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The Asymptotics of Almost Alternating Permutations

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The goal of this paper is to study the asymptotic behavior of almost alternating permutations, that is, permutations that are alternating except for a finite number of exceptions. Let $\beta(l_1, \ldots, l_k)$ denote the number of permutations which consist of l_1 ascents, l_2 descents, l_3 ascents, and so on. By combining the Viennot triangle and the boustrophedon transform, we obtain the exponential generating function for the numbers $\beta(L, 1^{n-m-1})$, where L is a descent–ascent list of size m. As a corollary we have $\beta(L, 1^{n-m-1}) \sim c(L) \cdot E_n$, where $E_n = \beta(1^{n-1})$ denotes the nth Euler number and $c(L)$ is a constant depending on the list L. Using these results and inequalities due to Ehrenborg–Mahajan, we obtain $\beta(1^a, 2, 1^b) \sim 2/\pi \cdot E_n$, when $\min(a, b)$ tends to infinity and where $n = a + b + 3$. From this result we obtain that the asymptotic behavior of $\beta(L_1, 1^a, L_2, 1^b, L_3)$ is the product of three constants depending respectively on the lists L_1, L_2, and L_3, times the Euler number $E_{a+b+m+1}$, where m is the sum of the sizes of the L_i's.

Key Words: boustrophedon transform; Euler number; Viennot triangle.

1. INTRODUCTION

A permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ in the symmetric group on n elements is called *alternating* if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$. The number of such permutations are enumerated by the nth Euler number, which has the asymptotic expression $4/\pi \cdot (2/\pi)^n \cdot n!$. The problem we study in this paper is the asymptotic behavior of the number of permutations that are almost alternating; that is, we allow permutations with a finite number of exceptions to the alternating property.

To explain the question better, let us review the notion of descent set and the encoding of sets as lists. For a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, the

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descent set of σ is the set

$$\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i-1}\}.$$

Observe that the descent set is a subset of $\{1, \ldots, n-1\}$. For a subset S of $\{1, \ldots, n-1\}$, we define the *descent set statistic* $\beta(S)$ by

$$\beta(S) = |\{\sigma \in \mathcal{S}_n : \ \mathrm{Des}(\sigma) = S\}|.$$

There are two symmetries in this statistic. For S a subset of $\{1, \ldots, n-1\}$, let $\overline{S}$ denote the complementary set $\{1, \ldots, n-1\} - S$ and S^* the reverse set $\{i : n - i \in S\}$. Then we have $\beta(S) = \beta(\overline{S}) = \beta(S^*)$.

A *run* in a subset S is a maximal sequence of consecutive integers either all in the subset S or all not in S. For instance, for $S = \{1, 2, 6\} \subseteq \{1, \ldots, 9\}$ we have four runs, namely, $\{1, 2\}$, $\{3, 4, 5\}$, $\{6\}$, and $\{7, 8, 9\}$. We may encode a subset S by writing down the lengths of the runs in a list L in the order they occur. In our example, we obtain the list $L = (2, 3, 1, 3)$. Observe that this list also encodes the complementary set $\{3, 4, 5, 7, 8, 9\}$. Thus define the descent set statistic $\beta(L)$ of a list L to be $\beta(S)$, where L is the list that encodes S and $\overline{S}$. As another example, observe

$$\beta(\underbrace{1, \ldots, 1}_{n-1}) = \beta(1^{n-1})$$

is the nth Euler number E_n.

It is a classic result that $\beta(S)$ is maximized on the two alternating sets, that is, corresponding to the list (1^{n-1}). Many authors have been attracted to this problem; see [1, 10–12, 14]. Ehrenborg and Mahajan developed more refined inequalities to prove a conjecture of Ira Gessel. Loosely stated, when $\beta(L)$ is maximized over lists of a fixed length, the maximum occurs when the runs (the entries in the list) are roughly the same size [4].

Thus the question at hand is to determine the asymptotic behavior of $\beta(L, 1^b)$ and $\beta(1^a, L, 1^b)$ when a and b approach infinity. The answer, given in Theorems 3.2 and 5.1, is that their asymptotics differ by a constant multiple of the Euler number.

The method to prove Theorem 3.2 is to combine the Viennot triangle for computing the descent set statistic $\beta(S)$ with the boustrophedon transform of Millar *et al.* [9]. However, the two-sided result in Theorem 5.1 is more intricate. The key is to understand the asymptotic behavior of $\beta(1^a, 2, 1^b)$. Theorem 5.4 in [4] offers a complete description of the inequalities holding between these values. These inequalities are the essential step for proving the asymptotic result that $\beta(1^a, 2, 1^b) \sim 2/\pi \cdot E_n$; see Theorem 4.1.

In Section 2 we review basic material on the descent set statistic and Euler numbers. In the next section we consider the one-sided list $(L, 1^b)$ and obtain a generating function answer using the boustrophedon transform. In Section 4 we consider the particular list $(1^a, 2, 1^b)$ which provides the essential piece to complete the study of the lists $(1^a, L, 1^b)$, $(L_1, 1^a, L_2)$, and $(L_1, 1^a, L_2, 1^b, L_3)$ in Section 6. We end with concluding remarks about further questions to study.

2. PRELIMINARIES

We briefly review the notion of lists. See [4] for more details. Let $L = (l_1, \ldots, l_k)$ be a list. The size of the list $|L|$ is the sum of the entries; that is, $|L| = l_1 + \cdots + l_k$. Let s_i denote the partial sum $l_1 + \cdots + l_i$. The list encodes two complementary subsets of $\{1, \ldots, |L|\}$, namely

$$\{1, \ldots, s_1, s_2 + 1, \ldots, s_3, \ldots\}$$

and

$$\{s_1 + 1, \ldots, s_2, s_3 + 1, \ldots, s_4, \ldots\}.$$

Since the descent set statistic of both these sets are equal, the notion of $\beta(L)$ is well defined. The concatenation of two lists L and K is written as (L, K) and the descent set statistic of this combined list is written as $\beta(L, K)$. The list L repeated a times is denoted by L^a. Thus the list $(1^3, L, 1^2, 2)$ is the list $(1, 1, 1, l_1, \ldots, l_k, 1, 1, 2)$. The *reverse* of the list L is $L^* = (l_k, \ldots, l_1)$. The *parity* of a list L is even if the size minus the number of entries of the list is an even integer; otherwise the parity is odd.

A useful tool for working with the descent set statistic is MacMahon's multiplication theorem [8, Article 159]. Let S be a subset of $\{1, \ldots, n - 1\}$ and let T be a subset of $\{1, \ldots, m - 1\}$. MacMahon's multiplication theorem states that

$$\binom{n+m}{n} \cdot \beta(S) \cdot \beta(T) = \beta(S \cup (T + n)) + \beta(S \cup \{n\} \cup (T + n)), \quad (2.1)$$

where $T + n$ denotes the shifted set $T + n = \{t + n : t \in T\}$. For lists this statement breaks into two identities. Let (L, l) be a list of size $n - 1$ and (k, K) be a list of size $m - 1$. Then

$$\binom{n+m}{n} \cdot \beta(L, l) \cdot \beta(k, K) = \beta(L, l + k + 1, K) + \beta(L, l, 1, k, K) \quad (2.2)$$

$$= \beta(L, l + 1, k, K) + \beta(L, l, k + 1, K). \quad (2.3)$$

Also observe that $\binom{n+1}{1} \cdot \beta(L, l) = \beta(L, l + 1) + \beta(L, l, 1)$.

To obtain the results of Section 4, we need to understand the values $\beta(1^a, 2, 1^b)$. Theorem 5.4 in [4] studies a more general situation, which we now state.

THEOREM 2.1 (Ehrenborg–Mahajan). *Let P be a palindrome; that is, $P = P^*$. Let k be a positive integer smaller than the first (and last) entry of P. Then we have the following string of inequalities.*

$$\beta(k, P, k^{n-1}) \geq \beta(k^3, P, k^{n-3}) \geq \cdots \geq \beta\left(k^{\lfloor n/2 \rfloor}, P, k^{\lceil n/2 \rceil}\right)$$
$$\geq \cdots \geq \beta(k^2, P, k^{n-2}) \geq \beta(P, k^n).$$

The special case of interest for us is when $k = 1$ and P is the list (2).

Recall that the nth Euler number is the value $\beta(1^{n-1})$. The exponential generating function for the Euler numbers is given by

$$\sum_{n \geq 0} E_n \cdot \frac{x^n}{n!} = \sec(x) + \tan(x).$$

By combining the following four equations appearing in [5, that is, (5) and (6) in Sect. 0.233, p. 9 and (5) and (9) in Sect. 1.411, p. 42], and letting $s(i)$ denote the sign $(-1)^{(i-1)/2}$, we obtain a series for the Euler numbers.

$$E_n = 2 \cdot \left(\frac{2}{\pi}\right)^{n+1} \cdot n! \cdot \sum_{\substack{i \geq 1 \\ i \text{ odd}}} \frac{1}{(s(i) \cdot i)^{n+1}}$$
$$= 2 \cdot \left(\frac{2}{\pi}\right)^{n+1} \cdot n! \cdot \left(1 + \frac{1}{(-3)^{n+1}} + \frac{1}{5^{n+1}} + \frac{1}{(-7)^{n+1}} + \cdots\right). \tag{2.4}$$

Observe that the sum $1 + 1/(-3)^{n+1} + 1/5^{n+1} + \cdots$ converges for all non-negative integers n. A probabilistic proof of Eq. (2.4) appears in [3]. By considering only the first term in (2.4), we obtain the following asymptotic expression for the Euler numbers E_n:

$$E_n \sim \frac{4}{\pi} \cdot \left(\frac{2}{\pi}\right)^n \cdot n!. \tag{2.5}$$

By including a few more terms from Eq. (2.4), we obtain quite good approximations for the Euler numbers. For instance, the approximation $E_{19} \approx 2 \cdot (2/\pi)^{20} \cdot 19! \cdot (1 + 1/3^{20})$ has an error term less than 1/3.

3. ONE-SIDED ALTERNATING

We now study the behavior of the number sequence $\beta(L, 1^{n-m-1})$. Using the boustrophedon transform, we first obtain the exponential generating function and then conclude the desired asymptotic result.

THEOREM 3.1. *For a list L of size m, there exist two polynomials $p_L(x)$ and $q_L(x)$ of degree at most $m-1$ such that*

$$\sum_{n\geq m+1} \beta(L, 1^{n-m-1}) \cdot \frac{x^n}{n!} = p_L(x) \cdot (\sec(x) + \tan(x)) - q_L(x).$$

To present a proof of Theorem 3.1, we will introduce some notation and present results due to Viennot [14] and Millar *et al.* [9]. Our notation will differ from what was used in [4]. Define the affine maps $\overrightarrow{\Sigma}_x$ and $\overleftarrow{\Sigma}_x$ from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ by

$$\overrightarrow{\Sigma}_x(\mathbf{x}) = (x, x + x_1, x + x_1 + x_2, \ldots, x + x_1 + \cdots + x_n),$$
$$\overleftarrow{\Sigma}_x(\mathbf{x}) = (x + x_1 + \cdots + x_n, \ldots, x + x_{n-1} + x_n, x + x_n, x).$$

Observe that the arrows indicate in which direction the entries are added. Let $\overrightarrow{\Sigma}$ and $\overleftarrow{\Sigma}$ denote the linear maps $\overrightarrow{\Sigma}_0$ and $\overleftarrow{\Sigma}_0$.

Let S be a subset of $\{1, \ldots, n-1\}$. Viennot developed the following method to compute $\beta(S)$. Define a sequence of vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$, where $\mathbf{v}_i \in \mathbb{R}^{i+1}$, by $\mathbf{v}_0 = (1)$ and

$$\mathbf{v}_i = \begin{cases} \overrightarrow{\Sigma}(\mathbf{v}_{i-1}) & \text{if } i \notin S, \\ \overleftarrow{\Sigma}(\mathbf{v}_{i-1}) & \text{if } i \in S. \end{cases}$$

Then $\beta(S)$ is equal to the sum of the entries of the vector $\mathbf{v}_{n-1}$. As an example consider $S = \{3, 5, 6\} \subseteq \{1, \ldots, 6\}$ in Fig. 1. The arrows in the figure show the direction of the addition.

Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The *boustrophedon transform* of this sequence is the sequence $b_0, b_1, b_2, \ldots$ defined as follows. Let $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ be a sequence of vectors such that $\mathbf{w}_i \in \mathbb{R}^{i+1}$, $\mathbf{w}_0 = (a_0)$, and

$$\mathbf{w}_i = \begin{cases} \overrightarrow{\Sigma}_{a_i}(\mathbf{w}_{i-1}) & \text{if } i \text{ odd}, \\ \overleftarrow{\Sigma}_{a_i}(\mathbf{w}_{i-1}) & \text{if } i \text{ even}. \end{cases}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & & & & & \\
 & & & & 0 & \rightarrow & 0 & \rightarrow & 1 & & & & \\
 & & & 1 & \leftarrow & 1 & \leftarrow & 1 & \leftarrow & 0 & & & \\
 & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 3 & & \\
 & 9 & \leftarrow & 9 & \leftarrow & 8 & \leftarrow & 6 & \leftarrow & 3 & \leftarrow & 0 & \\
35 & \leftarrow & 26 & \leftarrow & 17 & \leftarrow & 9 & \leftarrow & 3 & \leftarrow & 0 & \leftarrow & 0
\end{array}$$

FIG. 1. Example of the Viennot triangle. Here $S = \{3, 5, 6\} \subseteq \{1, \ldots, 6\}$ and $\beta(S) = 35 + 26 + 17 + 9 + 3 + 0 + 0 = 90$.

$$
\begin{array}{c}
1 \\
0 \rightarrow 1 \\
0 \leftarrow 0 \leftarrow -1 \\
1 \rightarrow 1 \rightarrow 1 \rightarrow 0 \\
0 \leftarrow -1 \leftarrow -2 \leftarrow -3 \leftarrow -3 \\
9 \rightarrow 9 \rightarrow 8 \rightarrow 6 \rightarrow 3 \rightarrow 0 \\
35 \leftarrow 26 \leftarrow 17 \leftarrow 9 \leftarrow 3 \leftarrow 0 \leftarrow 0
\end{array}
$$

FIG. 2. The boustrophedon transform of the sequence $1, 0, -1, 1, -3, 9, 0, \ldots$.

Let b_i be the number at the opposite end of a_i in $\mathbf{w}_i$; that is, if i is even then b_i is the first entry of $\mathbf{w}_i$; otherwise it is the last entry of $\mathbf{w}_i$. The boustrophedon transform is best visualized as building a triangular array, where the rows are added in alternate directions. For an example, see Fig. 2.

Millar *et al*. [9] proved the generating function relation

$$\sum_{n \geq 0} b_n \cdot \frac{x^n}{n!} = (\sec(x) + \tan(x)) \cdot \sum_{n \geq 0} a_n \cdot \frac{x^n}{n!}. \tag{3.1}$$

In [4] there is a generalized version of the boustrophedon transform where the direction of addition is not necessarily alternating. This extends both the boustrophedon transform of Millar *et al*. and the Viennot triangle.

We now prove Theorem 3.1.

Proof (Theorem 3.1). There are two subsets of $\{1, \ldots, m+1\}$ corresponding to the list L, namely the two complementary subsets. Let S be the subset such that if $m+1$ is even then $m+1 \in S$; otherwise $m+1 \notin S$. Let T be the set

$$T = S \cup \{i : i \geq m+2 \text{ and } i \text{ is even}\}.$$

The values $\beta(L, 1^{n-m-1})$ can now be computed using the Viennot triangle. Let $\mathbf{v}_0 = (1)$ and

$$\mathbf{v}_i = \begin{cases} \overrightarrow{\Sigma}(\mathbf{v}_{i-1}) & \text{if } i \notin T, \\ \overleftarrow{\Sigma}(\mathbf{v}_{i-1}) & \text{if } i \in T. \end{cases}$$

Observe that the value $\beta(L, 1^{n-m-1})$ appears as the largest entry in the vector $\mathbf{v}_n$.

We now derive the values $\beta(L, 1^{n-m-1})$ from the boustrophedon transform by constructing the appropriate triangular array backward. Define the two difference maps $\overrightarrow{\delta}$ and $\overleftarrow{\delta}$ from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ by

$$\overrightarrow{\delta}(\mathbf{x}) = (x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}),$$

$$\overleftarrow{\delta}(\mathbf{x}) = (x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n).$$

Observe that $\overrightarrow{\delta}(\overrightarrow{\Sigma}_x(\mathbf{x})) = \overleftarrow{\delta}(\overleftarrow{\Sigma}_x(\mathbf{x})) = \mathbf{x}$.

For $i \geq m$ let $\mathbf{w}_i = \mathbf{v}_i$ and for $0 \leq i \leq m-1$ let

$$\mathbf{w}_i = \begin{cases} \overrightarrow{\delta}(\mathbf{w}_{i+1}) & \text{if } i \text{ is even,} \\ \overleftarrow{\delta}(\mathbf{w}_{i+1}) & \text{if } i \text{ is odd.} \end{cases}$$

Observe that the sequence of vectors $\mathbf{w}_0, \mathbf{w}_1, \ldots$ forms the triangular array of the boustrophedon transform.

Since $\overrightarrow{\delta}(\mathbf{x}) = -\overleftarrow{\delta}(\mathbf{x})$, the vector $\mathbf{w}_i$ only differs from $\mathbf{v}_i$ by a sign. In fact, we have

$$\mathbf{w}_i = (-1)^{|(S \,\triangle\, A_{m+1}) \cap \{i+1, \ldots, m\}|} \cdot \mathbf{v}_i,$$

where $A_{m+1} = \{2, 4, \ldots\} \cap \{1, \ldots, m\}$ and $\triangle$ denotes the symmetric difference. Let $p_0, \ldots, p_{m-1}$ and $q_0, \ldots, q_{m-1}$ be defined by

$$p_i = \begin{cases} \alpha(\mathbf{w}_i) & \text{if } i \text{ is odd,} \\ \omega(\mathbf{w}_i) & \text{if } i \text{ is even,} \end{cases} \quad \text{and} \quad q_i = \begin{cases} \omega(\mathbf{w}_i) & \text{if } i \text{ is odd,} \\ \alpha(\mathbf{w}_i) & \text{if } i \text{ is even,} \end{cases}$$

where α selects the first entry in the vector and ω the last entry. Set $p_L(x) = p_0 + p_1 \cdot x + \cdots + p_{m-1} \cdot x^{m-1}/(m-1)!$ and $q_L(x) = q_0 + \cdots + q_{m-1} \cdot x^{m-1}/(m-1)!$. By Eq. (3.1), we obtain the result. ∎

THEOREM 3.2. *Let L be a list of size m with $p_L(x) = p_0 + \cdots + p_{m-1} \cdot x^{m-1}/(m-1)!$. Then*

$$\beta(L, 1^{n-m-1}) = \sum_{i=0}^{m-1} \binom{n}{i} \cdot p_i \cdot E_{n-i},$$

for $n \geq m+1$. Moreover, the asymptotic behavior of $\beta(L, 1^{n-m-1})$ is given by

$$\beta(L, 1^{n-m-1}) \sim p_L(\pi/2) \cdot E_n,$$

as n tends to infinity.

Proof. The first statement follows by expanding the generating function appearing in Theorem 3.1. The asymptotic behavior follows from $\binom{n}{i} \cdot E_{n-i} \sim (\pi/2)^i/i! \cdot E_n$. ∎

Throughout we denote the constant $p_L(\pi/2)$ appearing in Theorem 3.2 by $c(L)$.

As an example, consider the list $L = (2, 1, 1, 2)$ and the corresponding set $S = \{3, 5, 6\}$. The Viennot triangle appears in Fig. 1 and the triangle corresponding to the boustrophedon transform in Fig. 2. Hence we obtain $p_{(2,1,1,2)}(x) = 1 - x^2/2! + x^3/3! - 3 \cdot x^4/4! + 9 \cdot x^5/5!$. Thus

$$c(2, 1, 1, 2) = 1 - 1/8 \cdot \pi^2 + 1/48 \cdot \pi^3 - 1/128 \cdot \pi^4 + 3/1280 \cdot \pi^5.$$

As a corollary to Theorem 3.2, we have the following result.

COROLLARY 3.1. *The value of* $\beta(1^a, 2, 1^{n-a-3})$ *is given by*

$$\beta(1^a, 2, 1^{n-a-3}) = \sum_{i=0}^{a+1} (-1)^{a+1-i} \cdot \binom{n}{i} \cdot E_i \cdot E_{n-i},$$

and the asymptotic behavior, as n approaches infinity, is given by

$$\beta(1^a, 2, 1^{n-a-3}) \sim \left(\sum_{i=0}^{a+1} (-1)^{a+1-i} \cdot \frac{E_i}{i!} \cdot \left(\frac{\pi}{2}\right)^i \right) \cdot E_n.$$

4. THE ASYMPTOTICS OF $\beta(1^A, 2, 1^B)$

In this section we will consider the asymptotic behavior of the expression $\beta(1^a, 2, 1^b)$ as both a and b approach infinity. The basic constants appearing in the resulting asymptotic expressions are the constants $c(1^a, 2)$. These were computed explicitly in Corollary 3.1 and we denote them by r_a; that is, $r_a = c(1^a, 2)$.

LEMMA 4.1. *For n an even integer we have*

$$\beta\left(1^{n/2-1}, 2, 1^{n/2-2}\right) = \frac{1}{2} \cdot \binom{n}{n/2} \cdot (E_{n/2})^2 \sim \frac{2}{\pi} \cdot E_n.$$

Proof. By MacMahon's multiplication theorem (2.3), we know that

$$\begin{aligned}\binom{n}{n/2} \cdot E_{n/2}^2 &= \beta\left(1^{n/2-1}, 2, 1^{n/2-2}\right) + \beta\left(1^{n/2-2}, 2, 1^{n/2-1}\right) \\ &= 2 \cdot \beta\left(1^{n/2-1}, 2, 1^{n/2-2}\right).\end{aligned}$$

Thus the asymptotic behavior of $\beta(1^{n/2-1}, 2, 1^{n/2-2})$ follows by applying Eq. (2.5). ∎

PROPOSITION 4.1. *For the constants* $r_a = c(1^a, 2)$, *we have the inequalities*

$$r_1 \geq r_3 \geq r_3 \geq \cdots \geq \frac{2}{\pi} \geq \cdots \geq r_4 \geq r_2 \geq r_0.$$

Proof. Let n be an even integer. By Theorem 2 we know that

$$\begin{aligned}\beta(1, 2, 1^{n-4}) \geq \beta(1^3, 2, 1^{n-6}) \geq \cdots &\geq \beta\left(1^{n/2-1}, 2, 1^{n/2-2}\right) \\ &\geq \cdots \geq \beta(1^2, 2, 1^{n-5}) \geq \beta(2, 1^{n-3}).\end{aligned}$$

Dividing this string of inequalities by the Euler number E_n and letting n approach infinity give the stated inequalities. ∎

As a consequence of Proposition 4.1, we have that

$$\lim_{\substack{a\to\infty \\ a \text{ is odd}}} r_a = \limsup_{a\longrightarrow\infty} r_a \quad \text{and} \quad \lim_{\substack{a\to\infty \\ a \text{ is even}}} r_a = \liminf_{a\longrightarrow\infty} r_a.$$

Observe that $r_{a-1} + r_a = E_{a+1}/(a+1)! \cdot (\pi/2)^{a+1}$, which tends to $4/\pi$ as a approaches infinity. Hence $\limsup_{a\to\infty} r_a + \liminf_{a\to\infty} r_a = 4/\pi$.

PROPOSITION 4.2. *The sequence $r_0, r_1, r_2, \ldots$ converges to the limit $2/\pi$.*

Proof. By the previous paragraph, it is enough to show that $r_1, r_3, r_5, \ldots$ converges to $2/\pi$. Let a be odd and let $k = (a+1)/2$. Then we have

$$\begin{aligned} \frac{\pi}{4} \cdot (r_a - 1) &= \frac{\pi}{4} \cdot \sum_{i=1}^{a+1} (-1)^i \cdot \frac{E_i}{i!} \cdot \left(\frac{\pi}{2}\right)^i \\ &= \sum_{j=1}^{k} \left(\frac{\pi}{4} \cdot \frac{E_{2j}}{(2j)!} \cdot \left(\frac{\pi}{2}\right)^{2j} - \frac{\pi}{4} \cdot \frac{E_{2j-1}}{(2j-1)!} \cdot \left(\frac{\pi}{2}\right)^{2j-1} \right). \end{aligned}$$

By Eq. (2.4) the term inside the sum can be written as

$$\sum_{\substack{i\geq 1 \\ i \text{ odd}}} \frac{1}{s(i) \cdot i^{2j+1}} - \sum_{\substack{i\geq 1 \\ i \text{ odd}}} \frac{1}{i^{2j}}.$$

Observe that these sums are absolutely convergent, and hence we can rearrange the terms as

$$\sum_{\substack{i\geq 1 \\ i \text{ odd}}} \left(\frac{1}{s(i) \cdot i^{2j+1}} - \frac{1}{i^{2j}} \right) = \sum_{\substack{i\geq 3 \\ i \text{ odd}}} \frac{1 - s(i) \cdot i}{s(i) \cdot i^{2j+1}}.$$

For $i \geq 3$, i odd, and $j \geq 1$, let

$$x_{i,j} = \frac{1 - s(i) \cdot i}{s(i) \cdot i^{2j+1}}.$$

Thus $\pi/4 \cdot (r_a - 1) = \sum_{j=1}^{k} \sum_{i\geq 3, i \text{ odd}} x_{i,j}$. We would like to compute the limit of this expression as k approaches infinity. Observe that

$$\begin{aligned} \sum_{\substack{i\geq 3 \\ i \text{ odd}}} \sum_{j\geq 1} |x_{i,j}| &\leq \sum_{\substack{i\geq 3 \\ i \text{ odd}}} \sum_{j\geq 1} \frac{i+1}{i^{2j+1}} \\ &= \sum_{\substack{i\geq 3 \\ i \text{ odd}}} \frac{1}{i \cdot (i-1)} < \infty. \end{aligned}$$

Hence the double series $\sum_{j\geq 1}\sum_{i\geq 3, i \text{ odd}} x_{i,j}$ is absolutely convergent. Again, we change the order of summation to give

$$\begin{aligned}
\lim_{a\longrightarrow\infty} \pi/4\cdot(r_a-1) &= \sum_{\substack{i\geq 3\\ i \text{ odd}}}\sum_{j\geq 1} x_{i,j}\\
&= \sum_{\substack{i\geq 3\\ i \text{ odd}}} \frac{1-s(i)\cdot i}{s(i)\cdot i}\cdot\sum_{j\geq 1}\frac{1}{i^{2j}}\\
&= -\sum_{\substack{i\geq 3\\ i \text{ odd}}} \frac{i-s(i)}{i}\cdot\frac{1}{i^2-1}.
\end{aligned}$$

This series is $-1/(2\cdot 3) - 1/(5\cdot 6) - 1/(6\cdot 7) - 1/(9\cdot 10) - \cdots$. We can rewrite it as

$$\begin{aligned}
&\frac{1}{2}-\sum_{n\geq 0}\left(\frac{1}{(4n+1)\cdot(4n+2)}+\frac{1}{(4n+2)\cdot(4n+3)}\right)\\
&=\frac{1}{2}-\sum_{n\geq 0}\left(\frac{1}{4n+1}-\frac{1}{4n+3}\right)\\
&=\frac{1}{2}-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\cdots\\
&=\frac{1}{2}-\frac{\pi}{4}=\frac{\pi}{4}\cdot\left(\frac{2}{\pi}-1\right).
\end{aligned}$$

Thus we conclude that the limit of the sequence $r_1, r_3, \ldots$ is $2/\pi$. ∎

THEOREM 4.1. *The asymptotic behavior of* $\beta(1^a, 2, 1^b)$ *is given by*

$$\beta(1^a, 2, 1^b) \sim \frac{2}{\pi}\cdot E_n \sim \frac{8}{\pi^2}\cdot\left(\frac{2}{\pi}\right)^n\cdot n!,$$

when $\min(a, b)$ *tends to infinity and where* $n = a + b + 3$.

Proof. Let c be an odd integer. By Theorem 2 we have that

$$\beta(1^c, 2, 1^{n-c-3}) \geq \beta(1^a, 2, 1^b),$$

for sufficiently large a, b, and n. Divide this inequality by E_n and let n approach infinity. We then have

$$r_c \geq \limsup_{n\longrightarrow\infty}\frac{\beta(1^a, 2, 1^b)}{E_n}.$$

This inequality holds for all odd integers c. Now let c approach infinity. We obtain

$$\frac{2}{\pi} \geq \limsup_{n\longrightarrow\infty}\frac{\beta(1^a, 2, 1^b)}{E_n}.$$

By considering even integers c we obtain the reverse bound for the quantity

$$\liminf_{n \longrightarrow \infty} \beta(1^a, 2, 1^b)/E_n,$$

and the desired statement is proved. ∎

5. TWO-SIDED ALTERNATING

We now consider the asymptotic behavior of lists having the following three forms: $(1^a, L, 1^b)$, $(L_1, 1^a, L_2)$, and $(L_1, 1^a, L_2, 1^b, L_3)$.

THEOREM 5.1. *For a list L of size m, there exists a constant $d(L)$ such that*

$$\beta(1^a, L, 1^b) \sim d(L) \cdot E_n,$$

as $\min(a, b)$ *tends to infinity and where $n = a + m + b + 1$.*

Proof. On the collection of all lists, define a graph G by the following edge relations:

(i) $(L, l + 1 + k, K) \sim (L, l, 1, k, K)$,

(ii) $(L, l + 1, k, K) \sim (L, l, 1 + k, K)$, and

(iii) $(1, L) \sim L \sim (L, 1)$.

Observe that if $L \sim K$ then the lists L and K have the same parity. We claim that the graph G has two connected components; that is, from every list L there is a path to either the list (1) or the list (2). By relation (ii) we can move an arbitrary list L to the form $(k, 1^b)$. Now repeatedly apply relation (i) in the form $(k, 1^b) \sim (k - 2, 1^{b+2})$ until each entry in the list is either 1 or 2. Finally, by the third relation (iii), we can remove all 1's, and we finish at either the list (1) or (2).

The proof is now by induction on the distance from the list L to one of the lists (1) and (2). The induction basis is $L = (1)$, where we directly have $d(1) = 1$, and $L = (2)$, where Theorem 4.1 implies $d(2) = 2/\pi$.

The induction step consists of three cases corresponding to the three types of edges of the graph. Consider the edges of type (i). By the MacMahon multiplication theorem, Eq. (2.2), we have

$$\begin{aligned}\binom{n}{a+m} \cdot \beta(1^a, L, l) \cdot \beta(k, K, 1^b) \\ = \beta(1^a, L, l + 1 + k, K, 1^b) + \beta(1^a, L, l, 1, k, K, 1^b), \qquad (5.1)\end{aligned}$$

where the list (L, l) has size $m - 1$. Observe that $\binom{n}{a} \cdot E_a \cdot E_b \sim 4/\pi \cdot E_n$, where a and b tend to infinity and $n = a + b$. By this asymptotic result and by Theorem 3.2, we have that the left-hand side of Eq. (5.1) is a constant times the Euler number E_n. Hence if $\beta(1^a, L, l+1+k, K, 1^b)$ has the asymptotic behavior of a constant times the Euler number E_n, then so does $\beta(1^a, L, l, 1, k, K, 1^b)$ and vice versa. The other two types of edges are handled similarly. ∎

THEOREM 5.2. *For a list L_1 of size m_1 and a list L_2 of size m_2, we have*

$$\beta(L_1, 1^a, L_2) \sim c(L_1) \cdot c(L_2^*) \cdot E_n,$$

as a tends to infinity, where $n = a + m_1 + m_2 + 1$, and the constants $c(\cdot)$ are from Theorem 3.2.

THEOREM 5.3. *Let L_1, L_2, and L_3 be three lists of sizes m_1, m_2, and m_3. We have*

$$\beta(L_1, 1^a, L_2, 1^b, L_3) \sim c(L_1) \cdot d(L_2) \cdot c(L_3^*) \cdot E_n,$$

as $\min(a, b)$ approaches infinity, where $n = a + b + m_1 + m_2 + m_3 + 1$, and the constants $c(\cdot)$ and $d(\cdot)$ are from Theorems 3.2 and 5.1.

The proof of these two theorems follows the pattern of the proof of Theorem 5.1. Define two new graphs G_l and G_r by the relations $\sim_l$ and $\sim_r$ by extending the relation $\sim$ as follows. First, $L \sim K$ implies both $L \sim_l K$ and $L \sim_r K$. The added properties are $(1, l, L) \sim_l (1 + l, L)$ and $(K, k, 1) \sim_r (K, k + 1)$. Observe that the two new graphs are connected.

To prove Theorem 5.3 consider the product graph $H = G_l \times G \times G_r$. This is the graph consisting of triplets of lists $[L_1; L_2; L_3]$. Two triplets $[L_1; L_2; L_3]$ and $[K_1; K_2; K_3]$ are adjacent if one of the following conditions are satisfied:

- $L_1 \sim_l K_1$, $L_2 = K_2$, and $L_3 = K_3$,
- $L_1 = K_1$, $L_2 \sim K_2$, and $L_3 = K_3$,
- $L_1 = K_1$, $L_2 = K_2$, and $L_3 \sim_r K_3$.

MacMahon's multiplication theorem will imply the induction step. That is, if a triplet in the graph H satisfies the asymptotic result then also will the neighbors of the triplet.

Similarly to prove Theorem 5.2 one considers the graph $H = G_l \times G_r$.

6. A FORMULA IN TERMS OF EULER NUMBERS

When the list L has even parity, we can give an explicit formula for $\beta(1^a, L, 1^b)$. The number of terms only depends on L and not a and b. To do this we introduce a third way to encode descent sets. This method compares the descent set with the alternating set. The resulting formula works for all descent sets, but it is the most effective for computing $\beta(1^a, L, 1^b)$ when L is an even list.

We begin to introduce some notation. Let A_n denote the alternating set $\{2, 4, 6, \ldots\} \cap \{1, \ldots, n-1\}$. A composition of the integer n with k parts is k nonnegative integers $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ such that $n = p_1 + p_2 + \cdots + p_k$. We denote the number of parts by $l(\mathbf{p})$. Let Γ_n denote the set of compositions of n. The set Γ_n has a natural partial order defined by the cover relation

$$(p_1, \ldots, p_i + p_{i+1}, \ldots, p_k) \leq (p_1, \ldots, p_i, p_{i+1}, \ldots, p_k).$$

This partial order gives the Boolean algebra on $n-1$ elements. Observe that the minimal element is (n), and the Möbius function is given by $\mu(\mathbf{q}, \mathbf{p}) = (-1)^{l(\mathbf{p})-l(\mathbf{q})}$.

For a composition $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ of the integer n, let $\langle \mathbf{p} \rangle$ denote the following subset of $\{1, \ldots, n-1\}$:

$$\langle \mathbf{p} \rangle = \langle p_1, p_2, \ldots, p_k \rangle = \{p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_{k-1}\} \triangle A_n.$$

Moreover, let $\beta\langle \mathbf{p} \rangle = \beta\langle p_1, p_2, \ldots, p_k \rangle$ denote the descent statistic $\beta(\langle \mathbf{p} \rangle)$. For instance, we have $\beta\langle n \rangle = E_n$.

As an example, consider

$$\langle 4, 3, 1, 4 \rangle = \{4, 7, 8\} \triangle \{2, 4, 6, 8, 10\} = \{2, 6, 7, 10\}.$$

This pattern is pictured in Fig. 3. Observe how the boxes have their lengths prescribed by the composition and how the pattern alternates in each box according to the alternating set A_{12}.

MacMahon's multiplication theorem, Eq. (2.1), can now be stated in terms of compositions. Let $\mathbf{p} = (p_1, \ldots, p_k)$ be a composition of n and

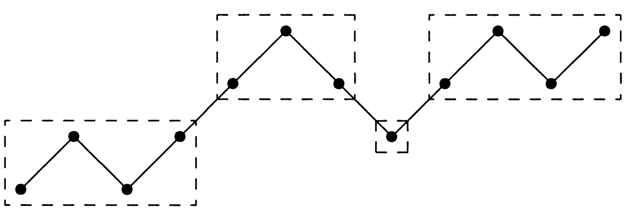

FIG. 3. The descent picture of the composition $\langle 4, 3, 1, 4 \rangle$.

let $\mathbf{q} = (q_1, \ldots, q_r)$ be a composition of m. Then

$$\binom{n+m}{n} \cdot \beta\langle\mathbf{p}\rangle \cdot \beta\langle\mathbf{q}\rangle = \beta\langle p_1, \ldots, p_k, q_1, \ldots, q_r\rangle + \beta\langle p_1, \ldots, p_k + q_1, \ldots, q_r\rangle. \tag{6.1}$$

For a composition $\mathbf{p}$ of n we define the factorial function $\mathbf{p}!$, the multinomial coefficient $\binom{n}{\mathbf{p}}$, and the product of the Euler numbers $E_{\mathbf{p}}$ by

$$\mathbf{p}! = p_1! \cdots p_k!, \binom{n}{\mathbf{p}} = \binom{n}{p_1, \ldots, p_k} = \frac{n!}{\mathbf{p}!}, \quad \text{and} \quad E_{\mathbf{p}} = E_{p_1} \cdots E_{p_k}.$$

PROPOSITION 6.1. *Let* $\mathbf{p}$ *be a composition of* n. *Then the descent set statistic* $\beta\langle\mathbf{p}\rangle$ *is given by*

$$\beta\langle\mathbf{p}\rangle = \sum_{\mathbf{q} \leq \mathbf{p}} (-1)^{l(\mathbf{p})-l(\mathbf{q})} \cdot \binom{n}{\mathbf{q}} \cdot E_{\mathbf{q}}.$$

Proof. Iterate MacMahon's multiplication theorem, Eq. (6.1), $l(\mathbf{p}) - 1$ number of times. We obtain $\binom{n}{\mathbf{p}} \cdot E_{\mathbf{p}} = \sum_{\mathbf{q} \leq \mathbf{p}} \beta\langle\mathbf{q}\rangle$. The proposition then follows by inverting this relation. ∎

As an example, consider the list $(1^a, 3, 2, 2, 1^b)$ and let $n = a + b + 8$. The composition corresponding to this list is $(a + 2, 3, 1, b + 2)$; see Fig. 3. By Proposition 6.1 we have

$$\begin{aligned}
\beta(1^a, 3, 2, 2, 1^b) &= \beta\langle a+2, 3, 1, b+2\rangle \\
&= \binom{n}{a+2, 3, 1, b+2} E_{a+2}E_3E_1E_{b+2} - \binom{n}{a+5, 1, b+2} E_{a+5}E_1E_{b+2} \\
&\quad - \binom{n}{a+2, 4, b+2} E_{a+2}E_4E_{b+2} + \binom{n}{a+6, b+2} E_{a+6}E_{b+2} \\
&\quad - \binom{n}{a+2, 3, b+3} E_{a+2}E_3E_{b+3} + \binom{n}{a+5, b+3} E_{a+5}E_{b+3} \\
&\quad + \binom{n}{a+2, b+6} E_{a+2}E_{b+6} - E_n.
\end{aligned}$$

This allows us to calculate the asymptotic expansion. Assume that $\mathbf{q} = (q_1, \ldots, q_r)$ is a composition of m and that $a + m + b$ equals n. Then we have that

$$\binom{n}{a, q_1, \ldots, q_r, b} \cdot E_a \cdot E_{\mathbf{q}} \cdot E_b \sim \frac{4}{\pi} \cdot \left(\frac{\pi}{2}\right)^m \cdot \frac{E_{\mathbf{q}}}{\mathbf{q}!} \cdot E_n.$$

as $\min(a, b)$ approaches infinity. Now we obtain the constant in the asymptotic expression for our example.

$$\begin{aligned}
d(3,2,2) &= \frac{4}{\pi}\left(\frac{\pi}{2}\right)^4 \frac{E_3 E_1}{3!1!} - \frac{4}{\pi}\left(\frac{\pi}{2}\right)\frac{E_1}{1!} \\
&\quad -\frac{4}{\pi}\left(\frac{\pi}{2}\right)^4 \frac{E_4}{4!} + \frac{4}{\pi} \\
&\quad -\frac{4}{\pi}\left(\frac{\pi}{2}\right)^3 \frac{E_3}{3!} + \frac{4}{\pi} \\
&\quad +\frac{4}{\pi} - 1 \\
&= \frac{\pi^3}{32} - \frac{\pi^2}{6} - 3 + \frac{12}{\pi}.
\end{aligned}$$

This method to compute $d(L)$ only works when the list L has even parity. The reason is for an odd list L the first alternating part in $(1^a, L, 1^b)$ is out of sync with the last alternating part.

The formula presented in Proposition 6.1 can be viewed as a dual formula with respect to the following expression for $\beta(S)$. See [13, Sect. 1.3], as well as Carlitz [2]. For $T = \{t_1 < t_2 < \cdots < t_{k-1}\}$ a subset of $\{1, \dots, n-1\}$, define

$$\alpha(T) = \binom{n}{t_1 - t_0, t_2 - t_1, \dots, t_k - t_{k-1}},$$

where $t_0 = 0$ and $t_k = n$. Then we have

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \alpha(T).$$

This formula is useful for computations when the set S has small cardinality. Compare this with Proposition 6.1, which is useful when $|S \triangle A_n|$ is small.

7. CONCLUDING REMARKS

We conjecture the following generalization of the results in Section 5.

Conjecture 7.1. Let $L_1, L_2, \dots, L_{k+1}$ be lists and let m be the sum of their sizes. The asymptotic behavior of $\beta(L_1, 1^{a_1}, L_2, 1^{a_2}, \dots, 1^{a_k}, L_{k+1})$ is $\beta(L_1, 1^{a_1}, L_2, 1^{a_2}, \dots, 1^{a_k}, L_{k+1}) \sim c(L_1) \cdot d(L_2) \cdots d(L_k) \cdot c(L_{k+1}{}^*) \cdot E_n$, as $\min(a_1, \dots, a_k)$ tends to infinity and where $n = 1 + m + \sum_{i=1}^{k} a_i$.

This conjecture follows from the following generalization of Theorem 4.1. The asymptotic behavior of $\beta(1^{a_1}, 2, 1^{a_2}, 2, \dots, 2, 1^{a_k})$ is given by

$$\beta(1^{a_1}, 2, 1^{a_2}, 2, \dots, 2, 1^{a_k}) \sim \left(\frac{2}{\pi}\right)^{k-1} \cdot E_n, \tag{7.1}$$

as $\min(a_1, \ldots, a_k)$ tends to infinity and we have $n = 2k - 1 + a_1 + \cdots + a_k$. The proof that Conjecture 7.1 follows from Eq. (7.1) is similar to the proofs in Section 5. However, armed alone with Theorem 4.1 we can only verify Conjecture 7.1 when at most one of the lists $L_2, \ldots, L_k$ has odd parity.

It is now natural to ask how we can extend our techniques to understand the asymptotic behavior of

$$\beta(L_1, M^{a_1}, L_2, M^{a_2}, \ldots, M^{a_k}, L_{k+1}),$$

where M is a list and M^a denotes the list repeated a times. Here we are assuming that $\min(a_1, \ldots, a_k)$ tends to infinity. We believe the asymptotic behavior of these values is closely related to the asymptotics of

$$\beta(M^{a_1+a_2+\cdots+a_k}).$$

As another special case, the numbers

$$E_{nk}^{(k)} = \beta\Big(\underbrace{k-1, 1, k-1, 1, \ldots, 1}_{n-1}, k-1\Big)$$

were studied by Leeming and MacLeod [6, 7]. They obtained the limit

$$\lim_{n\to\infty} \frac{\left(E_{nk}^{(k)}\right)^{1/kn}}{n} = c_k,$$

where $4/(\pi e) \le c_k \le 1$ for $k \ge 2$ and $\lim_{k\to\infty} c_k = 1$. They also have a multitude of congruence relations for the values $E_{nk}^{(k)}$.

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