A geometric approach to acyclic orientations*

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Abstract

The set of acyclic orientations of a connected graph with a given sink has a natural poset structure. We give a geometric proof of a result of Jim Propp: this poset is the disjoint union of distributive lattices.

Let G be a connected graph on the vertex set $[\underline{n}] = \{0\} \cup [n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$. Let P denote the collection of acyclic orientations of G, and let P_0 denote the collection of acyclic orientations of G with 0 as a sink. If Ω is an orientation in P with the vertex i as a source, we can obtain a new orientation Ω' with i as a sink by *firing* the vertex i, reorienting all the edges adjacent to i towards i. The orientations Ω and Ω' agree away from i.

A *firing sequence* from Ω to Ω' in P consists of a sequence $\Omega = \Omega_1, \ldots, \Omega_{m+1} = \Omega'$ of orientations and a function $F : [m] \longrightarrow [\underline{n}]$ such that for each $i \in [m]$, the orientation Ω_{i+1} is obtained from Ω_i by firing the vertex $F(i)$. We will abuse language by calling F itself a firing sequence. We make P into a preorder by writing $\Omega \leq \Omega'$ if and only if there is a firing sequence from Ω to Ω'. From the definition it is clear that P is reflexive and transitive. While P is only a preorder, P_0 is a poset. By finiteness, antisymmetry can be verified by showing that firing sequences in P_0 cannot be arbitrarily long. This is a consequence of the fact that neighbors of the distinguished sink 0 cannot fire. The proof depends on the following lemma.

Lemma 1. *Let $F : [m] \longrightarrow [n]$ be a firing sequence for the graph G. If i and j are adjacent vertices in G, then*

$$|F^{-1}(i)| \leq |F^{-1}(j)| + 1.$$

Proof. A vertex can fire only if it is a source. Firing the vertex i reverses the orientation of its edge to the vertex j. Hence the vertex i cannot fire again until the orientation is again reversed, which can only happen by firing j. □

As a corollary, firing sequences have bounded length, implying that P_0 is a poset.

Corollary 2. *The preorder P_0 of acyclic orientations with a distinguished sink is a poset.*

Proof. Let $F : [m] \longrightarrow [n]$ be a firing sequence. By iterating the lemma, $|F^{-1}(i)| \leq d(0, i) - 1$, so

$$m = \sum_{i \in [n]} |F^{-1}(i)| \leq \sum_{i \in [n]} (d(0, i) - 1).$$

Hence firing sequences cannot be arbitrarily long, implying that P_0 is antisymmetric. □

*To appear in *Order*.

For a real number a, let $\lfloor a \rfloor$ denote the largest integer less than or equal to a. Similarly, let $\lceil a \rceil$ denote the least integer greater than or equal to a. Finally, let $\{a\}$ denote the fractional part of the real number a, that is, $\{a\} = a - \lfloor a \rfloor$. (It will be clear from the context if $\{a\}$ denotes the fractional part or the singleton set.) Observe that the range of the function $x \longmapsto \{x\}$ is the half open interval $[0, 1)$.

Let $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(G)$ be the *periodic graphic arrangement* of the graph G, that is, $\widetilde{\mathcal{H}}$ is the collection of all hyperplanes of the form

$$x_i = x_j + k,$$

where ij is an edge in the graph G and k is an integer. This hyperplane arrangement cuts $\mathbb{R}^{n+1}$ into open regions. Note that each region is translation-invariant in the direction $(1, \ldots, 1)$. Let C denote the complement of $\widetilde{\mathcal{H}}$, that is,

$$C = \mathbb{R}^{n+1} \setminus \bigcup_{H \in \widetilde{\mathcal{H}}} H.$$

Define a map $\varphi : C \longrightarrow P$ from the complement of the periodic graphic arrangement to the preorder of acyclic orientations as follows. For a point $x = (x_0, \ldots, x_n)$ and an edge ij observe that $\{x_i\} \neq \{x_j\}$ since the point does not lie on any hyperplane of the form $x_i = x_j + k$. Hence orient the edge ij towards i if $\{x_i\} < \{x_j\}$ and towards j if the inequality is reversed. This defines the orientation $\varphi(x)$. Also note that this is an acyclic orientation, since no directed cycles can occur.

Let H_0 be the coordinate hyperplane $\{x \in \mathbb{R}^{n+1} \; : \; x_0 = 0\}$. The map φ sends points of the intersection $C_0 = C \cap H_0$ to acyclic orientations in P_0.

The real line $\mathbb{R}$ is a distributive lattice; meet is minimum and join is maximum. Since $\mathbb{R}^{n+1}$ is a product of copies of $\mathbb{R}$, it is also a distributive lattice, with meet and join given by componentwise minimum and maximum. That is, given two points in $\mathbb{R}^n$, say $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$, their meet and join are given by

$$x \wedge y = (\min(x_0, y_0), \ldots, \min(x_n, y_n))$$

and

$$x \vee y = (\max(x_0, y_0), \ldots, \max(x_n, y_n))$$

respectively.

Lemma 3. *Each region R in the complement C of the periodic graphic arrangement $\widetilde{\mathcal{H}}$ is a distributive sublattice of $\mathbb{R}^{n+1}$. Hence the intersection $R \cap H_0$, which is a region in C_0, is also a distributive sublattice of $\mathbb{R}^{n+1}$.*

Proof. Since each region R is the intersection of slices of the form

$$T = \{x \in \mathbb{R} \; : \; x_i + k < x_j < x_i + k + 1\},$$

it is enough to prove that each slice is a sublattice of $\mathbb{R}^{n+1}$. Let x and y be two points in the slice T. Then $\min(x_i, y_i) + k = \min(x_i + k, y_i + k) < \min(x_j, y_j) < \min(x_i + k + 1, y_i + k + 1) = \min(x_i, y_i) + k + 1$, implying that $x \wedge y$ also lies in the slice T. A dual argument shows that the slice T is closed under the join operation. Thus the region R is a sublattice. Since distributivity is preserved under taking sublattices, it follows that R is a distributive sublattice of $\mathbb{R}^{n+1}$. $\square$

In the remainder of this paper we let R be a region in C_0.

Lemma 4. *Consider the restriction $\varphi|_R$ of the map φ to the region R. The inverse image of an acyclic orientation in P_0 is of the form:*

$$R \cap \left(\{0\} \times \prod_{i=1}^{n} [a_i, a_i + 1) \right),$$

where each a_i is an integer. That is, the inverse image of an orientation is the intersection of the region R with a half-open lattice cube. Hence the inverse image is a sublattice of $\mathbb{R}^{n+1}$.

Proof. Assume that x and y lie in the region R. Define the integers a_i and b_i by $a_i = \lfloor x_i \rfloor$ and $b_i = \lfloor y_i \rfloor$. Hence the coordinate x_i lies in the half-open interval $[a_i, a_i + 1)$ and the coordinate y_i lies in the half-open interval $[b_i, b_i + 1)$. Lastly, assume that $\varphi|_R$ maps x and y to the same acyclic orientation. The last condition implies that, for every edge ij, $0 \leq x_i - a_i < x_j - a_j < 1$ is equivalent to $0 \leq y_i - b_i < y_j - b_j < 1$. Consider an edge that is directed from j to i. Since x and y both lie in the region R, there exists an integer k such that $x_i + k < x_j < x_i + k + 1$ and $y_i + k < y_j < y_i + k + 1$. Now we have that $a_j - a_i < x_j - x_i < k + 1$. Furthermore, observe that $x_j - a_j - 1 < 0 \leq x_i - a_i$. Hence $a_j - a_i > x_j - x_i - 1 > k - 1$. Since $a_j - a_i$ is an integer, the two bounds implies that $a_j - a_i = k$. By similar reasoning we obtain that $b_j - b_i = k$.

Hence for every edge ij we know that $a_j - a_i = b_j - b_i$. Since $a_0 = b_0 = 0$ and the graph G is connected we obtain that $a_i = b_i$ for all vertices i. □

Lemma 5. *The restriction $\varphi|_R : R \longrightarrow P_0$ is a poset homomorphism, that is, for two points y and z in the region R such that $y \leq z$ the order relation $\varphi(y) \leq \varphi(z)$ holds.*

Proof. Since the region R is convex, the line segment from y to z is contained in R. Let a point x move continuously from y to z along this line segment and consider what happens with the associated acyclic orientations $\varphi(x)$. Note that each coordinate x_i is non-decreasing. When the point x crosses a hyperplane of the form $x_i = p$ where p is an integer, observe that the value $\{x_i\}$ approaches 1 and then jumps down to 0. Hence the vertex i switches from being a source to being a sink, that is, the vertex i fires.

Observe that two adjacent nodes i and j cannot fire at the same time, since the intersection of the two hyperplanes $x_i = p$ and $x_j = q$ is contained in the hyperplane $x_i = x_j + (p - q)$ which is not in the region R.

Hence we obtain a firing sequence from the acyclic orientation $\varphi(y)$ to $\varphi(z)$, proving that $\varphi(y) \leq \varphi(z)$. □

Lemma 6. *Let x be a point in the region R. Let Ω' be an acyclic orientation comparable to $\Omega = \varphi(x)$ in the poset P_0. Then there exists a point z in the region of R as x such that $\varphi(z) = \Omega'$.*

Proof. It is enough to prove this for cover relations in the poset P. We begin by considering the case when Ω' covers Ω in P. Thus Ω' is obtained from Ω by firing a vertex i.

First pick a positive real number λ such that $\{x_j\} < 1 - \lambda$ for each nonzero vertex j. Let y be the point $y = x + \lambda \cdot (0, 1, \ldots, 1)$. Observe that y belongs to the same region R and that φ maps y to the same acyclic orientation as the point x.

Since i is a source in Ω, the value $\{y_i\}$ is larger than any other value $\{y_j\}$ for vertexes j adjacent to the vertex i. Let z be the point with coordinates $z_j = y_j$ for $j \neq i$ and $z_i = \lceil y_i \rceil + \lambda/2$. Observe that moving from y to the point z we do not cross any hyperplanes of the form $x_i = x_j + k$. Hence the point z also belongs to region R.

However, we did cross a hyperplane of the form $x_i = p$, corresponding to firing the vertex i. Hence we have that $\varphi(z) = \Omega'$. Now we can iterate this argument to extend to the general case when $\Omega < \Omega'$.

The case when Ω' is covered by Ω is done similarly. However this case is easier since one can skip the middle step of defining the point y. Hence this case is omitted. $\square$

A connected component of a finite poset is a weakly connected component of its associated comparability graph. That is, a finite poset is the disjoint union of its connected components.

Lemma 7. *Let Q be a connected component of the poset of acyclic orientations P_0. Then there exists a region R in C_0 such that the map φ maps R onto the component Q.*

Proof. Let Ω be an orientation in the component Q. Since φ is surjective we can lift Ω to a point x in C_0. Say that the point x lies in the region R. It is enough to show that every orientation Ω' in Q can be lifted to a point in R. The two orientations Ω and Ω' are related by a sequence in Q of orientations $\Omega = \Omega_1, \Omega_2, \ldots, \Omega_k = \Omega'$ such that Ω_i and Ω_{i+1} are comparable. By iterating Lemma 6 we obtain points x_i in R such that $\varphi(x_i) = \Omega_i$. In particular, $\varphi(x_k) = \Omega'$. $\square$

Proposition 8. *Let Q be a connected component of the poset of acyclic orientations P_0. Then the component Q as a poset is a lattice. Moreover, let R be a region of C_0 that maps onto Q by φ. Then the poset map $\varphi|_R : R \longrightarrow Q$ is a lattice homomorphism.*

Proof. The previous discussion showed that we can lift the component Q to a region R. Consider two acyclic orientations Ω and Ω'. We can lift them to two points x and y in R, that is, $\varphi(x) = \Omega$ and $\varphi(y) = \Omega'$. Since $\varphi|_R$ is a poset map we obtain that $\varphi(x \wedge y)$ is a lower bound for Ω and Ω'. It remains to show that the lower bound is unique.

Assume that Ω'' is a lower bound of Ω and Ω'. By Lemma 6 we can lift Ω'' to an element z in R such that $z \leq x$. Similarly, we can lift Ω'' to an element w in R such that $w \leq y$. That is we have that $\varphi(z) = \varphi(w) = \Omega''$. Now by Lemma 4 we have that $\varphi(z \wedge w) = \Omega''$. But since $z \wedge w$ is a lower bound of both x and y we have that $z \wedge w \leq x \wedge y$. Now applying φ we obtain that $\varphi(x \wedge y)$ is the greatest lower bound, proving that the meet is well-defined. A dual argument shows that the join is well-defined, hence Q is a lattice.

Finally, we have to show that $\varphi|_R$ is a lattice homomorphism. Let x and y be two points in the region R. By Lemma 6 we can lift the inequality $\varphi(x) \wedge \varphi(y) \leq \varphi(x)$ to obtain a point z in R such that $z \leq x$ and $\varphi(z) = \varphi(x) \wedge \varphi(y)$. Similarly, we can lift the inequality $\varphi(x) \wedge \varphi(y) \leq \varphi(y)$ to obtain a point w in R such that $w \leq y$ and $\varphi(w) = \varphi(x) \wedge \varphi(y)$. By Lemma 4 we know that $\varphi(z \wedge w) = \varphi(x) \wedge \varphi(y)$. But $z \wedge w$ is a lower bound of both x and y, so $\varphi(x) \wedge \varphi(y) = \varphi(z \wedge w) \leq \varphi(x \wedge y)$. But since $\varphi(x \wedge y)$ is a lower bound of both $\varphi(x)$ and $\varphi(y)$ we have $\varphi(x \wedge y) \leq \varphi(x) \wedge \varphi(y)$. Thus the map $\varphi|_R$ preserves the meet operation. The dual argument proves that $\varphi|_R$ preserves the join operation, proving that it is a lattice homomorphism. $\square$

Combining these results we can now prove the result of Propp [7].

Theorem 9. *Each connected component of the poset of acyclic orientations P_0 is a distributive lattice.*

Proof. It is enough to recall that $\mathbb{R}^{n+1}$ is a distributive lattice and each region R is a sublattice. Furthermore, the image under a lattice morphism of a distributive lattice is also distributive. $\square$

Observe that the minimal element in each connected component Q is an acyclic orientation with the unique sink at the vertex 0. Greene and Zaslavsky [4] proved that the number of such orientations is given by the sign -1 to the power one less than the number of vertices times the linear coefficient in the chromatic polynomial of the graph G. Gebhard and Sagan gave several proofs of this result [3]. A geometric proof of this result can be found in [2], where the authors view the graphical hyperplane arrangement on a torus and count the regions on the torus.

That the connected components are confluent, that is, each pair of elements has a lower and an upper bound, can also be shown by analyzing chip-firing games [1]. Is there a geometric way to prove the confluency of chip-firing? More discussions relating these distributive lattice with chip-firing can be found in [5, 6].

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