Combinatorics, Commutative Algebra and Topology of Simplicial Complexes  
(C-CATS-C)

For a polytope $P$ of dimension $n$, the $f$-vector $(f_0, \ldots, f_{n-1})$ counts the number of $i$-dimensional faces in the polytope. For example, the $f$-vector of an icosahedron is $(12, 30, 20)$.

Already there are many basic questions one can ask about the $f$-vector, such as:

- For fixed dimension and fixed number of vertices, how small can the entries of the $f$-vector be?
- For fixed dimension and fixed number of vertices, how large can the entries of the $f$-vector be?
- Given a vector of entries, is it an $f$-vector of a simplicial polytope (or more generally, of a simplicial complex)?

The proofs of the first two questions, known as the Lower and Upper Bound Theorems, are very geometric. Already the third result, due to Kruskal-Katona, suggests some of the algebraic tools later developed to answer deeper questions about polytopes. We will discuss these three questions during the first third of the course. The middle third will serve as an introduction to commutative algebra techniques for studying polytopes. During the last part of the course, we will show how a noncommutative polynomial called the $cd$-index encodes the flag data of a polytope and how it can be used to prove further results about polytopes and subdivisions of manifolds.

COURSE OUTLINE:

- Introduction to convex polytopes
- Kruskal-Katona Theorem
- Upper and Lower Bound Theorems
- A Friendly Introduction to Commutative Algebra
- The Stanley-Reisner Ring
- Reisner’s Topological Criterion
- Upper Bound Theorem for Spheres
- Discrete Morse theory
- Flag Vectors, the $cd$-index and manifolds

TEXTBOOK:

WEBPAGE:
http://www.math.uky.edu/~jrge/714/