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On Apolarity and Generic Canonical Forms

Richard Ehrenborg*

Department of Mathematics, White Hall, Cornell University, Ithaca, New York 14853-7901

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The notion of $\mathscr{S}$-algebra is introduced. The theory of apolarity and generic canonical forms for polynomials is generalized to $\mathscr{S}$-algebras over the complex field $\mathbb{C}$. We apply this theory to the problem of finding the essential rank of general, symmetric, and skew-symmetric tensors. Upper bounds for the essential ranks are found by different combinatorial coverings. © 1999 Academic Press

1. INTRODUCTION

The theory of apolarity was first developed by Clebsch, Lasker, Richmond, Sylvester, and Wakeford [10, 17, 20]. They were first interested in studying homogeneous polynomials of degree p and in q variables, and in expressing them as sums of pth powers of linear terms. The problem is to minimize the number of pth powers which are required in such a sum. For instance, a result due to Sylvester is that a generic homogeneous polynomial in two variables of degree $2n - 1$ may be written as a sum with n terms of $(2n - 1)$st powers. This is only true for a generic polynomial; for instance $x^2 y$ cannot be written as the sum of two cubes.

They then focused on the more general problem of finding canonical ways of expressing a generic homogeneous polynomial. For example, it was

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discovered that a generic quaternary cubic (four variables and degree 3) may be written in the form $h_1 h_2 h_3 + h_4 h_5 h_6$, where the h_i's are linear terms. In [3] the main theorems on apolarity and generic canonical forms are presented for homogeneous polynomials, and numerous examples are included.

In the present work we extend the theory of apolarity and generic canonical forms to a more general setting. We introduce the notion of $\mathscr{S}$-algebra, which is a vector space A together with a set $\mathscr{S}$ of multilinear forms on the space A. This idea generalizes the notion of an algebra. We then consider the space of all functions, $A\{x_1, \dots, x_d\}$, that may be constructed on the space A to itself by the multilinear forms in the set $\mathscr{S}$. These functions behave like polynomials. We consider an important class of derivations on the space $A\{x_1, \dots, x_d\}$, namely the polarizations. The polarizations satisfy many properties, including some chain rules which play a crucial role in the proof of our main theorem.

Recall that symmetric tensors $\mathrm{Sym}(W)$ correspond to polynomials in n variables, where n is the dimension of W. The classical notion of apolarity can then be formulated as follows. Given two symmetric tensors $f \in \mathrm{Sym}_p(W^*) = \mathrm{Sym}_p(W)^*$ and $g \in \mathrm{Sym}_r(W)$, where $r \leq p$, we say that g is apolar to f if for all $h \in \mathrm{Sym}_{p-r}(W)$ we have that $\langle f \mid g \cdot h \rangle = 0$. We restate this definition by considering the linear map ϕ: $\mathrm{Sym}_{p-r}(W) \to \mathrm{Sym}_p(W)$ defined by $\phi(h) = g \cdot h$. Now we say that the linear map ϕ is apolar to $f \in \mathrm{Sym}_p(W^*)$ if for all $h \in \mathrm{Sym}_{p-r}(W)$ we have that $\langle f \mid \phi(h) \rangle = 0$. It is in this setting that we are able to generalize the idea of apolarity; see Definition 4.4.

In Section 5 we state the main theorem on generic canonical forms and apolarity. In short, the problem of determining whether a form $p \in A\{x_1, \dots, x_d\}$ in an $\mathscr{S}$-algebra A is canonical may be solved by finding certain elements in the vector space and showing that a certain linear equation system does not have a non-trivial solution. The concepts of apolarity and polarizations are important in finding this equation system.

We apply the above theory to the study of the rank of general tensors, symmetric tensors, and skew-symmetric tensors. In Section 6 we consider general tensors, which correspond to multi-dimensional matrices. The problem of finding upper bounds for the essential rank of general tensors has a combinatorial interpretation. Namely, an upper bound for the number of rooks required to cover a multidimensional chess board is also an upper bound for the essential rank of tensors. Similar combinatorial bounds are found in Section 7 for symmetric and skew-symmetric tensors. With the help of Steiner triple systems we find explicit bounds for the essential rank of symmetric and skew-symmetric tensors of degree 3. In the concluding remarks we suggest a conjecture which gives a duality between symmetric and skew-symmetric tensors.

2. POLYNOMIAL FUNCTIONS

DEFINITION 2.1. An $\mathcal{S}$-algebra A is a vector space together with a set $\mathcal{S}$ of multilinear forms on A. That is, for each $M \in \mathcal{S}$ there exists a positive integer $k(M)$ such that

$$M: \underbrace{A \times \cdots \times A}_{k(M)} \to A,$$

is a multilinear form.

When there is no risk of confusion we will write k instead of $k(M)$. An example of an $\mathcal{S}$-algebra is any associative algebra A. In this case $\mathcal{S}$ consists of a single bilinear form, which is the product on the algebra. But in the general instance of $\mathcal{S}$-algebra we do not assume any relations among the multilinear forms in the set $\mathcal{S}$.

In this article we will consider $\mathcal{S}$-algebras over the complex field $\mathbb{C}$. We assume that A carries a topology such that the induced topology on every finite dimensional linear subspace is Euclidean. Moreover, we also assume that $\mathcal{S}$ is a set of continuous multilinear forms on the linear space A.

We define $A\{x_1, \ldots, x_d\}$ to be the set of all functions in the variables $x_1, \ldots, x_d$ that we may construct by using the multilinear forms in the set $\mathcal{S}$. More formally

DEFINITION 2.2. Let $A\{x_1, \ldots, x_d\}$ be the smallest set that contains the vector space A and the variables $x_1, \ldots, x_d$, and which is closed under

1. linear combinations: if $p, q \in A\{x_1, \ldots, x_d\}$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha \cdot p + \beta \cdot q \in A\{x_1, \ldots, x_d\}$,
2. compositions with the multilinear forms in the set $\mathcal{S}$: if $M \in S$, $k = k(M)$ and $p_1, \ldots, p_k \in A\{x_1, \ldots, x_d\}$ then $M(p_1, \ldots, p_k) \in A\{x_1, \ldots, x_d\}$.

The elements of $A\{x_1, \ldots, x_d\}$ are called *polynomials* in the d variables $x_1, \ldots, x_d$.

Observe that an element of $A\{x_1, \ldots, x_d\}$ may be constructed from A and the variables $x_1, \ldots, x_d$ in a finite number of steps by the two rules in Definition 2.2. Hence we can prove statements about elements in $A\{x_1, \ldots, x_d\}$ by induction.

In the language of universal algebra the set $A\{x_1, \ldots, x_d\}$ is the polynomial clone of d-ary operations closed under the functions in $\mathcal{S}$, addition, and multiplication with scalars in $\mathbb{C}$; see [15].

LEMMA 2.3. *There is a unique evaluation map, denoted by* eval, *from* $A\{x_1, \ldots, x_d\} \times A^d$ *to* A *such that for* $(a_1, \ldots, a_d) \in A^d$

1. $\text{eval}(x_i; a_1, \ldots, a_d) = a_i$,
2. $\text{eval}(a; a_1, \ldots, a_d) = a$ *for* $a \in A$,

3. $\text{eval}(\alpha \cdot p + \beta \cdot q; a_1, \ldots, a_d) = \alpha \cdot \text{eval}(p; a_1, \ldots, a_d) + \beta \cdot \text{eval}(q; a_1, \ldots, a_d)$ *for* $p, q \in A\{x_1, \ldots, x_d\}$ *and* $\alpha, \beta \in \mathbb{C}$,

4. $\text{eval}(M(p_1, \ldots, p_k); a_1, \ldots, a_d) = M(\text{eval}(p_1; a_1, \ldots, a_d), \ldots, \text{eval}(p_k; a_1, \ldots, a_d))$ *for* $M \in \mathscr{S}$, $k = k(M)$, *and* $p_1, \ldots, p_k \in A\{x_1, \ldots, x_d\}$.

To avoid confusion we will sometimes write $\text{eval}(p; x_1 \leftarrow a_1, \ldots, x_d \leftarrow a_d)$ instead of the shorter (but still correct) $\text{eval}(p; a_1, \ldots, a_d)$.

We may say that two polynomials p and q in $A\{x_1, \ldots, x_d\}$ are equivalent if for all $a_1, \ldots, a_d \in A$ we have that

$$\text{eval}(p; a_1, \ldots, a_d) = \text{eval}(q; a_1, \ldots, a_d).$$

That is, p and q behave in the same way as functions, but they may consist of different expressions.

3. POLARIZATIONS

In this section we introduce an important class of linear maps on the linear space $A\{x_1, \ldots, x_d\}$. These maps generalize the concept of derivations on an algebra. This development will be done very much in the spirit of the theory of supersymmetric algebra; see, for instance, [7].

DEFINITION 3.1. A polarization D_{t, x_i} is a linear map from the polynomials $A\{x_1, \ldots, x_d\}$ to the polynomials $A\{t, x_1, \ldots, x_d\}$, such that

1. $D_{t, x_i}(a) = 0$ for $a \in A$,
2. $D_{t, x_i}(x_i) = t$,
3. $D_{t, x_i}(x_j) = 0$ for $j \neq i$,
4. $D_{t, x_i}(M(p_1, \ldots, p_k)) = \sum_{j=1}^{k} M(p_1, \ldots, p_{j-1}, D_{t, x_i}(p_j), p_{j+1}, \ldots, p_k)$, for $M \in \mathscr{S}$ and $k = k(M)$.

Observe that condition 4 in the definition generalizes the formula for the derivation of a product.

We continue this section by showing that polarizations satisfy certain chain rules. The proofs of these chain rules will follow by induction on the elements in $A\{x_1, \ldots, x_d\}$.

PROPOSITION 3.2. *Let* $p \in A\{x\}$ *and* $q \in A\{y\}$. *We may consider the element* $\text{eval}(p; x \leftarrow q)$ *as an element of* $A\{y\}$. *Moreover, the polarization of this element by* $D_{u, y}$ *is given by*

$$D_{u, y} \text{eval}(p; x \leftarrow q) = \text{eval}\big(D_{t, x} p; t \leftarrow D_{u, y}(q), x \leftarrow q\big).$$

Proof. The proof is by induction on $p \in A\{x\}$. It is easy to check that it holds for $p = a \in A$ and for $p = x$. Moreover, both sides are linear in p.

What remains to show is the induction step. Namely, given that the result holds for $p_1, \dots, p_k \in A\{x\}$, then it also holds for any $p = M(p_1, \dots, p_k)$, where M is a multilinear form in $\mathscr{S}$ such that $k(M) = k$. We have that

$$\begin{aligned} D_{u,y}&\operatorname{eval}\big(M(p_1, \dots, p_k); x \leftarrow q\big) \\ &= D_{u,y}\big(M\big(\operatorname{eval}(p_1; x \leftarrow q), \dots, \operatorname{eval}(p_k; x \leftarrow q)\big)\big) \\ &= \sum_{i=1}^{k} M\big(\operatorname{eval}(p_1; x \leftarrow q), \dots, D_{u,y}(\operatorname{eval}(p_i; x \leftarrow q)), \\ &\qquad\qquad \dots, \operatorname{eval}(p_k; x \leftarrow q)\big) \\ &= \sum_{i=1}^{k} M\big(\operatorname{eval}(p_1; x \leftarrow q), \dots, \operatorname{eval}\big(D_{t,x} p_i; t \leftarrow D_{u,y}(q), x \leftarrow q\big), \\ &\qquad\qquad \dots, \operatorname{eval}(p_k; x \leftarrow q)\big) \\ &= \sum_{i=1}^{k} \operatorname{eval}\big(M(p_1, \dots, D_{t,x} p_i, \dots, p_k); t \leftarrow D_{u,y}(q), x \leftarrow q\big) \\ &= \operatorname{eval}\big(D_{t,x} M(p_1, \dots, p_k); t \leftarrow D_{u,y}(q), x \leftarrow q\big) \\ &= \operatorname{eval}\big(D_{t,x} p; t \leftarrow D_{u,y}(q), x \leftarrow q\big). \end{aligned}$$

Hence the induction is complete. ∎

We say that a function ϕ from the complex numbers $\mathbb{C}$ to the vector space A is differentiable if for all $\alpha \in \mathbb{C}$ the limit

$$\lim_{h \to 0} \frac{\phi(\alpha + h) - \phi(\alpha)}{h}$$

exists. When this limit exists, we denote it with $(\partial/\partial\alpha)\phi(\alpha) = \phi'(\alpha)$. Let M be a multilinear form in $\mathscr{S}$, with $k = k(M)$. Let ϕ_i, for $i = 1, \dots, k$, be differentiable functions from $\mathbb{C}$ to A. We claim that $M(\phi_1, \dots, \phi_k)$ is also a differentiable function from $\mathbb{C}$ to A. This is easy to check with the following computation.

$$\begin{aligned} &\frac{M(\phi_1(\alpha + h), \dots, \phi_k(\alpha + h)) - M(\phi_1(\alpha), \dots, \phi_k(\alpha))}{h} \\ &\quad = \sum_{i=1}^{k} M\bigg(\phi_1(\alpha + h), \dots, \phi_{i-1}(\alpha + h), \frac{\phi_i(\alpha + h) - \phi_i(\alpha)}{h}, \\ &\qquad\qquad \phi_{i+1}(\alpha), \dots, \phi_k(\alpha)\bigg). \end{aligned}$$

Since M is continuous the above expression converges when $h \to 0$, and the limit is

$$\sum_{i=1}^{k} M\big(\phi_1(\alpha), \dots, \phi_{i-1}(\alpha), \phi_i'(\alpha), \phi_{i+1}(\alpha), \dots, \phi_k(\alpha)\big).$$

Observe that this expression is like the derivation of a product.

We also have the following chain rule for polarizations.

PROPOSITION 3.3. *Let $p \in A\{x_1, \dots, x_d\}$ and let ϕ_i be a differentiable function from $\mathbb{C}$ to A for $i = 1, \dots, d$. Then*

$$\frac{\partial}{\partial\alpha}\operatorname{eval}(p; \phi_1(\alpha), \dots, \phi_d(\alpha)) = \sum_{i=1}^{d} \operatorname{eval}\big(D_{t, x_i} p; \phi_i'(\alpha), \phi_1(\alpha), \dots, \phi_d(\alpha)\big).$$

Proof. Just as in the proof of Proposition 3.2, we proceed by induction on the elements p of $A\{x_1, \dots, x_d\}$. It is easy to prove the basis for induction, that is, when $p = a \in A$ and $p = x_i$. Notice that, as in the previous proposition, both sides are linear in p. It remains to show the induction step. Given that the result holds for $p_1, \dots, p_k \in A\{x\}$, then it also holds for any $p = M(p_1, \dots, p_k)$, where M is a multilinear form in $\mathscr{S}$ such that $k(M) = k$. Then

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\operatorname{eval}\big(M(p_1, \dots, p_k); \phi_1(\alpha), \dots, \phi_d(\alpha)\big) \\
&\quad = \frac{\partial}{\partial\alpha} M\big(\operatorname{eval}(p_1; \phi_1(\alpha), \dots, \phi_d(\alpha)), \\
&\qquad\qquad \dots, \operatorname{eval}(p_k; \phi_1(\alpha), \dots, \phi_d(\alpha))\big) \\
&\quad = \sum_{j=1}^{k} M\Big(\operatorname{eval}(p_1; \phi_1(\alpha), \dots, \phi_d(\alpha)), \\
&\qquad\qquad \dots, \frac{\partial}{\partial\alpha}\operatorname{eval}\big(p_j; \phi_1(\alpha), \dots, \phi_d(\alpha)\big), \\
&\qquad\qquad \dots, \operatorname{eval}(p_k; \phi_1(\alpha), \dots, \phi_d(\alpha))\Big)
\end{aligned}$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{d} M\Big(\mathrm{eval}(p_1; \phi_1(\alpha), \dots, \phi_d(\alpha)),$$
$$\dots, \mathrm{eval}\big(D_{t,x_i} p_j; \phi_i'(\alpha), \phi_1(\alpha), \dots, \phi_d(\alpha)\big),$$
$$\dots, \mathrm{eval}(p_k; \phi_1(\alpha), \dots, \phi_d(\alpha))\Big)$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{k} \mathrm{eval}\big(M(p_1, \dots, D_{t,x_i} p_j, \dots, p_k); \phi_i'(\alpha), \phi_1(\alpha), \dots, \phi_d(\alpha)\big)$$

$$= \sum_{i=1}^{d} \mathrm{eval}\left(\sum_{j=1}^{k} M(p_1, \dots, D_{t,x_i} p_j, \dots, p_k); \phi_i'(\alpha), \phi_1(\alpha), \dots, \phi_d(\alpha)\right)$$

$$= \sum_{i=1}^{d} \mathrm{eval}\big(D_{t,x_i} M(p_1, \dots, p_k); \phi_i'(\alpha), \phi_1(\alpha), \dots, \phi_d(\alpha)\big).$$

Here the induction is complete. ∎

When A is a commutative algebra, we may express the polarizations in terms of classical derivations. That A is a commutative algebra means that $\mathscr{S}$ consists of one bilinear form that is symmetric (commutative), fulfills the associative law, and has a unit element 1. For an algebra we denote this unique bilinear form by $\cdot$.

Definition 3.4. For a commutative algebra A we define the derivation $D_{x_i}\colon A\{x_1, \dots, x_d\} \to A\{x_1, \dots, x_d\}$ to be the linear map satisfying

1. $D_{x_i}(a) = 0$ for $a \in A$,
2. $D_{x_i}(x_i) = 1$,
3. $D_{x_i}(x_j) = 0$ for $j \neq i$,
4. $D_{x_i}(p \cdot q) = D_{x_i}(p) \cdot q + p \cdot D_{x_i}(q)$.

Proposition 3.5. *Let A be a commutative algebra. Then for any element $p \in A\{x_1, \dots, x_d\}$ and for $a, a_1, \dots, a_d \in A$ we have that*

$$\mathrm{eval}\big(D_{t,x_i} p; a, a_1, \dots, a_d\big) = a \cdot \mathrm{eval}\big(D_{x_i} p; a_1, \dots, a_d\big).$$

We omit the proof, since it is a straightforward induction argument.

4. HOMOGENEOUS POLYNOMIALS AND APOLARITY

Definition 4.1. Let $V, W_1, \dots, W_d$ be linear subspaces of A. We say that an element p in the set $A\{x_1, \dots, x_d\}$ is homogeneous with respect to the linear spaces $V, W_1, \dots, W_d$, if for all $w_1 \in W_1, \dots, w_d \in W_d$ we have $\mathrm{eval}(p; w_1, \dots, w_d) \in V$.

PROPOSITION 4.2. *Let V be a subspace of A of dimension n and let W_j be a subspace of A of dimension n_j for $j = 1, \dots, d$. Let $p \in A\{x_1, \dots, x_d\}$ be homogeneous with respect to the linear spaces $V, W_1, \dots, W_d$. Let $u_1, \dots, u_n$ be a basis of V and $z_{j,1}, \dots, z_{j,n_j}$ be a basis of W_j.* $\mathrm{Par} = \{(j, i) : 1 \le j \le d, 1 \le i \le n_j\}$. *Thus for $k \in \mathrm{Par}$, z_k is a base vector in W_j for some index j. Now there exist polynomials $\phi_i \in \mathbb{C}[\alpha_k]_{k \in \mathrm{Par}}$ in the variables $\alpha_{j,i}$ such that*

$$\sum_{i=1}^{n} \phi_i(\alpha_k)_{k \in \mathrm{Par}} u_i = \mathrm{eval}\left(p; \sum_{i=1}^{n_1} \alpha_{1,i} z_{1,i}, \dots, \sum_{i=1}^{n_d} \alpha_{d,i} z_{d,i} \right).$$

Proof. The right-hand side consists of multilinear forms. Hence we may use the multilinear property and linearly expand the expression. In doing this we obtain a linear combination of elements of the form

$$\mathrm{eval}(p; z_{1,m_1}, \dots, z_{d,m_d}),$$

where $1 \le m_j \le n_j$. Moreover, the coefficients in this linear combination will be polynomials in the α's.

Since p is homogeneous, we know that each of the elements $\mathrm{eval}(p; z_{1,m_1}, \dots, z_{d,m_d})$ lies in V, and thus can be expanded in the basis $u_1, \dots, u_n$. By composing these two expansions the result follows. ∎

PROPOSITION 4.3. *Assume that the polynomial $p \in A\{x_1, \dots, x_d\}$ is homogeneous with respect to the linear spaces $V, W_1, \dots, W_d$. Then the polynomial $D_{t,x_i}(p)$ is homogeneous with respect to the linear spaces V, W_i, $W_1, \dots, W_d$.*

Proof. Since p is homogeneous with respect to the linear spaces $V, W_1, \dots, W_d$, we know that the element $\mathrm{eval}(p; w_1, \dots, w_i + \alpha \cdot w, \dots, w_d)$, lies in V, where $\alpha \in \mathbb{C}$, $w \in W_i$, and $w_j \in W_j$ for $j = 1, \dots, d$. Take the partial derivative in the variable α of this element. The derivative will also take values in V. By Proposition 3.3 the derivative is equal to

$$\begin{aligned} &\frac{\partial}{\partial \alpha} \mathrm{eval}(p; w_1, \dots, w_i + \alpha \cdot w, \dots, w_d) \\ &\qquad = \mathrm{eval}\left(D_{t,x_i p}; \frac{\partial}{\partial \alpha}(w_i + \alpha \cdot w), w_1, \dots, w_i + \alpha \cdot w, \dots, w_d \right) \\ &\qquad = \mathrm{eval}\big(D_{t,x_i} p; w, w_1, \dots, w_i + \alpha \cdot w, \dots, w_d \big). \end{aligned}$$

Now by setting $\alpha = 0$ the conclusion follows. ∎

We now introduce the concept of apolarity in its general setting.

DEFINITION 4.4. Let V and W be vector spaces and let f be a linear map from W to V. We say $L \in V^*$ is apolar to f if for all $w \in W$

$$\langle L | f(w) \rangle = 0.$$

One important class of linear maps on which we consider the apolarity condition is given by the following.

PROPOSITION 4.5. *Let $p \in A\{x_1, \dots, x_d\}$ and let $a_1, \dots, a_d \in A$. Then the following map from A to A is linear*

$$y \mapsto \operatorname{eval}\big(D_{t, x_i} p; y, a_1, \dots, a_d\big).$$

Proof. The proof is by induction on $p \in A\{x_1, \dots, x_d\}$. When $p = a \in A$ then $D_{t, x_i} p = 0$, so the map is the zero map, which is linear. When $p = x_j$ then $D_{t, x_i} p = \delta_{i, j} t$. So $y \mapsto \delta_{i, j} y$, which is linear. Hence the two base cases are done. The linear combination of two linear maps is a linear map. The remaining case to consider is when $p = M(p_1, \dots, p_k)$, where $M \in \mathscr{S}$ and $k = k(M)$. Assume it holds for $p_1, \dots, p_k$. Then we have that

$$\begin{aligned}
&\operatorname{eval}\big(D_{t, x_i} M(p_1, \dots, p_k); y, a_1, \dots, a_d\big) \\
&\quad = \operatorname{eval}\left(\sum_{j=1}^{k} M\big(p_1, \dots, D_{t, x_i} p_j, \dots, p_k\big); y, a_1, \dots, a_d \right) \\
&\quad = \sum_{j=1}^{k} M\big(\operatorname{eval}(p_1; a_1, \dots, a_d), \dots, \operatorname{eval}\big(D_{t, x_i} p_j; y, a_1, \dots, a_d\big), \\
&\qquad\qquad \dots, \operatorname{eval}(p_k; a_1, \dots, a_d)\big).
\end{aligned}$$

Each term in this expression is a linear map. Since the sum of linear maps is a linear map, the proof is done. ∎

5. GENERIC CANONICAL FORMS

DEFINITION 5.1. Let V be a finite dimensional linear space. We say that a generic element $v \in V$ has a property P, if the set of all elements in V that has this property forms a dense set in V, where V has the Euclidean topology.

THEOREM 5.2. *Let $V, W_1, \dots, W_d$ be finite dimensional linear subspaces of the algebra A. Let p in $A\{x_1, \dots, x_d\}$ be homogeneous with respect to the linear spaces $V, W_1, \dots, W_d$. A generic element $v \in V$ can be written in the form*

$$v = \operatorname{eval}(p; w_1, \dots, w_d)$$

for some $w_1, \ldots, w_d$ *if and only if there exist* $w'_1, \ldots, w'_d$ *so that there is no nonzero dual element in* V^* *which is apolar to the linear map*

$$y_j \mapsto \operatorname{eval}\bigl(D_{t, x_j} p; y_j, w'_1, \ldots, w'_d\bigr)$$

relative to W_j, *for all* $1 \leq j \leq d$.

Observe that Proposition 4.5 shows that the map $y_j \mapsto \operatorname{eval}(D_{t, x_j} p;\ y_j, w'_1, \ldots, w'_d)$ is linear, and that Proposition 4.3 guarantees that it maps the linear space W_j into V.

The proof of the theorem requires the following two propositions. Proofs of these two propositions may be found in [3]. Let $\overline{\mathbb{C}(x_1, \ldots, x_q)}$ be the field of all algebraical functions in the variables $x_1, \ldots, x_q$.

PROPOSITION 5.3. *Let* $p_1(x_1, \ldots, x_q), \ldots, p_r(x_1, \ldots, x_q) \in \overline{\mathbb{C}(x_1, \ldots, x_q)}$, *where* $r \leq q$. *Then the algebraic functions* $p_1, \ldots, p_r$ *are algebraically independent if and only if the matrix*

$$\left(\frac{\partial p_i}{\partial x_j}\right)_{1 \leq i \leq r,\ 1 \leq j \leq q}$$

has full rank.

PROPOSITION 5.4. *Let* $p_1(x_1, \ldots, x_q), \ldots, p_r(x_1, \ldots, x_q) \in \overline{\mathbb{C}(x_1, \ldots, x_q)}$, *where* $r \leq q$. *Let* $P\colon \mathbb{C}^q \to \mathbb{C}^r$ *be defined by*

$$P(x_1, \ldots, x_q) = \bigl(p_1(x_1, \ldots, x_q), \ldots, p_r(x_1, \ldots, x_q)\bigr).$$

Then the algebraic functions $p_1, \ldots, p_r$ *are algebraically independent if and only if the range of the map* P *is dense in* $\mathbb{C}^r$.

Proof of Theorem 5.2. Let $\dim(V) = n$ and $\dim(W_j) = n_j$ for $j = 1, \ldots, d$. Let $u_1, \ldots, u_n$ be a basis for V, and similarly let $z_{j,1}, \ldots, z_{j,n_j}$ be a basis for W_j. Thus an element $w_j \in W_j$ can be written in the form

$$w_j = \sum_{i=1}^{n_j} \alpha_{j,i} z_{j,i},$$

where $\alpha_{j,i} \in \mathbb{C}$. We will call the coefficients $\alpha_{j,i}$ *parameters*. Set $\text{Par} = \{(j, i) : 1 \leq j \leq d,\ 1 \leq i \leq n_j\}$. Thus a parameter is of the form α_k, where $k \in \text{Par}$. Observe that the number of parameters is $|\text{Par}| = n_1 + \cdots + n_d$.

We will begin to prove the necessary implication of the theorem. Hence assume that a generic element v of V can be written in the form $\operatorname{eval}(p; w_1, \ldots, w_d)$. By counting coefficients on the left hand side and parameters on the right hand side, we obtain the inequality $n \leq |\mathrm{Par}| = n_1 + \cdots + n_d$.

By Proposition 4.2 we can expand

$$\sum_{i=1}^{n} \phi_i(\alpha_k)_{k \in \mathrm{Par}} u_i = \mathrm{eval}\left(p; \sum_{i=1}^{n_1} \alpha_{1,i} z_{1,i}, \ldots, \sum_{i=1}^{n_d} \alpha_{d,i} z_{d,i}\right),$$

where ϕ_i are polynomials for $i = 1, \ldots, n$.

Consider the map $\Phi\colon \mathbb{C}^{\mathrm{Par}} \to \mathbb{C}^n$ defined by

$$\Phi\big((\alpha_k)_{k \in \mathrm{Par}}\big) = \big(\phi_i(\alpha_k)_{k \in \mathrm{Par}}\big)_{1 \leq i \leq n},$$

where the coordinates of $\mathbb{C}^{\mathrm{Par}}$ are indexed by the set Par.

The assumption is that the range of the map Φ is dense in $\mathbb{C}^n$. By Proposition 5.4 we infer that the n polynomials ϕ_i are algebraically independent. Hence, by Proposition 5.3, the matrix

$$\left(\frac{\partial \phi_i}{\partial \alpha_k}\right)_{1 \leq i \leq n,\, k \in Par} \tag{1}$$

has full rank, where rows are indexed by i and the columns by the set Par.

Since the matrix (1) has full rank, we can choose values for the parameters such that the matrix (1) still has full rank. Denote these values we choose for the parameters by γ_k for $k \in \mathrm{Par}$. Let

$$w_j' = \sum_{i=1}^{n_j} \gamma_{j,i} z_{j,i}.$$

Thus $w_j' \in W_j$.

Moreover, we know that the matrix

$$\left(\left[\frac{\partial \phi_i}{\partial \alpha_k}\right]_{\alpha_m = \gamma_m}\right)_{1 \leq i \leq n,\, k \in \mathrm{Par}}$$

has full rank. Hence the columns of the matrix span the linear space $\mathbb{C}^n$. But $\mathbb{C}^n$ is isomorphic to V. Via this isomorphism we get

$$\left(\frac{\partial \phi_i}{\partial \alpha_k}\right)_{1 \leq i \leq n} \mapsto \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial \alpha_k} u_i = \frac{\partial v}{\partial \alpha_k}.$$

Thus, the elements

$$\left[\frac{\partial v}{\partial \alpha_k}\right]_{\alpha_m = \gamma_m}$$

span the linear space V.

Hence there is no nonzero functional $L \in V^*$ such that

$$\left\langle L \,\middle|\, \left[\frac{\partial v}{\partial \alpha_k}\right]_{\alpha_m = \gamma_m} \right\rangle = 0$$

for all $k \in \text{Par}$.

Each of the parameters will only occur in one of the vectors $w_1, \ldots, w_d$. The parameter $\alpha_{j,i}$ occurs only in w_j. In particular, we have

$$\frac{\partial w_j}{\partial \alpha_{j,i}} = z_{j,i}.$$

Hence by the chain rule, Proposition 3.3, we conclude that

$$\begin{aligned}
\frac{\partial v}{\partial \alpha_{j,i}} &= \frac{\partial}{\partial \alpha_{j,i}} \operatorname{eval}(p; w_1, \ldots, w_d) \\
&= \operatorname{eval}\left(D_{t, x_j} p; \frac{\partial w_j}{\partial \alpha_{j,i}}, w_1, \ldots, w_d\right) \\
&= \operatorname{eval}\big(D_{t, x_j} p; z_{j,i}, w_1, \ldots, w_d\big).
\end{aligned}$$

Observe that

$$\Big[\operatorname{eval}\big(D_{t, x_j} p; z_{j,i}, w_1, \ldots, w_d\big)\Big]_{\alpha_m = \gamma_m} = \operatorname{eval}\big(D_{t, x_j} p; z_{j,i}, w'_1, \ldots, w'_d\big).$$

Thus we can write our condition as follows: there is no nonzero functional $L \in V^*$ such that

$$\Big\langle L \,\Big|\, \operatorname{eval}\big(D_{t, x_j} p; z_{j,i}, w'_1, \ldots, w'_d\big) \Big\rangle = 0$$

for all $1 \le j \le d$ and $1 \le i \le n_j$. Observe that the above expression is linear in $z_{j,i}$ and recall that the elements $z_{j,1}, \ldots, z_{j,n_j}$ form a basis for W_j. Hence the statement above is equivalent to the condition that there is no nonzero functional $L \in V^*$ such that

$$\Big\langle L \,\Big|\, \operatorname{eval}\big(D_{t, x_j} p; y_j, w'_1, \ldots, w'_d\big) \Big\rangle = 0$$

for all $1 \leq j \leq d$ and for all $y_j \in W_j$.

Thus we have proven that there is no nonzero element in V^* apolar to all the maps

$$y_j \mapsto \operatorname{eval}\left(D_{t,x_j} p; y_j, w'_1, \ldots, w'_d\right)$$

relative to W_j for $j = 1, \ldots, s$. This proves the necessary condition of the theorem.

To prove the sufficient condition, we can trace the equivalences in the necessary part in opposite direction. We begin by assuming that there is no nonzero element in V^* apolar to all the maps

$$y_j \mapsto \operatorname{eval}\left(D_{t,x_j} p; y_j, w'_1, \ldots, w'_d\right)$$

relative to W_j for $j = 1, \ldots, d$. This implies that the elements

$$\operatorname{eval}\left(D_{t,x_j} p; z_{j,i}, w'_1, \ldots, w'_d\right),$$

where $1 \leq j \leq d$ and $1 \leq i \leq n_j$, span the linear space V. Hence $n_1 + \cdots + n_d$ elements span a linear space of dimension n. Thus $n \leq n_1 + \cdots + n_d$. By the identities above, we can rewrite the above elements, and by using the isomorphism between V and $\mathbb{C}^n$ we get that the matrix

$$\left(\left[\frac{\partial \phi_i}{\partial \alpha_k}\right]_{\alpha_m = \gamma_m}\right)_{1 \leq i \leq n,\, k \in \mathrm{Par}}$$

has rank d. Since $n \leq n_1 + \cdots + n_d$ the above matrix has full rank. Thus the matrix

$$\left(\frac{\partial \phi_i}{\partial \alpha_k}\right)_{1 \leq i \leq n,\, k \in \mathrm{Par}},$$

where we remove the values of the β's, cannot have lower rank. But the rank cannot increase so the last matrix has full rank also.

By Proposition 5.3 we know that the polynomials $\phi_1, \ldots, \phi_d$ are algebraically independent. Proposition 5.4 implies that the range of the map $\Phi: \mathbb{C}^{\mathrm{Par}} \to \mathbb{C}^d$ is dense and the theorem follows. ∎

When the $\mathscr{S}$-algebra is a commutative and associative algebra with a unit element, then the linear maps in the statement of Theorem 5.2 reduce to

$$y_j \mapsto y_j \cdot \operatorname{eval}\left(D_{x_j} p; w'_1, \ldots, w'_d\right),$$

where D_{x_j} is the derivation in the variable x_j; see Proposition 3.5. Moreover, if the algebra A is the algebra of polynomials with complex coefficients, and W_j is the linear space of homogeneous polynomials of degree d_j, then Theorem 5.2 reduces to the result presented in [3].

6. THE ESSENTIAL RANKS OF MULTI-DIMENSIONAL MATRICES

Let $W_1, \ldots, W_d$ be vector spaces over the complex numbers such that $\dim(W_i) = n_i$. Consider the linear space $V = W_1 \otimes W_2 \otimes \cdots \otimes W_d$. Assume that $z_{j,1}, \ldots, z_{j,n_j}$ is a basis for W_j. Then an element $v \in V$ may be written as

$$v = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} a_{i_1, i_2, \ldots, i_d} \cdot z_{1,i_1} \otimes z_{2,i_2} \otimes \cdots \otimes z_{d,i_d}.$$

Thus the element v corresponds to an $n_1 \times n_2 \times \cdots \times n_d$ matrix $(a_{i_1, i_2, \ldots, i_d})_{1 \le i_j \le n_j}$. Hence V is isomorphic to the linear space of $n_1 \times n_2 \times \cdots \times n_d$ matrices whose entries are complex numbers. We will now extend the idea of rank of a two dimensional matrix to the multi-dimensional case.

DEFINITION 6.1. An element $v \in W_1 \otimes \cdots \otimes W_d$ has rank k if k is the smallest integer m such that there exist $w_{1,1}, \ldots, w_{1,m} \in W_1$, $\ldots, w_{d,1}, \ldots, w_{d,m} \in W_d$, such that

$$v = \sum_{i=1}^{m} w_{1,i} \otimes \cdots \otimes w_{d,i}.$$

A question to study is: What is the maximal rank of the elements v in $W_1 \otimes \cdots \otimes W_d$? Clearly an upper bound for the maximal rank is $(\max(n_1, \ldots, n_d))^{-1} \cdot \prod_{j=1}^{d} n_j$. For instance, when $d = 2$ it is well-known that this bound is reached, namely the maximal rank of $W_1 \otimes W_2$ is $\min(n_1, n_2)$.

Also, when $d = 3$ and $n_1 = n_2 = n_3 = 2$ then the maximal rank of $W_1 \otimes W_2 \otimes W_3$ is 3; see [5]. An example of an element that has rank 3 is $z_{1,2} \otimes z_{2,1} \otimes z_{3,1} + z_{1,1} \otimes z_{2,2} \otimes z_{3,1} + z_{1,1} \otimes z_{2,1} \otimes z_{3,2}$. Instead of considering the maximal rank, we will continue with a related question.

DEFINITION 6.2. The linear space $W_1 \otimes \cdots \otimes W_d$ has essential rank k if k is the smallest integer m such that the set of elements v in $W_1 \otimes \cdots \otimes W_d$ of rank at most m forms a dense set in the Euclidean topology.

Observe that the maximal rank of $W_1 \otimes \cdots \otimes W_d$ might be different from the essential rank of $W_1 \otimes \cdots \otimes W_d$. For instance, when $n_1 = n_2 = n_3 = 2$, the essential rank of $W_1 \otimes W_2 \otimes W_3$ is 2, which is different from the maximal rank 3. For more on this last example, see [5].

To find upper bounds for the essential rank of the linear space $W_1 \otimes \cdots \otimes W_d$ we will consider the combinatorial problem of rook coverings. Let $A_1, \dots, A_d$ be finite sets and let $A_1 \times \cdots \times A_d$ be the Cartesian product of these sets.

DEFINITION 6.3. A rook covering of $A_1 \times \cdots \times A_d$ is a subset R of $A_1 \times \cdots \times A_d$ such that for all $(a_1, \dots, a_d)$ in $A_1 \times \cdots \times A_d$ there exist $(r_1, \dots, r_d) \in R$ such that $(a_1, \dots, a_d)$ and $(r_1, \dots, r_d)$ differ in at most one place. That is,

$$\left|\{i : a_i \neq r_i\}\right| \leq 1.$$

We call a rook covering R perfect if for all $(a_1, \dots, a_d) \in A_1 \times \cdots \times A_d$ there exists a unique $(r_1, \dots, r_d) \in R$ with the above conditions.

Let $n_1, \dots, n_d$ be positive integers. We will say that there is a rook covering of $n_1 \times \cdots \times n_d$ of cardinality N, if there is a rook covering of $A_1 \times \cdots \times A_d$ of cardinality N, where A_i has size n_i for $i = 1, \dots, d$.

The problem of finding small rook coverings is a well studied problem, see [8, 13, 18].

PROPOSITION 6.4. *Let W_j be a vector space over the complex numbers of dimension n_j for $j = 1, \dots, d$. If there is a rook covering of $n_1 \times \cdots \times n_d$ of cardinality N, then the essential rank of $V = W_1 \otimes \cdots \otimes W_d$ is less than or equal to N.*

Proof. Let A be the vector space defined as

$$A = W_1 \oplus \cdots \oplus W_d \oplus V.$$

Since A is finite dimensional, let the topology of A be the Euclidean. Define

$$M\colon \underbrace{A \times \cdots \times A}_{d} \to A$$

to be a multilinear form, such that

$$M(a_1, \dots, a_d) = \begin{cases} a_1 \otimes \cdots \otimes a_d & \text{if } a_j \in W_j, \\ 0 & \text{otherwise}, \end{cases}$$

and extend M by linearity. It is easy to see that M is continuous.

Clearly the polynomial $p \in A\{x_{1,1}, \dots, x_{1,N}, \dots, x_{d,1}, \dots, x_{d,N}\}$ defined by

$$p = \sum_{i=1}^{N} M(x_{1,i}, \dots, x_{d,i})$$

is homogeneous with respect to

$$V, \underbrace{W_1, \dots, W_1}_{N}, \dots, \underbrace{W_d, \dots, W_d}_{N}.$$

We would like to show that a generic element of V can be written in the form

$$\operatorname{eval}(p; w_{1,1}, \dots, w_{1,N}, \dots, w_{d,1}, \dots, w_{d,N}), \tag{2}$$

where $w_{j,i} \in W_j$. Observe that $D_{t,x_{j,i}}(p) = M(x_{1,i}, \dots, x_{j-1,i}, t, x_{j+1,i}, \dots, x_{d,i})$. By Theorem 5.2 it is enough to prove that there exist $w'_{j,i} \in W_j$ for $j = 1, \dots, d$ and $i = 1, \dots, N$ such that there is no nonzero dual element $L \in V^*$ apolar to all the linear maps

$$y_{j,i} \mapsto M(w'_{1,i}, \dots, w'_{j-1,i}, y_{j,i}, w'_{j+1,i}, \dots, w'_{d,i}), \tag{3}$$

where $y_{j,i} \in W_j$. In order to do this, let $z_{j,1}, \dots, z_{j,n_j}$ be a basis for W_j. Similarly let $z^*_{j,1}, \dots, z^*_{j,n_j}$ be the dual basis for W_j^*. That is, $\langle z^*_{j,i} \mid z_{j,k} \rangle = \delta_{i,k}$. Let A_j be the set $\{1, \dots, n_j\}$. Let R be a rook covering of $A_1 \times \cdots \times A_d$ of cardinality N. Thus let

$$R = \{(r_{1,i}, \dots, r_{d,i}) : 1 \leq i \leq N\}.$$

Choose $w'_{j,i} = z_{j,r_{j,i}}$.

Assume that L in V^* is apolar to all the linear maps in (3). We can write the dual element L in terms of the dual basis.

$$L = \sum_{(i_1, \dots, i_d) \in A_1 \times \cdots \times A_d} \beta_{i_1, \dots, i_d} \cdot z^*_{1,i_1} \otimes \cdots \otimes z^*_{d,i_d}.$$

Consider an element $(i_1, \dots, i_d) \in A_1 \times \cdots \times A_d$. Since R is a rook covering, there is an element $(r_{1,k}, \dots, r_{d,k}) \in R$ that differs in at most one coordinate from $(i_1, \dots, i_d)$. Let the coordinate where $(i_1, \dots, i_d)$ and $(r_{1,k}, \dots, r_{d,k})$ differ be j. (If $(i_1, \dots, i_d) = (r_{1,k}, \dots, r_{d,k})$ then choose j arbitrarily.) Thus we know that L is apolar to the linear map

$$y_{j,k} \mapsto M(z_{1,r_{1,k}}, \dots, z_{j-1,r_{j-1,k}}, y_{j,k}, z_{j+1,r_{j+1,k}}, \dots, z_{d,r_{d,k}}).$$

Let $y_{j,k}$ take the value of z_{j,i_j}. Hence

$$\begin{aligned} 0 &= \left\langle L \middle| M(z_{1,r_{1,k}}, \ldots, z_{j-1,r_{j-1,k}}, z_{j,i_j}, z_{j+1,r_{j+1,k}}, \ldots, z_{d,r_{d,k}}) \right\rangle \\ &= \left\langle L \middle| z_{1,i_1} \otimes \cdots \otimes z_{j-1,i_{j-1}} \otimes z_{j,i_j} \otimes z_{j+1,i_{j+1}} \otimes \cdots \otimes z_{d,i_d} \right\rangle \\ &= \beta_{i_1,\ldots,i_d}. \end{aligned}$$

We conclude that all coefficients of L vanish. Thus we know that $L = 0$. Theorem 5.2 implies that a generic element of V can be written in the form (2). Thus the linear space $V = W_1 \otimes \cdots \otimes W_d$ has essential rank less than or equal to N. ∎

The smallest number of rooks covering the set $n \times n \times n$ is $\lceil n^2/2 \rceil$. This was stated as a problem in the Soviet Olympiad 1971. See also [2, Problem 39]. Thus we obtain:

COROLLARY 6.5. *Let W be a vector space over the complex numbers of dimension n. Then $W^{\otimes 3}$ has essential rank less than or equal to $\lceil n^2/2 \rceil$.*

An error correcting code that corrects one error may be viewed as a perfect rook covering. Such a code is the Hamming codes [11]. They have the following parameters. Let q be a prime power, and let $d = (q^k - 1)/(q - 1)$, where k is a positive integer. Then there is a perfect rook covering of

$$q^d = \underbrace{q \times \cdots \times q}_{d}$$

of size q^{d-k}. We then obtain:

COROLLARY 6.6. *Let W be a vector space over the complex numbers of dimension q, where q is a prime power. Let $d = (q^k - 1)/(q - 1)$, where k be a positive integer. Then $W^{\otimes d}$ has essential rank less than or equal to q^{d-k}.*

PROPOSITION 6.7. *If there is a rook covering of $n_1 \times \cdots \times n_d$ using N rooks, then there is a rook covering of $(m \cdot n_1) \times \cdots \times (m \cdot n_d)$ using $m^{d-1} \cdot N$ rooks.*

Proof. Let A_i be a set of size n_i and let $B_i = A_i \times \{0, 1, \ldots, m-1\}$. Let R be a rook covering of $A_1 \times \cdots \times A_d$ of cardinality N. We would like to find a rook covering of the set $B_1 \times \cdots \times B_d$. Consider the set R' defined as

$$\begin{aligned} \{ ((a_1, p_1), \ldots, (a_d, p_d)) \in B_1 \times \cdots \times B_d \\ : (a_1, \ldots, a_d) \in R,\, p_1 + \cdots + p_d \equiv 0 \pmod{m} \}. \end{aligned}$$

It is a direct verification that R' is a rook covering of $B_1 \times \cdots \times B_d$ with cardinality $m^{d-1} \cdot N$. ∎

The linear analogue to Proposition 6.7 is:

PROPOSITION 6.8. *Let $U, W_1, \ldots, W_d$ be vector spaces over the complex numbers of dimensions $m, n_1, \ldots, n_d$, respectively. If the linear space $W_1 \otimes \cdots \otimes W_d$ has essential rank N, then the linear space $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$ has essential rank at most $m^{d-1} \cdot N$.*

Proof. Let $z_{j,1}, \ldots, z_{j,n_j}$ be a basis of W_j, and let $z^*_{j,1}, \ldots, z^*_{j,n_j}$ be the dual basis of W_j^*. Let I denote the set $\{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_d\}$.

We know that a generic element of $W_1 \otimes \cdots \otimes W_d$ can be written in the form

$$\sum_{i=1}^{N} w_{1,i} \otimes \cdots \otimes w_{d,i},$$

where $w_{j,i} \in W_j$. By Theorem 5.2 this canonical form implies that there exist $w'_{j,i} \in W_j$, $1 \leq j \leq d$ and $1 \leq i \leq N$, such that there is no nonzero element in $(W_1 \otimes \cdots \otimes W_d)^*$ apolar to all the maps

$$t_{j,i} \mapsto w'_{1,i} \otimes \cdots \otimes t_{j,i} \otimes \cdots \otimes w'_{d,i},$$

where $t_{j,i} \in W_j$. Hence the images of these linear maps span the space $W_1 \otimes \cdots \otimes W_d$. This is equivalent to the statement that for all $(i_1, \ldots, i_d) \in I$ there exist $y_{j,i} \in W_j$ for $1 \leq j \leq d$ and $1 \leq i \leq N$, such that

$$\sum_{j=1}^{d} \sum_{i=1}^{N} w'_{1,i} \otimes \cdots \otimes y_{j,i} \otimes \cdots \otimes w'_{d,i} = z_{i_1} \otimes \cdots \otimes z_{i_d}. \tag{4}$$

Let $u_1, \ldots, u_m$ be a basis of U. Consider the following subset of $\{0, 1, \ldots, m-1\}^d$.

$$P = \left\{ (p_1, \ldots, p_d) \in \{0, 1, \ldots, m-1\}^d : p_1 + \cdots + p_d \equiv 0 \pmod{m} \right\}.$$

The cardinality of P is m^{d-1}. Define the elements $v'_{j,i,\mathbf{p}} \in W_j \otimes U$, where $j = 1, \ldots, d$, $i = 1, \ldots, N$, and $\mathbf{p} \in P$, by

$$v'_{j,i,\mathbf{p}} = w'_{j,i} \otimes u_{p_j}.$$

Thus there are $d \cdot N \cdot m^{d-1}$ such elements. Consider now the linear maps

$$t_{j,i,\mathbf{p}} \mapsto v'_{1,i,\mathbf{p}} \otimes \cdots \otimes t_{j,i,\mathbf{p}} \otimes \cdots \otimes v'_{d,i,\mathbf{p}}, \tag{5}$$

where $t_{j,i,\mathbf{p}} \in W_j \otimes U$. We would like to show that the images of these linear maps span the linear space $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$. Choose $(i_1, \ldots, i_d) \in I$ and $(q_1, \ldots, q_d) \in \{0, 1, \ldots, m-1\}^d$. This choice corresponds to the basis element

$$\left(z_{i_1} \otimes u_{q_1}\right) \otimes \cdots \otimes \left(z_{i_d} \otimes u_{q_d}\right)$$

of $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$. As observed before, we can find $y_{j,i} \in W_j$ for $1 \le j \le d$ and $1 \le i \le N$, such that Eq. (4) is satisfied. Let $\mathbf{p}^j = (p_1^j, \ldots, p_d^j)$ be the element of P such that

$$p_k^j = \begin{cases} q_k & \text{if } k \neq j, \\ q_k - (q_1 + \cdots + q_d) \pmod{m} & \text{if } k = j. \end{cases}$$

Thus $\mathbf{q}$ and $\mathbf{p}^j$ only differ in the jth coordinate. Let

$$\widehat{y_{j,i}} = y_{j,i} \otimes u_{q_j}.$$

The element $v'_{1,i,\mathbf{p}^j} \otimes \cdots \otimes \widehat{y_{j,i}} \otimes \cdots \otimes v'_{d,i,\mathbf{p}^j}$ lies in the image of one of the maps in (5). Consider now the sum of these elements.

$$\begin{aligned} &\sum_{i=1}^{N} \sum_{j=1}^{d} v'_{1,i,\mathbf{p}^j} \otimes \cdots \otimes \widehat{y_{j,i}} \otimes \cdots \otimes v'_{d,i,\mathbf{p}^j} \\ &\quad = \sum_{i=1}^{N} \sum_{j=1}^{d} \left(w'_{1,i} \otimes u_{p_1^j}\right) \otimes \cdots \otimes \left(y_{j,i} \otimes u_{q_j}\right) \otimes \cdots \otimes \left(w'_{d,i} \otimes u_{p_d^j}\right) \\ &\quad = \sum_{i=1}^{N} \sum_{j=1}^{d} \left(w'_{1,i} \otimes u_{q_1}\right) \otimes \cdots \otimes \left(y_{j,i} \otimes u_{q_j}\right) \otimes \cdots \otimes \left(w'_{d,i} \otimes u_{q_d}\right). \end{aligned}$$

Observe that the two linear spaces $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$ and $W_1 \otimes \cdots \otimes W_d \otimes U^{\otimes d}$ are naturally isomorphic. Let Φ be this isomorphism, that is,

$$\Phi: W_1 \otimes \cdots \otimes W_d \otimes U^{\otimes d} \to (W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U).$$

View the isomorphism Φ as a reordering of the terms. Now, the above element in $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$ can be written as

$$\begin{aligned}
&\Phi\left(\sum_{i=1}^{N} \sum_{j=1}^{d} w'_{1,i} \otimes \cdots \otimes y_{j,i} \otimes \cdots \otimes w'_{d,i} \otimes u_{q_1} \otimes \cdots \otimes u_{q_j} \otimes \cdots \otimes u_{q_d}\right) \\
&\quad = \Phi\left(\left(\sum_{i=1}^{N} \sum_{j=1}^{d} w'_{1,i} \otimes \cdots \otimes y_{j,i} \otimes \cdots \otimes w'_{d,i}\right) \otimes u_{q_1} \otimes \cdots \otimes u_{q_d}\right) \\
&\quad = \Phi\left(z_{i_1} \otimes \cdots \otimes z_{i_d} \otimes u_{q_1} \otimes \cdots \otimes u_{q_d}\right) \\
&\quad = \left(z_{i_1} \otimes u_{q_1}\right) \otimes \cdots \otimes \left(z_{i_d} \otimes u_{q_d}\right).
\end{aligned}$$

But this is the particular basis element we chose in $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$. Hence we conclude that the images of the linear maps (5) span the space $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$. By Theorem 5.2 it follows that the linear space $(W_1 \otimes U) \otimes \cdots \otimes (W_d \otimes U)$ has essential rank at most $m^{d-1} \cdot N$. ∎

7. ESSENTIAL RANK OF SYMMETRIC AND SKEW-SYMMETRIC TENSORS

Let W be a vector space over the complex numbers of dimension n. Recall that $\mathrm{Sym}(W)$ is the algebra of symmetric tensors over W. That is, $\mathrm{Sym}(W)$ is isomorphic to the algebra of polynomials in n variables with complex coefficients. Similarly, $\mathrm{Ext}(W)$ is the exterior algebra on W. The product in $\mathrm{Sym}(W)$ is denoted by $\cdot$ and the product in $\mathrm{Ext}(W)$ is denoted by $\wedge$. Both these algebras are graded, and we may write

$$\mathrm{Ext}(W) = \bigoplus_{d \geq 0} \mathrm{Ext}_d(W) \quad \text{and} \quad \mathrm{Sym}(W) = \bigoplus_{d \geq 0} \mathrm{Sym}_d(W).$$

Observe that

$$\dim(\mathrm{Ext}_d(W)) = \binom{n}{d} \qquad \text{and} \qquad \dim(\mathrm{Sym}_d(W)) = \left\langle {n \atop d} \right\rangle,$$

where $\binom{n}{d}$ is the number of ways to select a subset of size d from an n-element set and where $\left\langle {n \atop d} \right\rangle$ is the number of ways to select a multisubset of cardinality d from an n-element set. We have that $\left\langle {n \atop d} \right\rangle = \binom{n+d-1}{d}$.

Similar to Definition 6.1 we may define the rank of a symmetric respectively skew-symmetric tensor.

DEFINITION 7.1. An element v in the linear space $\mathrm{Ext}_d(W)$ has rank k if k is the smallest integer m such that there exist $y_{j,i} \in W$ for $1 \le j \le d$ and $1 \le i \le m$ such that

$$v = \sum_{i=1}^{m} y_{1,i} \wedge \cdots \wedge y_{d,i}.$$

Similarly, an element v in the linear space $\mathrm{Sym}_d(W)$ has rank k if k is the smallest integer m such that there exist $y_{j,i} \in W$ for $1 \le j \le d$ and $1 \le i \le m$ such that

$$v = \sum_{i=1}^{m} y_{1,i} \cdots y_{d,i}.$$

When $d = 2$ the maximal ranks that occur in $\mathrm{Ext}_2(W)$ and $\mathrm{Sym}_2(W)$ are well-known.

PROPOSITION 7.2. *Let W be a vector space over the complex numbers of dimension n. Then the maximal rank of* $\mathrm{Ext}_2(W)$ *is* $\lfloor n/2 \rfloor$ *and the maximal rank of* $\mathrm{Sym}_2(W)$ *is* $\lceil n/2 \rceil$.

Proof. The first result is a well-known result in exterior algebra; see, for example, [4]. An element in $\mathrm{Sym}_2(W)$ corresponds to a polynomial homogeneous of degree 2 and in n variables. It is well-known from linear algebra that such a polynomial may be written as a sum of n squares (or less) of linear forms. By the identity $a^2 + b^2 = (a + i \cdot b) \cdot (a - i \cdot b)$, we may combine $2 \cdot \lfloor n/2 \rfloor$ squares into $\lfloor n/2 \rfloor$ products. ∎

Similar to Definition 6.2 we have:

DEFINITION 7.3. The linear space $\mathrm{Ext}_d(W)$ ($\mathrm{Sym}_d(W)$) has essential rank k if k is the smallest integer m such that the set of elements v in $\mathrm{Ext}_d(W)$ ($\mathrm{Sym}_d(W)$) that has rank less than or equal to m form a dense set in the Euclidean topology.

When $\dim(W) = 6$ it is known that $\mathrm{Ext}_3(W)$ has essential rank 2; see [7]. Also when $\dim(W) = 4$ the essential rank of $\mathrm{Sym}_3(W)$ is equal to 2; see, for instance, [3, Corollary 4.10].

For a set A let the set $\binom{A}{d}$ be the set of all subsets of cardinality d of A. Similarly, let $\left\langle {A \atop d} \right\rangle$ be the set of all multisubsets of cardinality d of A. Observe that we have

$$\left| \binom{A}{d} \right| = \binom{|A|}{d} \qquad \text{and} \qquad \left| \left\langle {A \atop d} \right\rangle \right| = \left\langle {|A| \atop d} \right\rangle.$$

DEFINITION 7.4. A covering C of $\binom{A}{d}$ is a subset C of $\binom{A}{d}$ such that for all $I \in \binom{A}{d}$ there exist $J \in C$ such that $|I \cap J| \ge d - 1$. Similarly, a *covering* C of $\left\langle {A \atop d} \right\rangle$ is a subset C of $\left\langle {A \atop d} \right\rangle$ such that for all $I \in \left\langle {A \atop d} \right\rangle$ there exist $J \in C$ such that $|I \cap J| \geq d - 1$.

We will ($\left\langle {A \atop d} \right\rangle$) say that there is a covering of $\binom{n}{d}$ ($\left\langle {n \atop d} \right\rangle$) if there exists a covering of $\binom{A}{d}$ ($\left\langle {A \atop d} \right\rangle$), where cardinality of A is n.

The problem of finding coverings of $\binom{n}{d}$ as small as possible is called the lotto problem [14]. Similarly, the problem of finding small coverings of $\left\langle {n \atop d} \right\rangle$ is called the lotto problem with replacements.

Similar to Proposition 6.4 we have:

PROPOSITION 7.5. *Let W be an n-dimensional vector space over the complex numbers $\mathbb{C}$. If there exists a covering of $\binom{n}{d}$ of size N then the essential rank of* $\mathrm{Ext}_d(W)$ *is less than or equal to N. If there exists a covering of $\left\langle {n \atop d} \right\rangle$ of size N then the essential rank of* $\mathrm{Sym}_d(W)$ *is less than or equal to N.*

The proof is similar to the proof of Proposition 6.4 and thus is omitted.

For the remaining part of this section we will consider the case when $d = 3$.

DEFINITION 7.6. A Steiner triple system on a non-empty set A is a subset S of $\binom{A}{3}$ such that for all pairs $P \in \binom{A}{2}$ there is a unique triple $Q \in S$ such that $P \subseteq Q$.

A necessary and sufficient condition for a Steiner triple system to exist on a set A of cardinality n is that $n \equiv 1, 3 \bmod 6$. Observe also that the size of a Steiner triple system is $\frac{1}{3} \cdot \binom{n}{2} = (n \cdot (n-1))/6$.

PROPOSITION 7.7. *Assume that there exists a Steiner triple system on an n-set and on an m-set. Then there exists a covering of $\binom{n+m}{3}$ of size $(n \cdot (n-1) + m \cdot (m-1))/6$. Similarly, there exists a covering of $\left\langle {n+m \atop 3} \right\rangle$ of size $(n \cdot (n+5) + m \cdot (m+5))/6$.*

Proof. Assume that A_1 and A_2 are two disjoint sets such that $|A_1| = n$ and $|A_2| = m$. Let S_i be a Steiner triple system on A_i for $i = 1, 2$. Then we claim that $S_1 \cup S_2$ is a covering of the set $\binom{A_1 \cup A_2}{3}$. Assume that $I \in \binom{A_1 \cup A_2}{3}$. Then there is an index i such that $|A_i \cap I| \geq 2$. Hence $A_i \cap I$ contains a pair P. Then we may find $x \in A_i$ such that $P \cup \{x\} \in S_i$.

For the multiset case, consider the set of multisets

$$\{\{x, x, x\} : x \in A_1 \cup A_2\} \cup S_1 \cup S_2,$$

which has cardinality $n + m + (n \cdot (n-1) + m \cdot (m-1))/6$. It is easy to see that this is a covering of $\left\langle {A_1 \cup A_2 \atop 3} \right\rangle$. ∎

COROLLARY 7.8. *Let n be an even positive integer.*

- *If $n \equiv 0, 8 \bmod 12$ then there exists a covering of $\binom{n}{3}$ of size $(n \cdot (n-2))/12 + 3$.*

- *If* $n \equiv 2, 6 \bmod 12$ *then there exists a covering of* $\binom{n}{3}$ *of size* $(n \cdot (n-2))/12$.
- *If* $n \equiv 4 \bmod 12$ *then there exists a covering of* $\binom{n}{3}$ *of size* $(n \cdot (n-2)+4)/12$.
- *If* $n \equiv 10 \bmod 12$ *then there exists a covering of* $\binom{n}{3}$ *of size* $(n \cdot (n-2)+4)/12 + 1$.

Proof. We will only prove the first statement. Since $n/2 \equiv 0, 4 \bmod 6$ we have that $n/2 - 3 \equiv 1, 3 \bmod 6$, and $n/2 + 3 \equiv 1, 3 \bmod 6$. Thus by Proposition 7.7, there exists a covering of size

$$\frac{(n/2-3)\cdot(n/2-4)+(n/2+3)\cdot(n/2+2)}{6} = \frac{n\cdot(n-2)}{12} + 3. \quad \blacksquare$$

COROLLARY 7.9. *Let W be an n-dimensional vector space over the complex numbers* $\mathbb{C}$, *where n is an even positive integer*.

- *If* $n \equiv 0, 8 \bmod 12$ *then the essential rank of* $\mathrm{Ext}_3(W)$ *is less than or equal to* $(n \cdot (n-2))/12 + 3$.
- *If* $n \equiv 2, 6 \bmod 12$ *then the essential rank of* $\mathrm{Ext}_3(W)$ *is less than or equal to* $(n \cdot (n-2))/12$.
- *If* $n \equiv 4 \bmod 12$ *then the essential rank of* $\mathrm{Ext}_3(W)$ *is less than or equal to* $(n \cdot (n-2)+4)/12$.
- *If* $n \equiv 10 \bmod 12$ *then the essential rank of* $\mathrm{Ext}_3(W)$ *is less than or equal to* $(n \cdot (n-2)+4)/12 + 1$.

In the case when the dimension of the linear space W is 8, Corollary 7.9 says that the essential rank of $\mathrm{Ext}_3(W)$ is less than or equal to 7. But in fact, we do even better, as we will see in the next lemma.

LEMMA 7.10. *Let W be a linear space of dimension* 8 *over the complex number* $\mathbb{C}$. *Then the linear space* $V = \mathrm{Ext}_3(W)$ *has essential rank less than or equal to* 4.

Proof. Let $z_1, \ldots, z_8$ be a basis for W. Let $z_1^*, \ldots, z_8^*$ be the dual basis for W^*. Then $z_i^* \wedge z_j^* \wedge z_k^*$, where $1 \le i < j < k \le 8$, form a basis for V^*.

Choose $w'_{1,i} = z_{2i-1}$, $w'_{2,i} = z_{2i}$, and $w'_{3,i} = z_{2i-2} - z_{2i+1}$, for $i = 1, 2, 3, 4$ and where indices are counted modulo 8. Assume that $L \in V^*$ is apolar to the 12 linear maps

$$y_{1,i} \mapsto y_{1,i} \wedge w'_{2,i} \wedge w'_{3,i} = y_{1,i} \wedge z_{2i} \wedge (z_{2i-2} - z_{2i+1}),$$
$$y_{2,i} \mapsto w'_{1,i} \wedge y_{2,i} \wedge w'_{3,i} = z_{2i-1} \wedge y_{2,i} \wedge (z_{2i-2} - z_{2i+1}),$$
$$y_{3,i} \mapsto w'_{1,i} \wedge w'_{2,i} \wedge y_{3,i} = z_{2i-1} \wedge z_{2i} \wedge y_{3,i}.$$

We can write

$$L = \sum_{1 \leq i < j < k \leq 8} \alpha_{\{i,j,k\}} \cdot z_i^* \wedge z_j^* \wedge z_k^*.$$

By using the fact that L is apolar to the third map with $y_{3,i} = z_k$, we get that $\alpha_{\{2i-1,2i,k\}} = 0$. Since L is apolar to the first map with $y_{1,i} = z_{2i-2}$, we have $\alpha_{\{2i-2,2i,2i+1\}} = 0$. Similarly, since L is apolar to the second map with $y_{2,i} = z_{2i-2}$, we have $\alpha_{\{2i-2,2i-1,2i+1\}} = 0$.

Since L is apolar to the first map with $y_{1,i} = z_{2i+2}$, we get

$$\begin{aligned}
0 &= \left\langle L \middle| z_{2i+2} \wedge z_{2i} \wedge (z_{2i-2} - z_{2i+1}) \right\rangle \\
&= \operatorname{sign}(2i+2, 2i, 2i-2) \cdot \alpha_{\{2i+2,2i,2i-2\}} \\
&\qquad - \operatorname{sign}(2i+2, 2i, 2i+1) \cdot \alpha_{\{2i+2,2i,2i+1\}} \\
&= \operatorname{sign}(2i+2, 2i, 2i-2) \cdot \alpha_{\{2i+2,2i,2i-2\}}.
\end{aligned}$$

Again, use the first map with $y_{1,i} = z_{2i+4}$.

$$\begin{aligned}
0 &= \left\langle L \middle| z_{2i+4} \wedge z_{2i} \wedge (z_{2i-2} - z_{2i+1}) \right\rangle \\
&= \operatorname{sign}(2i+4, 2i, 2i-2) \cdot \alpha_{\{2i+4,2i,2i-2\}} \\
&\qquad - \operatorname{sign}(2i+4, 2i, 2i+1) \cdot \alpha_{\{2i+4,2i,2i+1\}} \\
&= -\operatorname{sign}(2i+4, 2i, 2i+1) \cdot \alpha_{\{2i+4,2i,2i+1\}}.
\end{aligned}$$

Use the first linear map with $y_{1,i} = z_{2i+3}$.

$$\begin{aligned}
0 &= \left\langle L \middle| z_{2i+3} \wedge z_{2i} \wedge (z_{2i-2} - z_{2i+1}) \right\rangle \\
&= \operatorname{sign}(2i+3, 2i, 2i-2) \cdot \alpha_{\{2i+3,2i,2i-2\}} \\
&\qquad - \operatorname{sign}(2i+3, 2i, 2i+1) \cdot \alpha_{\{2i+3,2i,2i+1\}} \\
&= \operatorname{sign}(2i+3, 2i, 2i-2) \cdot \alpha_{\{2i+3,2i,2i-2\}}.
\end{aligned}$$

By symmetry, use the second map, with $y_{2,i} = z_{2i-3}$, $y_{2,i} = z_{2i+3}$, and $y_{2,i} = z_{2i+4}$, to conclude that $\alpha_{2i-1,2i-3,2i+1} = 0$, $\alpha_{2i-1,2i+3,2i-2} = 0$, and $\alpha_{2i-1,2i+4,2i+1} = 0$. It is easy to see that we have shown that all coefficients of L vanish, and thus L is equal to 0. By Theorem 5.2, $\sum_{i=1}^{4} w_{i,1} \wedge w_{i,2} \wedge w_{i,3}$ is a generic canonical form for the linear space $\mathrm{Ext}_3(W)$, and thus the space has essential rank 4. ∎

The following corollary is straightforward to obtain.

COROLLARY 7.11. *Let n be an even positive integer.*

- *If* $n \equiv 0, 8 \mod 12$ *then there exists a covering of* $\langle {n \atop 3} \rangle$ *of size* $(n \cdot (n + 10))/12 + 3$.
- *If* $n \equiv 2, 6 \mod 12$ *then there exists a covering of* $\langle {n \atop 3} \rangle$ *of size* $(n \cdot (n + 10))/12$.
- *If* $n \equiv 4 \mod 12$ *then there exists a covering of* $\langle {n \atop 3} \rangle$ *of size* $(n \cdot (n + 10) + 4)/12$.
- *If* $n \equiv 10 \mod 12$ *then there exists a covering of* $\langle {n \atop 3} \rangle$ *of size* $(n \cdot (n + 10) + 4)/12 + 1$.

By Proposition 7.5 these upper bounds on covering $\langle {n \atop 3} \rangle$ give us upper bounds on the essential rank of $\mathrm{Sym}_3(W)$. But we are able to press these bounds down a little by the following proposition.

PROPOSITION 7.12 (R. E. Losonczy and J. Losonczy). *Let n and m be two positive integers such that there exists a Steiner triple system on sets of cardinalities n and m. Let W be a vector space over the complex number* $\mathbb{C}$ *of dimension* $n + m$. *Then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to*

$$\frac{n \cdot (n-1) + m \cdot (m-1)}{6} + \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil.$$

Proof. Let A_1 and A_2 be two disjoint sets of cardinalities n and m. Let A be the union of these two sets, that is, $A = A_1 \cup A_2$. Let S_i be a Steiner triple system on A_i. Hence $|S_1| = (n \cdot (n-1))/6$ and $|S_2| = (m \cdot (m - 1))/6$. Let T_i be a set of three element subsets of A_i such that $A_i =$

$\bigcup_{I \in T_i} I$. We would like the size of T_i to be as small as possible. Hence we may assume that $|T_i| = \lceil |A_i|/3 \rceil$. We will now prove that

$$\sum_{I \in S_1} a_{1,I} \wedge a_{2,I} \wedge a_{3,I} + \sum_{i \in S_2} b_{1,I} \wedge b_{2,I} \wedge b_{3,I}$$
$$+ \sum_{I \in T_1} c_{1,I} \wedge c_{2,I} \wedge c_{3,I} + \sum_{I \in T_2} d_{1,I} \wedge d_{2,I} \wedge d_{3,I}$$

is a generic canonical form for $\operatorname{Sym}_3(W)$, where $a_{j,I}, \ldots, d_{j,I} \in W$. Let $\{z_i\}_{i \in A_1 \cup A_2}$ be a basis for W and let $\{z_i^*\}_{i \in A_1 \cup A_2}$ be the dual basis for W^*.

Let $a'_{j,I} = z_x$, where x is the jth element in the triple $I \in S_1$. Similarly, let $b'_{j,I} = z_x$, where x is the jth element in the triple $I \in S_2$. Now we choose $c'_{j,I}$ and $d'_{j,I}$ more carefully. For $\{x_1, x_2, x_3\} = I \in T_1$ let $c'_{1,I} = (z_{x_1} + z_{x_2})$, $c'_{2,I} = (z_{x_1} + z_{x_3})$, and $c'_{3,I} = (z_{x_2} + z_{x_3})$. The same pattern for the $d'_{j,I}$: for $\{x_1, x_2, x_3\} = I \in T_2$ let $d'_{1,I} = (z_{x_1} + z_{x_2})$, $d'_{2,I} = (z_{x_1} + z_{x_3})$, and $d'_{3,I} = (z_{x_2} + z_{x_3})$.

Let $L \in \operatorname{Sym}_3(W^*)$ and assume that L is apolar to all the linear maps described by Theorem 5.2. Write L in terms of the dual basis

$$L = \sum_{\{i,j,k\}} \beta_{\{i,j,k\}} \cdot z_i^* \cdot z_j^* \cdot z_k^*,$$

where the sum ranges over all three element multisubsets of A.

We claim that for $i, j \in A_1$ and $k \in A$ we have that $\beta_{\{i,j,k\}} = 0$. This is true since we may find a linear map $y \mapsto z_i \cdot z_j \cdot y$, such that L is apolar to this linear map. By symmetry, for $i, j \in A_2$ and $k \in A$, we have $\beta_{\{i,j,k\}} = 0$. We conclude that $\beta_I = 0$ when I is a set, that is, when there are no repetitions in I. Hence any square free monomial in L has a vanishing coefficient.

Let $\{i, j, k\} \in T_1$. Hence L is apolar to the map $y \mapsto (z_i + z_j) \cdot (z_i + z_k) \cdot y$. For $h \in A$ we have that

$$\begin{aligned} 0 &= \left\langle L \middle| (z_i + z_j) \cdot (z_i + z_k) \cdot z_h \right\rangle \\ &= \left\langle L \middle| z_i^2 \cdot z_h + z_i \cdot z_j \cdot z_h + z_i \cdot z_k \cdot z_h + z_j \cdot z_k \cdot z_h \right\rangle \\ &= \beta_{\{i,i,h\}} + \beta_{\{i,j,h\}} + \beta_{\{i,k,h\}} + \beta_{\{j,k,h\}} \\ &= \beta_{\{i,i,h\}}. \end{aligned}$$

Since T_1 contains triplets, which contain all the elements of A_1, we have that for all $i \in A_1$ and $h \in A$ the coefficient $\beta_{\{i,i,h\}}$ vanishes. By symmetry, and by working with T_2 we obtain for $i, h \in A$ that $\beta_{\{i,i,h\}} = 0$. Hence all the monomials in L vanish and we conclude that L is equal to zero. By Theorem 5.2 the conclusion follows. ∎

COROLLARY 7.13. *Let W be an n-dimensional vector space over the complex numbers $\mathbb{C}$, where n is an even positive integer.*

- *If* $n \equiv 0 \bmod 12$ *then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to* $(n \cdot (n+2))/12 + 3$.
- *If* $n \equiv 2 \bmod 12$ *then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to* $(n \cdot (n+2) + 4)/12 + 1$.
- *If* $n \equiv 4 \bmod 12$ *then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to* $(n \cdot (n+2))/12 + 1$.
- *If* $n \equiv 6 \bmod 12$ *then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to* $(n \cdot (n+2))/12$.
- *If* $n \equiv 8 \bmod 12$ *then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to* $(n \cdot (n+2) + 4)/12 + 4$.
- *If* $n \equiv 10 \bmod 12$ *then the essential rank of* $\mathrm{Sym}_3(W)$ *is less than or equal to* $(n \cdot (n+2))/12 + 2$.

8. CONCLUDING REMARKS

It would be interesting to know if Theorem 5.2 has more applications than the one presented in this article and in [3]. So far, all the examples seen have been over algebras such as the tensor algebra, the algebra of polynomials, and the exterior algebra. Thus the general notion of $\mathscr{S}$-algebra has not been used yet.

The problem of finding the essential rank of $\mathrm{Ext}_d(W)$ has been considered recently by J. Losonczy [14]. He has found a geometrical interpretation of this essential rank. Moreover, he has worked out more upper bounds for the essential rank of $\mathrm{Ext}_d(W)$.

In Section 7 we considered symmetric and skew-symmetric tensors. They correspond to sets and multisets, which are dual to each other. There may be a deeper relationship between these tensors, as the following conjecture suggests.

CONJECTURE 8.1. *Let n and d be non-negative integers and let $m = n + d - 1$. Let V and W be vector spaces over the complex numbers $\mathbb{C}$ of dimensions n and m. Is the essential rank of* $\mathrm{Sym}_d(V)$ *equal to the essential rank of* $\mathrm{Ext}_d(W)$?

This hypothesis holds for small cases, for instance, when $d \leq 2$, and when $n \leq 2$. When $n = 4$ and $d = 3$ the results that $\mathrm{Sym}_3(V)$ and $\mathrm{Ext}_3(W)$ have essential rank 2 are known; see the comment after Definition 7.3. When $n = 6$ and $d = 3$ Corollary 7.13 implies that $\mathrm{Sym}_3(V)$ has essential rank at most 4. By counting parameters, it is easy to conclude that the essential rank of $\mathrm{Sym}_3(V)$ is equal to 4. Lemma 7.10 says that the essential rank of $\mathrm{Ext}_3(W)$ is at most 4.

If this conjecture is true, it suggests a duality between the linear spaces $\mathrm{Sym}_d(V)$ and $\mathrm{Ext}_d(W)$. A very vague hint for this duality is that $\dim(\mathrm{Sym}_d(V)) = \dim(\mathrm{Ext}_d(W))$.

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