moreover, our method appears to us to have more flexibility and to be more natural from a pde standpoint than either of the above methods.

1 Notation and Results

As above $\mathbb{R}^n$ denotes Euclidean n space with inner product : $\langle x, y \rangle = \sum_{i=1}^n x_i\, y_i,\ x, y \in \mathbb{R}^n$ and dx denotes Lebesgue measure on $\mathbb{R}^n$. Let $B(x, r) = \{y : |y - x| < r\}$, when $x \in \mathbb{R}^n$ and $r > 0$. Let $\bar{E}$, ∂E, $|E|$, denote the closure, boundary, and outer Lebesgue n measure of the set E. For a fixed positive integer N and E a Lebesgue n measurable set, let $L^p(E)$, $1 \leq p \leq \infty$, denote the usual space of Lebesgue measurable functions from E into $\mathbb{R}^N$ which are p th power integrable, with norm denoted by $\| \cdot \|_p$. If O is a bounded open set, m a positive integer, and $1 \leq p \leq \infty$, then $W_{m,p}(O)$ will denote the Banach space of functions in $L^p(O)$ whose distributional derivatives up to order m are also in $L^p(O)$ with norm

$$\|u\|_{m,p} \;=\; \|u\|_{m,p,O} \;=\; \sum_{|\alpha| \leq m} \|\partial^\alpha u / \partial x^\alpha\|$$

In this display $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, $|\alpha| = \sum \alpha_i$ is the length of α, and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. We say that $u \in W^{\rm loc}_{m,p}(O)$ if $u \in W_{m,p}(\Omega)$ whenever Ω is an open set with $\bar{\Omega} \subset O$. Let $C_0^\infty(O)$ be the infinitely differentiable functions with compact support in O and put $\dot{W}_{m,p}(O)$ equal to the closure of $C_0^\infty(O)$ in the norm of $W_{m,p}(O)$. Next let S denote the set of all n tuples of multi-indexes of length less than or equal to m and let card S denote the cardinality of S. Similarly S_k, $1 \leq k \leq m$, will denote the set of all n tuples of multi-indexes of length k with cardinality, card S_k. Let $P = \Pi_{\sigma \in S} \mathbb{R}^N$ denote the cartesian product of card S copies of $\mathbb{R}^N$ indexed by $\sigma \in S$. A similar interpretation holds for $P_k = \Pi_{\sigma \in S_k} \mathbb{R}^N$. If $u \in W_{m,p}(O)$, we let $D^m u : O {\rightarrow} P$ be the derivatives of u of order less than or equal to m considered as a vector function with $N \cdot$ card S components. Also, $\partial^k u\ : O {\rightarrow} P_k$ will stand for the vector function of k th derivatives of u with $N \cdot$ card S_k components.

Following Elcrat and Meyers[EM] , we consider nonlinear elliptic systems. For this purpose suppose that O, m, N are as above, $1 < p < \infty$, $0 < \epsilon < 1$, and $A = (A_\sigma)_{\sigma \in S}$, where A_σ : $O \times P {\rightarrow} \mathbb{R}^N$, for each $\sigma \in S$. Let $u = (u_1, ..., u_N) \in W^{\rm loc}_{m,r}(O)$ and assume that each component of $A(x, D^m u(x))$, $x \in O$, is Lebesgue n measurable with the following properties:

$$\gamma \sum_{\sigma \in S_m} \langle\, A_\sigma(x, D^m u(x)),\ \ \partial^\sigma u\ (x)\,\rangle \;\geq |\ \partial^m u\ (x)|^p - a(x), \tag{1.1}$$

a.e in O, where $a > 0 \in L^{1+\epsilon}(O)$, and for $\sigma \in S_i$, $0 \leq i \leq m$,

$$|A_\sigma(x, D^m u(x))| \leq a_i(x) \mid \partial^m u\ (x)|^{p-1} + b_i(x),\ a.e\,, \tag{1.2}$$

where $a_m = 1$, $a_i \in L^{p_i}(O)$ for $0 \leq i \leq m-1$, and $b_i \in L^{q_i}(O)$ for $1 \leq i \leq m$. Here $p_i = \max\{p, \frac{n}{(m-i)}\} + \epsilon$ for $0 \leq i \leq m-1$ while if $p' = \frac{p}{p-1}$ is the exponent conjugate to p, then $q_i = \max\{1, \frac{np'}{(n+p'(m-i))}\} + \epsilon$ for $0 \leq i \leq m$. We remark that if r, $p - \frac{1}{2} \leq r \leq p$, is chosen near enough p then from (1.2) and Hölder's inequality, we see that $A_\sigma(x, D^m u(x))$ is locally Lebesgue integrable on O for each $\sigma \in S$. With this restriction on r assume that

$$\begin{aligned} &\int_O \langle\, A(x, D^m u(x)), D^m \phi(x) \,\rangle \, dx \\ &\qquad = \sum_{\sigma \in S} \int_O \langle\, A_\sigma(x, D^m u(x)),\ \ \partial^\sigma \phi\ (x) \,\rangle \, dx = 0 \end{aligned} \tag{1.3}$$

whenever $\phi = (\phi_1, \ldots, \phi_N) \in C_0^\infty(O)$. We prove

Theorem 1. *Let $u \in W_{m,r}^{\,loc}\ (O)$ and A be as in (1.1)-(1.3). Then there exists for fixed $n, m, N, \epsilon, \gamma$ a positive continuous function $\delta = \delta(n, m, N, \epsilon, \gamma, p)$ of p on $(1, \infty)$ such that if $r = p - \delta$, then $u \in W_{m,p+\delta}^{loc}$, so u is a classical weak solution to (1.3) .*

We remark that Elcrat and Meyer's [EM] proved Theorem 1 with $p - \delta$ replaced by p. For applications of Theorem 1 see [EM] or [Gi, ch 5].

To outline our proof we consider a global version of Theorem 1 in a simple case. Suppose in Theorem 1 that $m = 1$, $O = \mathbb{R}^n$, $N = 1$, and $a_0 \equiv a \equiv b_i \equiv 0$, for $i = 0, 1$. We shall show there exists $\delta = \delta(n, \gamma, p) > 0$, such that if $u \in W_{1,p-\delta}(\mathbb{R}^n)$, then $u \equiv 0$. To this end for given $\lambda > 0$ let

$$E(\lambda) = \{x \in \mathbb{R}^n : M(|\ \partial^1 u\ |)(x) \leq \lambda\}$$

where M is the symmetric Hardy Littlewood Maximal function of $|\ \partial^1 u\ |$ on balls (see section 2 for a definition). Using Sobolev type estimates it is easily shown that u is Lipschitz on $E(\lambda)$ with norm at most $c\lambda$, where c is a positive constant. From this fact, the fact that $E(\lambda)$ is closed, and Whitney's extension theorem we see there exists a Lipschitz function $v = v_\lambda$ on $\mathbb{R}^n$ with $v = u$ on $E(\lambda)$ and Lip norm at most, $c\lambda$. It is easily checked that $\partial^1 v \in L^q(\mathbb{R}^n)$, for $q \geq p - \delta$. Indeed,

$$\begin{aligned} \int_{E(\lambda)} |\ \partial^1 v\ |^q(x) dx &= \int_{E(\lambda)} |\ \partial^1 u\ |^q(x)\, dx \\ &\leq (c\lambda)^{q+\delta-p} \int_{E(\lambda)} |\ \partial^1 u\ |^{p-\delta}(x) dx \end{aligned}$$

while on $\mathbb{R}^n \setminus E(\lambda)$,

$$\int_{\mathbb{R}^n \setminus E(\lambda)} |\ \partial^1 v\ |^q(x)\, dx \leq c^q \lambda^q\, |\mathbb{R}^n \setminus E(\lambda)| < \infty,$$

as we see from weak type estimates and the fact that $M(|\ \partial^1 u\ |) \in L^{p-\delta}(\mathbb{R}^n)$ (by the Hardy Littlewood Maximal Theorem). Now the conjugate exponent to $\frac{p-\delta}{p-1}$ is $s = \frac{p-\delta}{1-\delta} > p - \delta$.

Using this fact, (1.2), $\partial^1 v \in L^s(\mathbb{R}^n)$, and approximating v by functions in $C_0^\infty(\mathbb{R}^n)$, we find from a standard argument that (1.3) holds for $m = 1$ with ϕ replaced by v.

Using (1.1)-(1.3) with ϕ replaced by v, the fact that $D^1 u = D^1 v$ on $E(\lambda)$, and $\| \partial^1 v \|_\infty \leq c\lambda$, we deduce

$$\begin{aligned}
\int_{E(\lambda)} | \partial^1 u |^p(x)\, dx &\leq c\int_{E(\lambda)} \textstyle\sum_{\sigma\in S} A_\sigma(x, Du(x))\ \partial^\sigma u\ dx \\
&= -c\int_{\mathbb{R}^n\setminus E(\lambda)} \textstyle\sum_{\sigma\in S} A_\sigma(x, Du(x))\ \partial^\sigma v\ dx \qquad (1.4)\\
&\leq c\lambda \int_{\mathbb{R}^n\setminus E(\lambda)} | \partial^1 u |^{p-1}(x)\, dx
\end{aligned}$$

We multiply both sides of (1.4) by $\lambda^{-(1+\delta)}$ and integrate from 0 to ∞. It is easily checked that both sides of the resulting inequality are finite since $u \in W_{1,p-\delta}(\mathbb{R}^n)$. Interchanging the order of integration we get

$$\begin{aligned}
&\int_0^\infty \lambda^{-(1+\delta)} \left(\int_{\{\, M(|\partial^1 u|)\leq\lambda\,\}} | \partial^1 u |^p(x)\, dx\right) d\lambda \\
&= \int | \partial^1 u |^p \left(\int_{M(|\partial^1 u|)}^\infty \lambda^{-(1+\delta)}\, d\lambda\right) \\
&= \delta^{-1} \int_{\mathbb{R}^n} | \partial^1 u |^p\, M(| \partial^1 u |)^{-\delta}\, dx \\
&\leq c\int_0^\infty \left(\int_{\{M(|\partial^1 u|)>\lambda\}} | \partial^1 u |^{p-1}\, dx\right) \lambda^{-\delta}\, d\lambda \qquad (1.5)\\
&= c\int_{\mathbb{R}^n} | \partial^1 u |^{p-1} \left(\int_0^{M(|\partial^1 u|)} \lambda^{-\delta}\, d\lambda\right) dx \\
&= \tfrac{c}{1-\delta} \int_{\mathbb{R}^n} | \partial^1 u |^{p-1}\, M(| \partial^1 u |)^{1-\delta} dx.
\end{aligned}$$

Thus for some $c = c(n, \gamma, p)$ we have

$$\begin{aligned}
&\delta^{-1}\int_{\mathbb{R}^n} | \partial^1 u |^p\, M(| \partial^1 u |)^{-\delta}\, dx \\
&\qquad \leq c\int_{\mathbb{R}^n} | \partial^1 u |^{p-1}\, M(| \partial^1 u |)^{1-\delta}\, dx. \qquad (1.6)
\end{aligned}$$

If $u \not\equiv 0$ we use the observation that $M(| \partial^1 u |)^{-\delta}$ is an A_p weight(see section 3) for δ sufficiently small(say $\delta \leq \delta_0$) with A_p constants depending only on p and n. It follows from this fact, (1.6),

and Muckenhoupt's
theorem (see [T, ch 9]):

$$\int_{\mathbb{R}^n} (Mf)^p\,\omega\,dx \;\le\; \tilde{c}\int_{\mathbb{R}^n} |f|^p\,\omega\,dx$$

where ω is an A_p weight and $\tilde{c}$ depends only on p and the A_p constants; that for $\delta \le \delta_0$

$$\begin{aligned}\delta^{-1}\int_{\mathbb{R}^n} M(|\ \partial^1 u\ |)^{p-\delta}dx \;&\le\; c\delta^{-1}\int_{\mathbb{R}^n} |\ \partial^1 u\ |^p\, M(|\ \partial^1 u\ |)^{-\delta}\,dx \\ &\le c\int_{\mathbb{R}^n} M(|\ \partial^1 u\ |)^{p-\delta}\,dx.\end{aligned} \tag{1.7}$$

Clearly, (1.7) implies for δ small enough that $\partial^1 u\ \equiv 0$. Since $u \in W_{1,p-\delta}(\mathbb{R}^n)$ it follows that $u \equiv 0$ which is a contradiction. Hence $u \equiv 0$. □

In section 2 we state and outline the proof of some lemmas involving Sobolev and reverse Hölder inequalities which will be used in the proof of Theorem 1. In section 3 we prove Theorem 1.

Theorem 1 has a parabolic analogue which we shall prove in a future paper. Finally, we would like to thank Tadeusz Iwaniec for helpful conversations concerning this problem.

2 Sobolev and Reverse Hölder Inequalities

If f is an integrable function on $\mathbb{R}^n$ and $B \subset \mathbb{R}^n$ is measurable, we let

$$f_B \;=\; |B|^{-1}\int_B f dx \;=\; ⨍_B f\,dx$$

$$Mf(x) \;=\; M^1 f(x) \;=\; \sup_{r>0} ⨍_{B(x,r)} |f|dx$$

$$M^k f(x) \;=\; M^{k-1}(Mf)(x), \text{ for } k \ge 2\,.$$

We shall need the following lemma.

Lemma 2.1 *Let k be a positive integer, $1 < q < \infty$, $x_0 \in \mathbb{R}^n, r > 0$, and $B = B(x_0, r)$. If $w \in W_{k,q}(B)$, $\partial^\alpha w\ )_B = 0$ for $0 \le |\alpha| \le k-1$, and $x \in B$, then there exists $c_1 = c_1(n,k,q)$ such that*

$$|w|(x) \;\le\; c_1 r^k\, M^k(|\ \partial^k v\ |\chi_B\,)(x) \tag{2.1a}$$

where χ_B is the characteristic function of B; while if $0 < s \le q$, and $ks < n$, there exists $\hat{c}_1 = \hat{c}_1(n,k,s)$ such that

$$|w|(x) \le \hat{c}_1 r^k \left(⨍_B [M^k(|\ \partial^k w\ |\chi_B)]^s dx\right)^{\frac{k}{n}}\ [M^k(\,|\ \partial^k w\ |\chi_B)]^{\frac{s}{s^*}}(x) \tag{2.1b}$$

$$\left(\fint_B | w\chi_B |^{s^*} dx\right)^{\frac{1}{s^*}} \le \hat{c}_1 r^k \left(\fint_B [M^k(| \partial^k w |\chi_B)]^s dx\right)^{\frac{1}{s}} \tag{2.1c}$$

where χ_B is the characteristic function of B and $s^* = \frac{ns}{n-ks}$.

Lemma 2.1 is well known, however for completeness we sketch a proof of this lemma along the lines of [H]. We first prove Lemma 2.1 when $k = 1$. In this case it follows from Morrey's lemma that for $x \in B$

$$\begin{aligned} w(x) &\le c\int_B | \partial^1 w ||x-y|^{1-n} dy \\ &= \int_{\{y\in B:|x-y|<\eta\}} \dots dx + \int_{\{y\in B:|x-y|\ge\eta\}} \dots dx \\ &= L_1 + L_2 . \end{aligned} \tag{2.2}$$

In (2.2), as well as in the rest of this section, $c \ge 2$ denotes a constant depending only on n, k, q, not necessarily the same at each occurrence. To estimate L_1 we write the integral defining L_1 as a sum over $\{y \in B : 2^{-(k+1)}\eta \le |x-y| \le 2^{-k}\eta\}$, $k = 0, 1, \dots$ and use the definition of the maximal function to get at $x \in B$

$$L_1 \le c\eta\, M(| \partial^1 w |\chi_B) . \tag{2.3}$$

From (2.2), (2.3), and the fact that $L_2 = 0$, when $\eta \ge 2r$, we see that (2.1a) is valid for $k = 1$. If $\eta \ge 2r$, we estimate L_2 using Hölder's inequality. We obtain

$$L_2 \le c\eta^{1-\frac{n}{s}} \left(\int_B | \partial^1 w |^s dx\right)^{\frac{1}{s}} . \tag{2.4}$$

If

$$\eta^{\frac{n}{s}} = \frac{\left(\int_B | \partial^1 w |^s dy\right)^{\frac{1}{s}}}{M(| \partial^1 w |\chi_B)}$$

then from (2.2)-(2.4) we see that (2.1b) is true when $k = 1$. Raising both sides of (2.1b) to the s^*, integrating, and taking $1/s^*$ powers of the resulting expression we find that (2.1c) is true when $k = 1$. Hence Lemma 2.1 is true when $k = 1$.

Assume by way of induction that Lemma 2.1 is true when $k = l$, a positive integer. If $k = l + 1$, then from (2.2), (2.3) with $\eta = 2r$ and (2.1a) of the induction hypothesis applied to $\partial^1 w$, we find that (2.1a) holds when $k = l + 1$. Also from (2.3) and (2.1b) of the induction hypothesis applied to $\partial^1 w$, we have at $x \in B$

$$\begin{aligned} L_1 &\le c\eta\, M(| \partial^1 w |\chi_B) \\ &\le c\eta r^{k-1} \left(\fint_B M^{k-1}(| \partial^k w |\chi_B)^s dx\right)^{\frac{k-1}{n}} M\left(M^{k-1}(| \partial^k w |\chi_B)^{\frac{s}{\bar{s}}}\right) \\ &\le c\eta r^{k-1} \left(\fint_B [M^{k-1}(| \partial^k w |\chi_B)]^s dx\right)^{\frac{k-1}{n}} [M^k(| \partial^k w |\chi_B)]^{\frac{s}{\bar{s}}} \end{aligned} \tag{2.5}$$

where $\bar{s} = \frac{ns}{n-(k-1)s}$ and we have used Hölder's inequality to get the last inequality. Applying (2.1c) of the induction hypothesis to $\partial^1 w$ and using Hölder's inequality once again, we obtain

$$
\begin{aligned}
L_2 &\leq c\eta^{1-\frac{n}{\bar{s}}} \left(\int_B \mid \partial^1 w \mid^{\bar{s}} dx\right)^{\frac{1}{\bar{s}}} \\
&\leq c\eta^{1-\frac{n}{s}} \left(\int_B [M^{k-1}(\mid \partial^k w \mid \chi_B)]^s \, dx\right)^{\frac{1}{s}} .
\end{aligned}
\tag{2.6}
$$

Choosing

$$
\eta^{\frac{n}{s}} = \frac{\left(\int_B [M^{k-1}(\mid \partial^k w \mid \chi_B)]^s \, dy\right)^{\frac{1}{s}}}{M^k(\mid \partial^k w \mid \chi_B)},
$$

in (2.5), (2.6), we deduce from (2.2) that (2.1b) is valid when $k = l + 1$. (2.1c) follows from (2.1b) as in the case $k = 1$. Hence by induction, Lemma 2.1 is true . □
We shall also need

Lemma 2.2 *Let k be a positive integer , $\lambda > 0$, $1 < q < \infty$, $x_0 \in \mathbb{R}^n$, and $r > 0$. If $h = (h_1, ..., h_N) \in W_{k,q}(\mathbb{R}^n)$, supp $h \subset \bar{B}(x_0, r)$, and*

$$
F(\lambda) = \{x : M^k(\mid \partial^k h \mid)(x) \leq \lambda\} \cap B(x_0, 2r) \neq \emptyset ,
$$

then $h|_{F(\lambda)}$ has an extension $H = H(\cdot, \lambda)$ to $\mathbb{R}^n$ satisfying,

$$
\begin{aligned}
&(i) \quad H = h \text{ on } F(\lambda) \text{ and supp } H \subset B(x_0, 4r), \\
&(ii) \quad H \in W_{k,\infty}(\mathbb{R}^n) \text{ with } \| \partial^\sigma H \|_\infty \leq c\lambda r^{k-|\sigma|}, 0 \leq |\sigma| \leq k, \\
&(iii) \quad \mid \partial^\sigma (H - h) \mid (x) \leq c\lambda d(x)^{k-|\sigma|} \text{ for } 0 \leq |\sigma| \leq k-1 \\
&\qquad \text{and a.e } x \in \mathbb{R}^n, \text{ where } d(x) \text{ denotes the} \\
&\qquad \text{distance from } x \text{ to } F(\lambda).
\end{aligned}
\tag{2.7}
$$

To outline the proof of Lemma 2.2, we first observe from the divergence theorem that

$$
\partial^\sigma h\ )_{B(x_o,2r)} = 0 \text{ when } 0 < |\sigma| \leq k .
$$

From this observation we see that the hypotheses of Lemma 2.1 are satisfied with $B = B(x_0, 2r), w = h - h_B$. Using (2.1a) with $|x - x_0| = \frac{3r}{2}$, and the fact that supp $h \subset \bar{B}(x_0, r)$ we get

$$
\begin{aligned}
|h(x) - h_B| &= |h_B| \leq cr^k M^k(\mid \partial^k h \mid)(x) \\
&\leq cr^{k-n} \int_{B(x_0,2r)} \mid \partial^k h \mid dx \leq c\lambda r^k,
\end{aligned}
$$

where the last inequality is a consequence of the fact that $F(\lambda) \neq \emptyset$. Next from the above inequality and (2.1a) with k replaced by $k - l$, $0 \leq l \leq k - 1$, it follows that

$$
\mid \partial^l h \mid (y) \leq c\lambda r^{k-l}, \, y \in F(\lambda) \cap B(x_0, 2r), 0 \leq l \leq k . \tag{2.8}
$$

If $z_0 \in F(\lambda)$ and $s > 0$, let $V = V(\cdot, z_0, s) = (V_1, ..., V_N)$ be such that $V_i, 1 \leq i \leq N$, is the unique polynomial of degree $k-1$ with

$$\partial^\sigma V_i\ )_{B(z_0,s)} \;=\; \partial^\sigma h_i\ )_{B(z_0,s)} \text{ for } 0 \leq |\sigma| \leq k-1,\ 1 \leq i \leq N\,. \tag{2.9}$$

From (2.9) and (2.1a) with $w = \ \partial^l (h - V)\ , x_0 = z_0, s = r$, and k replaced by $k - l$, we see that

$$|\ \partial^l (h - V)\ |(y) \;\leq\; c\lambda s^{k-l}\,, y \in F(\lambda) \cap B(z_0, s),\ 0 \leq l \leq k-1. \tag{2.10}$$

Let $W_i, 1 \leq i \leq N$, be the Taylor polynomial of degree $k-1$ defined relative to $z_0,\ h_i,$ and put $W = (W_1, ..., W_N)$. Using (2.10) we deduce that

$$|\ \partial^l (V - W)\ |(y) \;\leq\; c\lambda s^{k-l}\,, \tag{2.11}$$

when $y \in B(z_0, s)$ and $0 \leq l \leq k-1$. From (2.11) and (2.10) we see that

$$|\ \partial^l (h - W)\ |(y) \;\leq\; c\lambda\, s^{k-l},\ y \in F(\lambda) \cap B(z_0, s),\ 0 \leq l \leq k. \tag{2.12}$$

Since s is arbitray, we conclude from (2.12) and (2.8) that $h|_{F(\lambda)}$ satisfies the hypotheses of the Whitney extension theorem (see [St, ch 6]) . Applying this theorem we get H satisfying (2.7)(i)-(iii) . □
Finally, we shall need an amended form of a theorem of Gehring [G].

Lemma 2.3 *Let $R > 0, 0 < \xi < 1, q > 1, \beta = (1+\xi)q,\ f \in L^\beta(B(x_0, R)),$ and $g \in L^q(B(x_0, R))$. Assume that whenever $x \in B(x_0, \frac{R}{2}),\ 0 < r \leq \frac{R}{16}$, we have*

$$\begin{aligned} \fint_{B(x,\frac{r}{16})} |g|^q dx \leq c_2 \left(\fint_{B(x,r)} |g| dx \right)^q &+ \fint_{B(x,r)} |f|^q\, dx \\ &+ \theta \fint_{B(x,r)} |g|^q\, dx \end{aligned} \tag{2.13}$$

for some $0 < \theta < 1$, and $2 \leq c_2 < \infty$. Then there exists $\eta = \eta(n, \theta, c_2, \xi, q) > 0$ which for fixed $n, \theta, c_2, \xi,$ is a continuous function of q on $(1, \infty)$, and $c_3 = c_3(n, \theta, c_2, \xi, q), 2 \leq c_3 < \infty$, such that if $\tau = q(1+\eta)$, then

$$\begin{aligned} &\left(\fint_{B(x_0,\frac{R}{32})} |g|^\tau dx \right)^{\frac{1}{\tau}} \\ &\leq c_3 \left(\fint_{B(x_0,\frac{R}{2})} |f|^\tau\, dx \right)^{\frac{1}{\tau}} + c_3 \left(\fint_{B(x_0,\frac{R}{2})} |g|^q\, dx \right)^{\frac{1}{q}}. \end{aligned} \tag{2.14}$$

For a proof of Lemma 2.3 (see [Gi, ch 5, Proposition 1.1]). This author does not state explicitly that η can be chosen as a continuous function of q on $(1, \infty)$(for fixed n, θ, c_2, ξ), but this statement can be deduced from a careful examination of the proof. Indeed, there are two parts to the proof. The first part consists of using (2.13) and Calderón-Zygmund type arguments to derive an inequality for a certain distribution function. It is easily checked that the constants in this inequality can be chosen independent of q on any compact subset of $(1,\infty)$ provided the above constants are fixed. The second part of the proof (see [Gi, ch.5, Lemma 1.2]) consists in showing that the derived inequality implies (2.14). Here an explicit value for η is given which depends on the constants in the derived inequality . It follows that η can be chosen independent of q on $(1,\infty)$ (when the other variables are fixed), so clearly η can be chosen continuous on $(1,\infty)$.

3 Proof of Theorem 1

In this section, $c \geq 2$, denotes a constant that depends only on $n, m, N, \epsilon, \gamma, p,$ not necessarily the same at each occurrence. Let u, A, O be as in Theorem 1 and suppose $B(x_0, R) \subset O$ for some $R \leq 1$. For fixed $y_0 \in B(x_0, \frac{R}{2})$ and $0 < \rho < \frac{R}{32}$, let $P = (P_1, ..., P_N)$ where $P_i = P_i(\cdot, y_0, 8\rho), 1 \leq i \leq N$, is the unique polynomial of degree $m-1$ in the coordinate variables satisfying

$$\partial^\sigma P_i\ )_{B(y_0,8\rho)} \ = \ \partial^\sigma u_i\ )_{B(y_0,8\rho)} \text{ for } 0 \leq |\sigma| \leq m-1\,. \tag{3.1}$$

Let $\phi \in C_0^\infty(B(y_0, 2\rho))$ with $\phi = 1$ on $B(y_0,\rho)$ and $||\ \partial^\sigma \phi\ ||_\infty \leq c\,\rho^{-|\sigma|}$ when $|\sigma| \leq m$. Put $\tilde{u} = (u-P)\phi$, and $E(\lambda) = \{x \in \mathbb{R}^n : M^m(|\ \partial^m \tilde{u}\ |) \leq \lambda\}$. If $F(\lambda) = E(\lambda) \cap B(y_0, 4\rho) \neq \emptyset$, we may apply Lemma 2.2 with $h = \tilde{u}, x_0 = y_0$, and $r = 2\rho$, to get an extension v of $\tilde{u}|_{F(\lambda)}$ to $\mathbb{R}^n$ satisfying (i)-(iii) with H replaced by v. Using Lemma 2.2, (1.2), and approximating v by smooth functions, we see that (1.3) holds with ϕ replaced by v. Define $\hat{A} = (\hat{A}_\sigma)_{\sigma \in S}$ where $\hat{A} : O \times P \to \mathbb{R}^N$, by $\hat{A}_\sigma = A_\sigma$, when $\sigma \in S \setminus S_m$ and $\hat{A}_\sigma = (0, ..., 0)$, when $\sigma \in S_m$. Then from (1.3) with $\phi = v$, (ii) of Lemma 2.2, and (1.2), we deduce

$$\begin{aligned}
&\int_{F(\lambda)} \langle\, A(x, D^m u(x)),\ D^m \tilde{u}(x)\,\rangle\, dx \\
&= -\int_{\mathbb{R}^n \setminus F(\lambda)} \langle\, A(x, D^m u(x)),\ D^m v(x)\,\rangle\, dx \\
&\leq \int_{\mathbb{R}^n \setminus F(\lambda)} \langle\, \hat{A}(x, D^m u(x)),\ D^m (\tilde{u} - v)(x)\,\rangle\, dx \\
&- \int_{\mathbb{R}^n \setminus F(\lambda)} \langle\, \hat{A}(x, D^m u(x)),\ D^m \tilde{u}(x)\,\rangle\, dx \\
&+ c\lambda \int_{\mathbb{R}^n \setminus F(\lambda)} (|\ \partial^m u\ |^{p-1} + b_m)\, dx \\
&= J_1(\lambda)\ +\ J_2(\lambda)\ +\ J_3(\lambda)\,.
\end{aligned} \tag{3.2}$$

If $x \in \mathbb{R}^n \setminus B(y_0, 3\rho)$, we observe that

$$\begin{aligned} M^m(|\,\partial^m \tilde{u}\,|)(x) \;&\le\; c\rho^{-n} \int_{B(y_0,4\rho)} M^{m-1}(\,|\,\partial^m \tilde{u}\,|\,)\,dx \;=\; \lambda_0 \\ &\le\; c \min_{B(y_0,8\rho)} M^m(|\,\partial^m \tilde{u}\,|)\,, \end{aligned} \tag{3.3}$$

since supp $\tilde{u} \subset B(y_0, 2\rho)$. Thus $F(\lambda) \neq \emptyset$ for $\lambda > \lambda_0$ and

$$\mathbb{R}^n \setminus E(\lambda) \;=\; \{x \in \mathbb{R}^n : M^m(|\,\partial^m \tilde{u}\,|)(x) \;>\; \lambda\,\} \;=\; \mathbb{R}^n \setminus F(\lambda) \tag{3.4}$$

for $\lambda_0 < \lambda < \infty$.

We multiply both sides of (3.2) by $\lambda^{-(1+\delta)}$ and integrate the resulting inequality over (λ_0, ∞). Interchanging the order of integration and using (3.4), we get as in (1.5), (1.6)

$$\begin{aligned} &\delta^{-1} K \\ &= \delta^{-1} \int_{\mathbb{R}^n \setminus E(\lambda_0)} M(|\,\partial^m \tilde{u}\,|)^{-\delta}\, \langle A(x, D^m u(x)),\; D^m \tilde{u}(x)\,\rangle\, dx \\ &\qquad + \delta^{-1}\, \lambda_0^{-\delta} \int_{E(\lambda_0)} \langle A(x, D^m u(x)),\; D^m \tilde{u}(x)\,\rangle\, dx \\ &\le \sum_{i=1}^{3} \int_{\lambda_0}^{\infty} \lambda^{-(1+\delta)}\, J_i(\lambda)\, d\lambda \;=\; \sum_{i=1}^{3} K_i\,, \end{aligned} \tag{3.5}$$

where K_i denotes the integral with J_i in its integrand for $1 \le i \le 3$. To estimate K_1, we observe from (1.2), (2.7)(iii), and (3.4) that for $\lambda \in (\lambda_0, \infty)$ we have

$$\begin{aligned} |J_1(\lambda)| \;&\le\; c \sum_{l=0}^{m-1} \int_{B(y_0,8\rho)\setminus E(\lambda)} [\,a_l\,|\,\partial^m u\,|^{p-1} \;+\; b_l\,]|\,\partial^l(\tilde{u} - v)\,|\,dx \\ &\le c\lambda \sum_{l=0}^{m-1} \int_{B(y_0,8\rho)\setminus E(\lambda)} [\,a_l\,|\,\partial^m u\,|^{p-1} \;+ b_l\,]\, d^{m-l}(x) dx\,. \end{aligned} \tag{3.6}$$

For $0 \leq l \leq m-1$, let

$$
\begin{aligned}
\alpha &= \tfrac{(p-1)\epsilon}{100n(p+\epsilon)^2} \\
\tilde{p} &= \tilde{p}(l) = 1 - \tfrac{(m-l)(1-\alpha)p}{n} \\
\gamma_1 &= \gamma_1(l) = [\int_{B(x_0,\frac{3R}{4})} \{M^{2m}(|\,\partial^m u\,|\chi_{B(x_0,\frac{3R}{4})})\}^{\frac{n(1-\alpha)}{m-l}}\,dx]^{\frac{m-l}{n}} \\
\gamma_2 &= \gamma_2(l) = [\int_{B(x_0,\frac{3R}{4})} \{M^{2m}(|\,\partial^m u\,|\chi_{B(x_0,\frac{3R}{4})})\}^{p(1-\alpha)}dx]^{\frac{m-l}{n}} \qquad (3.7a) \\
\tau_1 &= \tau_1(l) = [\int_{B(x_0,\frac{3R}{4})} \{M^{m}(|\,\partial^m \tilde{u}\,)\}^{\frac{n(1-\alpha)}{m-l}}\,dx]^{\frac{m-l}{n}} \\
\tau_2 &= \tau_2(l) = [\int_{B(x_0,\frac{3R}{4})} \{M^{m}(|\,\partial^m \tilde{u}\,|)\}^{p(1-\alpha)}dx]^{\frac{m-l}{n}}
\end{aligned}
$$

Then from (3.6) and weak type estimates we see for $\lambda \in (\lambda_0, \infty)$ and $0 \leq l \leq m-1$ that

$$
\begin{aligned}
\lambda\, d^{m-l}(x) &\leq c\lambda\,|B(y_0, 8\rho) \setminus E(\lambda)|^{\frac{m-l}{n}} \\
&\leq c \min\{\lambda^\alpha\, \tau_1,\ \lambda^{\tilde{p}}\, \tau_2\} \leq c \min\{\lambda^\alpha \gamma_1,\ \lambda^{\tilde{p}}\, \gamma_2\}\,.
\end{aligned}
\qquad (3.7b)
$$

To get the last line in (3.7b) we have used the inequality

$$
|\,\partial^m \tilde{u}\,| \leq cM^m(|\,\partial^m u\,|\chi_{B(y_0,4\rho)}) \qquad (3.8)
$$

which follows from Lemma 2.1 as in (2.8) . We note for $0 \leq l \leq m-1$ from the Hardy - Littlewood maximal theorem that $\min[\,\gamma_1(l),\, \gamma_2(l)\,] < \infty$. Using (3.7b) in (3.6) we conclude that

$$
|J_1(\lambda)| \leq c\sum_{l=0}^{m-1} e_l(\lambda) \qquad (3.9)
$$

where

$$
e_l(\lambda) = \int_{B(y_o,8\rho)\setminus E(\lambda)} (\,a_l|\,\partial^m u\,|^{p-1} + b_l\,) \min\ \{\,\lambda^\alpha \gamma_1(l)\,,\, \lambda^{\tilde{p}}\, \gamma_2(l)\,\}\,dx.
$$

Let $p_l = \max\{p, \frac{n}{m-l}\} + \epsilon$, $0 \leq l \leq m-1$, and $q_l = \max\{1, \frac{np'}{n+p'(m-l)}\} + \epsilon$, $0 \leq l \leq m$, be as in section 1 . To estimate K_1 we consider two cases. First if $p_l = p + \epsilon$, then $q_l = 1 + \epsilon$ and we estimate $e_l(\lambda)$, by choosing $\lambda^\alpha\, \gamma_1(l)$ in the minimum . Integrating over (λ_0, ∞), we obtain when $\alpha^2 > 4\delta$,

$$\begin{aligned}
&\int_{\lambda_0}^{\infty} \lambda^{-(1+\delta)}\, e_l(\lambda)\, d\lambda \\
&\quad \leq\ c\gamma_1 \int_{B(y_0,8\rho)\setminus E(\lambda_0)} \left[\int_0^{M^m(|\partial^m \tilde{u}|)} \lambda^{\alpha-\delta-1} d\lambda \right] \\
&\qquad\qquad\qquad \cdot\, (a_l|\ \partial^m u\ |^{p-1} + b_l)\, dx \\
&\quad \leq c\gamma_1 \int_{B(y_0,8\rho)} [M^m(|\ \partial^m \tilde{u}\ |)]^{\alpha-\delta}\, (a_l|\ \partial^m u\ |^{p-1} + b_l)\, dx.
\end{aligned} \tag{3.10}$$

Second, if $p_l > p + \epsilon$, then $p_l = \frac{n}{m-l} + \epsilon$, $q_l = \frac{np'}{n+p'(m-l)} + \epsilon$, and we estimate $e_l(\lambda)$ by choosing $\gamma_2(l)\, \lambda^{\tilde{p}}$ in the minimum . We get for $0 < 4\delta < \alpha^2$,

$$\begin{aligned}
&\int_{\lambda_0}^{\infty} \lambda^{-(1+\delta)}\, e_l(\lambda)\, d\lambda \\
&\quad \leq\ c\,\gamma_2 \int_{B(y_0,8\rho)\setminus E(\lambda_0)} \left[\int_0^{M^m(|\partial^m \tilde{u}|)} \lambda^{\tilde{p}-\delta-1} d\lambda \right] \\
&\qquad\qquad\qquad \cdot\, (a_l|\ \partial^m u\ |^{p-1} + b_l)\, dx \\
&\quad \leq c\gamma_2 \int_{B(y_0,8\rho)} [M^m(|\ \partial^m \tilde{u}\ |)]^{\tilde{p}-\delta}\, (a_l|\ \partial^m u\ |^{p-1} + b_l)\, dx.
\end{aligned} \tag{3.11}$$

In the first case we put

$$f_l\ =\ c\gamma_1(l)\, M^{2m}(|\ \partial^m u\ |\chi_{B(x_0,\frac{3R}{4})})^{\alpha-\delta}\, (a_l\,|\ \partial^m u\ |^{p-1} + b_l)$$

and in the second case we set

$$f_l\ =\ c\gamma_2(l)\, M^{2m}(|\ \partial^m u\ |\chi_{B(x_0,\frac{3R}{4})})^{\tilde{p}-\delta}\, (a_l|\ \partial^m u\ |^{p-1} + b_l)$$

when $x \in B(x_0, \frac{3R}{4})$. Define $F_1 = F_{1,\delta}$ by $F_1^{p-\delta} = \sum_{l=0}^{m-1} f_l$. Summing (3.11), using (3.8), Hölder's inequality, the Hardy - Littlewood maximal theorem, and a ballpark estimate, we conclude that

$$\begin{aligned}
&(a) \qquad K_1\ \leq \int_{B(y_0,8\rho)} F_1^{p-\delta}\, dx\,, \\
&(b) \qquad F_1\ \in L^{p+\alpha}(B(x_0, \tfrac{3R}{4}))\,.
\end{aligned} \tag{3.12}$$

Next we consider J_2. From (1.2) we see as in (3.6) that for $\lambda \in (\lambda_0, \infty)$

$$
\begin{aligned}
|J_2(\lambda)| &\leq c\sum_{l=0}^{m-1} \int_{B(y_0,8\rho)\setminus E(\lambda)} [\, a_l \mid \partial^m u \mid^{p-1} + b_l]| \, \partial^l \tilde{u} \mid dx \\
&= \sum_{l=0}^{m-1} e_l' \, .
\end{aligned}
\tag{3.13}
$$

To estimate e_l' we again consider two cases. If $p_l = p + \epsilon, 0 \leq l \leq m-1$,we note for $0 < \beta < \frac{1}{2}$, that

$$
\begin{aligned}
⨍_{B(y_0,4\rho)} M^m(\mid \partial^m \tilde{u} \mid)^\beta \, dx &\leq c\, [⨍_{B(y_0,4\rho)} M^{m-1}(\mid \partial^m \tilde{u} \mid) \, dx]^\beta \\
&\leq c\, \lambda_0^\beta \, ,
\end{aligned}
\tag{3.14}
$$

as follows from weak type estimates for the maximal function. Let α be as in (3.7a). Then from (3.14) with $\beta = \frac{\alpha n}{m-l}$, and (2.1b) with $s = \beta$, $w = \mid \partial^l \tilde{u} \mid, k = m - l,$ we find for $\lambda \in (\lambda_0, \infty)$ and $x \in B(y_0, 8\rho)$ that

$$
\begin{aligned}
\mid \partial^l \tilde{u} \mid^\alpha &\leq c\rho^{\alpha(m-l)} \, \lambda^{\alpha^2} \, M^m(\mid \partial^m \tilde{u} \mid)^{\alpha-\alpha^2} \\
&\leq c\lambda^{\alpha^2} \, M^m(\mid \partial^m \tilde{u} \mid)^{\alpha-\alpha^2}
\end{aligned}
\tag{3.15}
$$

since $32\rho < R < 1$. Also from (2.1b) with $s = \frac{n(1-\alpha)}{m-l}$, we deduce at $x \in B(y_0, 8\rho)$ that

$$
\mid \partial^l \tilde{u} \mid^{1-\alpha} \leq c\gamma_1^{1-\alpha} \, M^m(\mid \partial^m \tilde{u} \mid)^{\alpha-\alpha^2} \tag{3.16}
$$

where $\gamma_1 = \gamma_1(l)$ is as in (3.7a). Multiplying the lefthand side of (3.15) and (3.16) together we get an estimate for $\mid \partial^l \tilde{u} \mid$. Using this estimate in (3.13) we find as in (3.10) for $0 < 4\delta < \alpha^2$ that

$$
\begin{aligned}
&\int_{\lambda_0}^{\infty} \lambda^{-(1+\delta)} \, e_l'(\lambda) d\lambda \\
&\leq c\gamma_1^{1-\alpha} \int_{B(y_o,8\rho)} [a_l \mid \partial^m u \mid^{p-1} + b_l] M^m(\mid \partial^l \tilde{u} \mid)^{2\alpha-\alpha^2-\delta} \, dx \, .
\end{aligned}
\tag{3.17}
$$

If $p_l > p+\epsilon$, we use (2.1b) with $s = p(1-\alpha), w$, and k as above to obtain at $x \in B(y_0, 8\rho)$ that

$$
\mid \partial^m \tilde{u} \mid^{1-\alpha} \leq c\gamma_2^{1-\alpha} \, [M^m(\mid \partial^m \tilde{u} \mid)]^{\tilde{p}(1-\alpha)} \tag{3.18}
$$

where $\tilde{p} = \tilde{p}(l), \gamma_2 = \gamma_2(l)$ are as in (3.7a). Multiplying the lefthand side of (3.15) and (3.18) together we get an estimate for $\mid \partial^l \tilde{u} \mid$ when $p_l > p + \epsilon$. Using this estimate we find as in (3.11) for $0 < 4\delta < \alpha^2$ that

$$\int_{\lambda_0}^{\infty} \lambda^{-(1+\delta)}\, e_l'(\lambda) d\lambda$$

$$\leq c\gamma_2^{1-\alpha} \int_{B(y_o,8\rho)} [\, a_l \mid \partial^m u \mid^{p-1} \,+\, b_l \,] \tag{3.19}$$

$$\cdot\, [M^m(\mid \partial^m \tilde{u} \mid)]^{\tilde{p}(1-\alpha)+\alpha-\delta}\, dx\,.$$

In the first case we let

$$\hat{f}_l \,=\, c\gamma_1^{1-\alpha}\, [\, a_l \mid \partial^m u \mid^{p-1} \,+\, b_l \,]\, M^{2m}(\mid \partial^m u \mid \chi_{B(x_0, \frac{3R}{4})})^{2\alpha-\alpha^2-\delta}$$

while in the second case we put

$$\hat{f}_l \,=\, c\gamma_2^{1-\alpha}\, [\, a_l \mid \partial^m u \mid^{p-1} \,+\, b_l \,]\, M^{2m}(\mid \partial^m u \mid \chi_{B(x_0, \frac{3R}{4})})^{[\tilde{p}(1-\alpha)+\alpha-\delta]}\,.$$

If $F_2 = F_{2,\delta}$ is defined by $F_2^{p-\delta} \,=\, \sum_{l=0}^{m-1}\, \hat{f}_l$, then as in (3.12) we see for c large enough that

$$\begin{aligned} &(a) \qquad\qquad K_2 \,\leq\, \int_{B(y_0,8\rho)} F_2^{p-\delta}\, dx, \\ &(b) \qquad\qquad F_2 \in L^{p+\alpha}(B(x_0, \tfrac{3R}{4}))\,. \end{aligned} \tag{3.20}$$

To handle K_3 we interchange the order of integration as previously to obtain

$$K_3 \,\leq\, c \int_{B(y_0,8\rho)} [\, \mid \partial^m u \mid^{p-1} + b_m \,]\, M^m(\mid \partial^m \tilde{u} \mid)^{1-\delta}\, dx. \tag{3.21}$$

Let $F_3^{p-\delta} \,=\, c\, b_m\, M^{2m}(\mid \partial^m u \mid \chi_{B(x_0, \frac{3R}{4})})^{1-\delta}$. Then from (3.21), (3.8), and the Hardy Littlewood maximal theorem, we find for $0 < 4\delta \leq \alpha^2$ and α as in (3.7a) that

$$\begin{aligned} &(a) \qquad\qquad K_3 \,\leq\, \int_{B(y_0,8\rho)} F_3^{p-\delta}\, dx \,+\, c \int_{B(y_o,8\rho)} \mid \partial^m u \mid^{p-\delta}\, dx \\ &(b) \qquad\qquad F_3 \in L^{p+\alpha}(B(x_0, \tfrac{3R}{4}))\,. \end{aligned} \tag{3.22}$$

From (3.22), (3.20), (3.12), and (3.5), we conclude that

$$K \,\leq\, \delta \sum_{i=1}^{3} \int_{B(y_0,8\rho)} F_i^{p-\delta} \,+ c\delta \int_{B(y_0,8\rho)} \mid \partial^m u \mid^{p-\delta}\, dx\,. \tag{3.23}$$

Next we estimate K from below . From (1.2), (3.8), and the fact that supp $\tilde{u} \subset B(y_0, 2\rho)$, we deduce

$$
\begin{aligned}
K = & \int_{\mathbb{R}^n \setminus E(\lambda_0)} M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} \langle A(x, D^m u(x)),\ D^m \tilde{u}(x) \rangle \, dx \\
& \qquad + \lambda_0^{-\delta} \int_{E(\lambda_0)} \langle A(x, D^m u(x)),\ D^m \tilde{u}(x) \rangle \, dx \\
\geq & -c \int_{B(y_0, 2\rho)} M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} [\sum_{l=0}^{m-1} (a_l |\ \partial^m u\ |^{p-1} + b_l) |\ \partial^l \tilde{u}\ |] dx \\
& - c \int_{E(\lambda_0)} M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} [\,|\ \partial^m u\ |^{p-1} + b_m] \,|\ \partial^m \tilde{u}\ | dx \\
& + \int_{B(y_0, 2\rho)} M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} \\
& \qquad \cdot \langle A(x, D^m u(x)) - \hat{A}(x, D^m u(x)),\ D^m \tilde{u}(x) \rangle \, dx \\
= & \ - L_1 - L_2 + L_3 \,.
\end{aligned}
\tag{3.24}
$$

To estimate L_1 we again consider two cases. If $p_l = p + \epsilon$ we note from (3.16) that

$$
|\ \partial^l \tilde{u}\ | \ \leq \ c\gamma_1 \, M^m(|\ \partial^m \tilde{u}\ |)^\alpha
$$

while if $p_l > p + \epsilon$, we see from (3.18) that

$$
|\ \partial^m \tilde{u}\ | \ \leq \ c\gamma_2 \, [M^m(|\ \partial^m \tilde{u}\ |)]^{\tilde{p}}
$$

where $\gamma_1 = \gamma_1(l), \gamma_2 = \gamma_2(l)$, and $\tilde{p} = \tilde{p}(l)$ are as in (3.7a). Using these inequalities for $|\ \partial^l \tilde{u}\ |$ and (3.8) in the sum defining L_1 we deduce for fixed $\delta \leq \frac{1}{4}\alpha^2$ and c large enough that

$$
L_1 \ \leq \int_{B(y_0, 8\rho)} F_1^{p-\delta} \, dx \,. \tag{3.25}
$$

As for L_2 suppose $0 < \eta \leq \frac{1}{2}$ and $|\ \partial^m u\ | \geq \eta^{-1}\lambda_0$ at $x \in E(\lambda_0)$. Then since $M^m(|\ \partial^m \tilde{u}\ |) \leq \lambda_0$ on $E(\lambda_0)$ and supp $\tilde{u} \subset B(y_0, 2\rho)$ we find at x that

$$
\begin{aligned}
& M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} [\,|\ \partial^m u\ |^{p-1} + b_m] \,|\ \partial^m \tilde{u}\ | \\
& \leq \ F_3^{p-\delta} + c\,\eta^{1-\delta} \,|\ \partial^m u\ |^{p-\delta} \,.
\end{aligned}
$$

On the other hand if $|\ \partial^m u\ | < \eta^{-1}\lambda_0$, then from the righthand inequality in (3.3) and (3.8) we deduce that

$$
\begin{aligned}
& M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} [\,|\ \partial^m u\ |^{p-1} + b_m] \,|\ \partial^m \tilde{u}\ | \\
& \leq \ F_3^{p-\delta} + c\,\eta^{1-p} \left(\fint_{B(y_0, 8\rho)} M^{2m}(|\ \partial^m u\ |\chi_{B(y_0, 4\rho)})^t dx \right)^{\frac{p-\delta}{t}}
\end{aligned}
$$

where $t = \frac{1+p}{2}$. Using these estimates in the integrand of L_2 and the Hardy - Littlewood maximal theorem, we get

$$
\begin{aligned}
L_2 \leq & \int_{B(y_0,8\rho)} F_3^{p-\delta} \quad dx + c\eta^{1-\delta} \int_{B(y_0,8\rho)} |\, \partial^m u \,|^{p-\delta} \, dx \\
& + c\eta^{1-p} \rho^n \left(\fint_{B(y_0,8\rho)} |\, \partial^m u \,|^t \, dx \right)^{\frac{p-\delta}{t}} .
\end{aligned}
\tag{3.26}
$$

Next we consider L_3. Let D_1 be the set of all $x \in B(y_0, 2\rho) \setminus B(y_0, \rho)$ such that

$$
M^m(|\, \partial^m \tilde{u} \,|)(x) \leq \delta \, M^m(|\, \partial^m u \,| \chi_{B(y_0,4\rho)})(x)
$$

and set $D_2 = B(y_0, 2\rho) \setminus [D_1 \cup B(y_0, \rho)]$. We write the integral in the definition of L_3 as a sum of integrals over $B(y_0, \rho)$, D_1, and D_2. In the integral over $B(y_0, \rho)$ we use (1.1) and the fact that $\phi \equiv 1$ in $B(y_0, \rho)$. In the integral over D_1 we use (1.2). Finally in the integral over D_2 we use (1.1) and the observation that if $|\sigma| = m$, then at $x \in D_2$

$$
|\, \partial^\sigma \tilde{u} \; - \; \partial^\sigma u \;\; \phi| \leq c\rho^{l-m} \sum_{l=0}^{m-1} |\, \partial^l (u - P) \,|.
$$

We get

$$
\begin{aligned}
L_3 = & \int_{B(y_0,2\rho)} M^m(|\, \partial^m \tilde{u} \,|)^{-\delta} \\
& \cdot \langle A(x, D^m u(x)) \; - \; \hat{A}(x, D^m u(x)), \; D^m \tilde{u}(x) \rangle \, dx \\
& \geq c^{-1} \int_{B(y_0,\rho)} M^m(|\, \partial^m \tilde{u} \,|)^{-\delta} \,|\, \partial^m u \,|^p \, dx \\
& - c \int_{D_1} M^m(|\, \partial^m \tilde{u} \,|)^{1-\delta} \, [\, |\, \partial^m u \,|^{p-1} \; + \; b_m \,] \, dx \\
& - c \sum_{l=0}^{m-1} \rho^{l-m} \int_{D_2} M^m(|\, \partial^m \tilde{u} \,|)^{-\delta} \; [|\, \partial^m u \,|^{p-1} \; + \; b_m] \\
& \qquad \cdot |\, \partial^l (u - P) \,| dx \\
& - \int_{B(y_0,2\rho)} a \, dx \\
& = I_1 \; - \; I_2 \; - \; I_3 \; - \; I_4 \, .
\end{aligned}
\tag{3.27}
$$

To estimate I_1 we observe from $\tilde{u} = u$ on $B(y_0, \rho)$ and (3.8) that at $x \in B(y_0, \frac{\rho}{2})$

$$M^m(|\, \partial^m \tilde{u}\, |) \le M^m(|\, \partial^m \tilde{u}\, |\chi_{B(y_0,\rho)}) + c \fint_{B(y_0,8\rho)} M^{m-1}(|\, \partial^m \tilde{u}\, |)\, dx$$

$$\le M^m(|\, \partial^m u\, |\chi_{B(y_0,\rho)}) + c^* \fint_{B(y_0,8\rho)} M^{2m}(|\, \partial^m u\, |\chi_{B(y_0,4\rho)})\, dx$$

provided $c^* = c^*(n, m, N)$ is large enough. Let,

$$G = \{x \in B(y_0, \tfrac{\rho}{2}) : M^m(|\, \partial^m u\, |\chi_{B(y_0,\rho)}) \ge c^* \fint_{B(y_0,8\rho)} M^{2m}(|\, \partial^m u\, |\chi_{B(x_0,4\rho)})\, dx\,\}\,.$$

Then from the above inequality we see that

$$M^m(|\, \partial^m \tilde{u}\, |) \le 2\, M^m(|\, \partial^m u\, |\chi_{B(y_0,\rho)}) \text{ on } G. \tag{3.28}$$

Before proceeding further, we show that $w = M^m(|\, \partial^m \tilde{u}\, |)^{-\delta}$ is an A_p weight when $2\delta \le p - 1$ with constants depending only on n. Indeed, for given $s > 0$ and $z_0 \in \mathbb{R}^n$, clearly

$$w(z_0) \le c \left[\fint_{B(z_0,s)} M^{m-1}(|\, \partial^m \tilde{u}\, |)\, dx\right]^{-\delta}$$

on $B(z_0, s)$ where c depends only on n. Also from (3.14) with $\beta = \frac{\delta}{p-1}$, we get

$$\fint_{B(z_0,s)} w^{-\frac{1}{p-1}} dx \le c \left[\fint_{B(z_0,s)} M^{m-1}(|\, \partial^m \tilde{u}\, |) dx\right]^{\frac{\delta}{p-1}}$$

where again c depends only on n when $2\delta \le p - 1$. Using these inequalities we deduce that

$$(\fint_{B(z_0,s)} w dx) \cdot (\fint_{B(z_0,s)} w^{-\frac{1}{p-1}}\, dx)^{p-1} \le c$$

which is the A_p condition of Muckenhoupt(see[T, ch 9]). Moreover from (3.14) it is not difficult to show that w is a doubling measure :

$$\fint_{B(z_0,2s)} w\, dx \le c \fint_{B(z_0,s)} w\, dx$$

for some $c > 0$ depending only on n.

Thus w satisfies the conditions of a theorem of Muckenhoupt mentioned in section 1. Using this theorem and (3.28) it follows for $2\delta \le p-1$ that

$$
\begin{aligned}
I_1 &= c^{-1} \int_{B(y_0,\rho)} M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} \,|\ \partial^m u\ |^p dx \\
&\ge c^{-1} \int_{B(y_0,\rho)} M^m(|\ \partial^m \tilde{u}\ |)^{-\delta} M^m(|\ \partial^m u\ |\chi_{B(y_0,\rho)})^p dx \\
&\ge c^{-1} \int_G M^m(|\ \partial^m u\ |\chi_{B(y_0,\rho)})^{p-\delta}\, dx \\
&= c^{-1} \int_{B(y_0,\frac{\rho}{2})} M^m(|\ \partial^m u\ |\chi_{B(y_0,\rho)})^{p-\delta}\, dx \\
&\qquad - c^{-1} \int_{B(y_0,\frac{\rho}{2})\setminus G} M^m(|\ \partial^m u\ |\chi_{B(y_0,\rho)})^{p-\delta}\, dx \qquad (3.29)\\
&\ge c^{-1} \int_{B(y_0,\frac{\rho}{2})} |\ \partial^m u\ |^{p-\delta}\, dx \\
&\qquad - c\,\rho^n \left(\fint_{B(y_0,8\rho)} M^{2m}(|\ \partial^m u\ |\chi_{B(x_0,4\rho)})\, dx \right)^{p-\delta} \\
&\ge c^{-1} \int_{B(y_0,\frac{\rho}{2})} |\ \partial^m u\ |^{p-\delta} dx - c\rho^n \left(\fint_{B(y_0,8\rho)} |\ \partial^m u\ |^t\, dx \right)^{\frac{p-\delta}{t}} .
\end{aligned}
$$

Next from the definition of D_1 and the Hardy - Littlewood maximal theorem, we see as in (3.21)-(3.22) that

$$
\begin{aligned}
I_2 &= c \int_{D_1} M^m(|\ \partial^m \tilde{u}\ |)^{1-\delta} \,[\,|\ \partial^m u\ |^{p-1} + b_m\,]\, dx \\
&\le c\delta^{1-\delta} \int_{B(y_0,8\rho)} |\ \partial^m u\ |^{p-\delta}\, dx + c \int_{B(y_0,8\rho)} F_3^{p-\delta}\, dx\,.
\end{aligned}
\qquad (3.30)
$$

To estimate I_3 we use Lemma 2.1 with $w = \partial^l (u-P)$, $k = m-l$, and as usual consider two cases. If $p_l = p+\epsilon$, we let $s = \frac{n(1-\alpha)}{m-l}$ in (2.1b) and obtain that

$$
|\ \partial^l (u-P)\ | \le c\rho^{m-l}\, \gamma_1 M^m(|\ \partial^m u\ |\chi_{B(y_0,8\rho)})^{\alpha}
$$

while if $p_l > p+\epsilon$, we let $s = p(1-\alpha)$ in (2.1b) and obtain that

$$
|\ \partial^l (u-P)\ | \le c\rho^{m-l} \gamma_2\, M^m(|\ \partial^m u\ |\chi_{B(y_0,8\rho)})^{\tilde{p}}.
$$

Set

$$
\tilde{f}_l = c\gamma_1(l) M^m(|\ \partial^m u\ |\chi_{B(x_0,\frac{3R}{4})})^{\alpha-\delta}\,(\,|\ \partial^m u\ |^{p-1} + b_m)
$$

when $p_l = p + \epsilon$ and put

$$\tilde{f}_l \,=\, c\gamma_2\, M^m(|\,\partial^m u\,|\chi_{B(x_0,\frac{3R}{4})})^{\tilde{p}-\delta}\,(|\,\partial^m u\,|^{p-1} + b_m)$$

when $p_l > p + \epsilon$. From the above inequalities we see that if $F_4 = F_{4,\delta}$ is defined by $F_4^{p-\delta} = \sum_{l=0}^{m-1} \tilde{f}_l$, then in either case we have

$$\begin{aligned} &(a) \qquad\qquad I_3 \,\le\, \int_{B(y_0,8\rho)} F_4^{p-\delta}\, dx\,, \\ &(b) \qquad\qquad F_4 \,\in\, L^{p+\alpha}(B(x_0,\tfrac{3R}{4}))\,. \end{aligned} \tag{3.31}$$

From (3.28)-(3.31), we conclude that

$$\begin{aligned} \fint_{B(y_0,\frac{\rho}{2})} |\,\partial^m u\,|^{p-\delta}\, dx \,\le\, c\rho^{-n}\, L_3 \,+ \fint_{B(y_0,8\rho)} [a \,+ \sum_{i=3}^{4} F_i^{p-\delta}\,]\, dx \\ +\; c\delta^{1-\delta}\fint_{B(y_0,8\rho)} |\,\partial^m u\,|^{p-\delta} dx + c(\fint_{B(y_0,8\rho)} |\,\partial^m u\,|^t dx\,)^{\frac{p-\delta}{t}}\,. \end{aligned} \tag{3.32}$$

Since $L_3 = L_1 + L_2 + K$ it follows from (3.32), (3.26), and (3.25) that

$$\begin{aligned} \fint_{B(y_0,\frac{\rho}{2})} |\,\partial^m u\,|^{p-\delta}\, dx \,&\le\, c\rho^{-n}\, K \,+ \fint_{B(y_0,8\rho)} [a \,+ \sum_{i=1}^{4} F_i^{p-\delta}\,]\, dx \\ &\quad + c(\delta^{1-\delta} \,+ \eta^{1-\delta})\fint_{B(y_0,8\rho)} |\,\partial^m u\,|^{p-\delta} dx \\ &\quad + c\eta^{1-p}(\fint_{B(y_0,8\rho)} |\,\partial^m u\,|^t dx\,)^{\frac{p-\delta}{t}}\,. \end{aligned} \tag{3.33}$$

which clearly gives an estimate for K from below. Using this estimate in (3.23), we get

$$\begin{aligned} \fint_{B(y_0,\frac{\rho}{2})} |\,\partial^m u\,|^{p-\delta}\, dx \,&\le \fint_{B(y_0,8\rho)} F^{p-\delta}\, dx \\ &\quad + c(\delta^{1-\delta} + \eta^{1-\delta})\fint_{B(y_0,8\rho)} |\,\partial^m u\,|^{p-\delta} dx \\ &\quad + c\eta^{1-p}(\fint_{B(y_0,8\rho)} |\,\partial^m u\,|^t dx\,)^{\frac{p-\delta}{t}}, \end{aligned} \tag{3.34}$$

where we have put $F \,=\, c\,[\sum_{i=1}^{4} F_i \,+\, a^{\frac{1}{p-\delta}}\,]$ for suitably large c. Let $\hat{c}$ denote the constant multiplying $\delta^{1-\delta} + \eta^{1-\delta}$ in (3.34) and put $\eta = (4\hat{c})^{-2}$. If $\delta_1 = \min\{\frac{1}{4}\alpha^2, \eta\}$, then from (3.34) we find for $\delta \le \delta_1$ and c^+ large enough that

$$
\begin{aligned}
\fint_{B(y_0,\frac{\rho}{2})} |\ \partial^m u\ |^{p-\delta}\, dx \ \le\ & \fint_{B(y_0,8\rho)} F^{p-\delta}\, dx \\
+\ \frac{1}{2}\fint_{B(y_0,8\rho)} |\ \partial^m u\ |^{p-\delta} dx\ +\ & c^+ \Big(\fint_{B(y_0,8\rho)} |\ \partial^m u\ \ |^t\, dx\,\Big)^{\frac{p-\delta}{t}},
\end{aligned}
\tag{3.35}
$$

where $t = \frac{1+p}{2}$ is as previously defined. In view of (3.35), we see for given $\delta \le \delta_1$ that the hypotheses of Lemma 2.3 are satisfied with $q = \frac{p-\delta}{t} > 1$, $g = |\ \partial^m u\ |^t$, $f = F^t$, $\theta = \frac{1}{2}$, $\xi = \frac{\alpha}{p}$, and $c_2 = c^+$. Since η in Lemma 2.3 is continuous as a function of q we may choose $\delta_2, 0 < \delta_2 \le \delta_1$ so small that

$$\eta(n, \tfrac{1}{2}, c^+, \xi, q)\ >\ \tfrac{1}{2}\eta(n, \tfrac{1}{2}, c^+, \xi, \tfrac{2p}{1+p}) = \eta_1$$

for $0 < \delta \le \delta_2$. Finally we fix $\delta = \min\{\frac{1}{4}\eta_1, \delta_2\}$. Applying Lemma 2.3 and using the continuity of η, we get Theorem 1 for this δ . $\Box$

REFERENCES

[EM] A. Elcrat and N. Meyers, *Some results on regularity for nonlinear elliptic sytems and quasiregular functions,* Duke Math. J. **42** (1975), 121 - 136.

[G] F. Gehring, *L^p integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265 - 277.

[Gi] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematical Studies **105**, Princeton University Press, 1983.

[GM] M. Giaquinta and G. Modica, *Regularity results for some classes of higher order nonlinear elliptic systems* , J. Reine Angew. Math. **311/312** (1979), 145-169.

[GS] M. Giaquinta and J. Souček, *Caccioppoli's inequality and Legendre -Hadamard condition*, Math. Ann.**270**(1985), no. 1, 105 - 107.

[HR] R. Hager and J. Ross, *A regularity theorem for linear second order elliptic divergence equations* , Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 283- 290.

[H] L. Hedberg, *On certain convolution inequalities,* Proc. Am. Math. Soc. **36**(1972), 505 - 510.

[I] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, to appear.

[S] J. Serrin, *Pathological solutions of elliptic equations*, Ann. Sc. Norm. Sup. Pisa **18** (1964), 385 - 387.

[St] E. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, 1970.

[T] A. Torchinsky, *Real variable methods in harmonic analysis,* Pure and Applied Mathematics **123**, Academic Press, 1986.