A generalization of combinatorial identities for stable discrete series constants

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Abstract

This article is concerned with the constants that appear in Harish-Chandra's character formula for stable discrete series of real reductive groups, although it does not require any knowledge about real reductive groups or discrete series. In Harish-Chandra's work the only information we have about these constants is that they are uniquely determined by an inductive property. Later Goresky–Kottwitz–MacPherson and Herb gave different formulas for these constants; see [GKM97, Theorem 3.1] and [Her00, Theorem 4.2]. In this article we generalize these formulas to the case of arbitrary finite Coxeter groups (in this setting, discrete series no longer make sense), and give a direct proof that the two formulas agree. We actually prove a slightly more general identity that also implies the combinatorial identity underlying the discrete series character identities of [Mor11, Proposition 3.3.1]. We also introduce a signed convolution of valuations on polyhedral cones in Euclidean space and show that the resulting function is a valuation. This gives a theoretical framework for the valuation appearing in [GKM97, Appendix A]. In Appendix B we extend the notion of 2-structures (due to Herb) to pseudo-root systems.

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1 Introduction

Although this paper deals exclusively with the combinatorics of real hyperplane arrangements and Coxeter complexes, it has its origin in the representation theory of real reductive groups and its connections with the cohomology of locally symmetric spaces, and in particular of Shimura varieties. We start by explaining some of this background. This explanation can be safely skipped by the reader not interested in Shimura varieties.

Let G be an algebraic group over $\mathbb{Q}$. To simplify the exposition, we assume that G is connected and semisimple. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and K be an open compact subgroup of $G(\mathbb{A}^\infty)$, where $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles of $\mathbb{Q}$. We consider the double quotient $X_K = G(\mathbb{Q})\backslash(G(\mathbb{R})\times G(\mathbb{A}^\infty))/(K_\infty \times K)$. This is a real analytic variety for K small enough, and the projective system $(X_K)_{K\subset G(\mathbb{A}^\infty)}$ has an action of $G(\mathbb{A}^\infty)$ by Hecke correspondences, which induces an action of the Hecke algebra at level K on the cohomology of X_K, for any reasonable cohomology theory.

We restrict our attention further to the case where the real Lie group $G(\mathbb{R})$ has a discrete series. This is the so-called "equal rank case" because it occurs if and only if the groups $G(\mathbb{R})$ and K_∞ have the same rank. Then the L^2-cohomology $H^*_{(2)}(X_K)$ is finite-dimensional, and Matsushima's formula, proved in this generality by Borel and Casselman [BC83], gives a description of this cohomology and of its Hecke algebra action in terms of discrete automorphic representations of G whose infinite component is a cohomological representation of $G(\mathbb{R})$, and in particular, either a discrete series or a special type of non-tempered representation.

Another cohomology of interest in this case is the intersection cohomology $IH^*(\overline{X}_K)$ of the minimal Satake compactification $\overline{X}_K$ of X_K. In order to study this cohomology, Goresky, Harder and MacPherson introduced in [GHM94] a family of cohomology theories called "weighted cohomologies" and showed that the two middle weighted cohomologies agree with $IH^*(\overline{X}_K)$ if X_K has the structure of a complex algebraic variety. This result was later generalized by Saper in [Sap18].

All the cohomology theories that we discussed have actions of the Hecke algebra, and the isomorphism of the previous paragraph is equivariant for this action. Zucker conjectured that there should be a Hecke-equivariant isomorphism between $H^*_{(2)}(X_K)$ and $IH(\overline{X}_K)$. This conjecture was proved by Looijenga [Loo88], Looijenga–Rapoport [LR91] and Saper–Stern [SS90] if X_K has the structure of a complex algebraic variety and by Saper [Sap18] in general. In particular, by comparing the formulas for the action of a Hecke operator on weighted cohomology (this was calculated by Goresky and MacPherson using topological methods in [GM03]) and on L^2-cohomology (this was calculated by Arthur using the Arthur–Selberg trace formula in [Art89]), we obtain a formula for averaged discrete series characters of the group $G(\mathbb{R})$. One of the goals of the paper [GKM97] of Goresky–Kottwitz–MacPherson was to prove this identity directly.

If moreover the space X_K is the set of complex points of a Shimura variety, then it descends to an algebraic variety over an explicit number field E known as the reflex field, as does the minimal Satake compactification, and so the intersection cohomology has a natural action of the absolute Galois group $\mathrm{Gal}(\overline{E}/E)$. We can further complicate the calculation by trying to calculate the trace on $IH^*(\overline{X}_K)$ of Hecke operators twisted by elements of the group $\mathrm{Gal}(\overline{E}/E)$, for example, powers of Frobenius maps. In the case where X_K is a Siegel modular variety, this was done by the second author in [Mor11]. It requires a slightly different character identity for averaged discrete series characters of $G(\mathbb{R})$, also involving discrete series characters of the endoscopic groups of G, and whose relationship with the Goresky–Kottwitz–MacPherson identity was not clear.

We return to a discussion of the current article.

In a previous article [EMR19], we investigate the character identity of [Mor11]. In particular we relate it to the geometry of the Coxeter complex of the symmetric group and give a simpler and more natural proof than the brute force calculation in the appendix of [Mor11]. The goal of the present article is to generalize the approach and methods of [EMR19] and to prove a combinatorial identity (Theorem 4.1.1) that implies the character formulas of [GKM97] and of [Mor11] (see Section 5). To obtain the character formula of [GKM97] from our results, we need to use Herb's formula for averaged discrete series characters (see for example [Her79] and [Her00]). We also generalize, in Corollary 2.2.7 and Lemma 2.3.4, the geometric result of [EMR19] (see Theorem 4.3 of that article). In fact, we prove an identity for all Coxeter systems with finite Coxeter groups, and not just the root systems that are generated by strongly orthogonal roots. The representation-theoretic interpretation of our identity in the general case is still unclear.

We now describe in more detail the different sections of the article.

In Section 2 we review some background material about real hyperplane arrangements and introduce our main geometric construction, which we call the *weighted complex*. The weighted complex is the set of all the faces of a fixed hyperplane arrangement that are on the nonnegative side of an auxiliary hyperplane H_λ. It contains what is known as the *bounded complex* in the theory of affine oriented matroids, and coincides with it if H_λ is in general position. We also prove that, under a hypothesis about the dihedral angles between the hyperplanes of the arrangement (Condition (A) in Subsection 2.2, which is always true in the Coxeter case), the weighted complex is shellable.

In Section 3 we introduce one half of our main identity (see Definition 3.1.4) and study its properties. In particular, we establish a recursion result in Proposition 3.2.2 which will be the heart of the proof of Theorem 4.1.1. Our treatment is inspired by that of Sections 1-2 and Appendix A of [GKM97], but we make no assumptions on the Coxeter system. In fact, we prove the recursion result for arbitrary hyperplane arrangements. The main technical tool is Theorem A.1.5, which generalizes [GKM97, Proposition A.4].

Section 4 contains the statement and proof of our main theorem (Theorem 4.1.1). We specialize to the case of Coxeter systems and introduce the second half of our main identity, generalizing a construction of Herb that we review in detail in Appendix B. The proof that both halves of the identity are equal follows the same lines as the proofs of the character formulas in the Goresky–Kottwitz–MacPherson and Herb articles: First, using the recursive expressions for both sides (Propositions 3.2.2 and 4.2.1), we reduce to the case where λ is dominant so that H_λ is as far as possible from the base chamber. Then we treat this case directly by a geometric argument based on the results of Section 2.

In Section 5 we explain how Theorem 4.1.1 implies the identities of [GKM97, Theorem 3.1] and of [EMR19, Theorem 6.4], and in Section 6 we include concluding remarks.

We finish with two appendices. Each can be read independently from the rest of the article. The goal of our Appendix A is to generalize [GKM97, Proposition A.4], which is a key part of the induction in the proof of our main theorem. In Appendix A of their article [GKM97], Goresky–Kottwitz–MacPherson show that a certain function, which they call $\psi_C(x, \lambda)$, is a *valuation* (see Definition A.1.1) on closed convex polyhedral cones, although they do not phrase it in these terms. We show that their function is a special case of a general construction that takes two valuations and produces a third one via a signed convolution. See Definition A.1.4 for the precise definition of this signed convolution.

In Appendix B we review the theory of 2-structures, due to Herb; see for example Herb's review article [Her00]. We believe that this will be useful to the reader for a number of reasons. The proofs of the fundamental results of this theory are somewhat scattered in the literature and sometimes left as exercises. Furthermore, we needed to slightly adapt a number of results so that they continue to hold for Coxeter systems that do not necessarily arise from a (crystallographic) root system.

2 Hyperplane arrangements

2.1 Background material

We fix a finite-dimensional $\mathbb{R}$-vector space V with an inner product $(\cdot, \cdot)$. If $\alpha \in V$, we write

$$H_\alpha = \{x \in V : (\alpha, x) = 0\}, \qquad H_\alpha^+ = \{x \in V : (\alpha, x) > 0\}, \qquad H_\alpha^- = \{x \in V : (\alpha, x) < 0\}.$$

We also denote by s_α the (orthogonal) reflection across the hyperplane H_α.

Let $(\alpha_e)_{e\in E}$ be a finite family of nonzero vectors in V. The corresponding *(central) hyperplane arrangement* is the family of hyperplanes $\mathcal{H} = (H_{\alpha_e})_{e\in E}$. Let V_0 be the intersection of all the hyperplanes, that is, $V_0 = \bigcap_{e\in E} H_{\alpha_e}$. We say that the arrangement $\mathcal{H}$ is *essential* if $V_0 = \{0\}$, which means that the family $(\alpha_e)_{e\in E}$ spans V.

Remark 2.1.1. We will mostly be interested in hyperplane arrangements that are essential and have no repeated hyperplanes, that is, where $\alpha_e \notin \mathbb{R}\alpha_f$ for distinct $e, f \in E$. However, some of the arrangements arising in the inductive formula of Proposition 3.2.2 might not satisfy these conditions, so we do not want to impose this restriction earlier than necessary.

Consider the map $s : V \longrightarrow \{+,-,0\}^E$ sending $x \in V$ to the family $(\mathrm{sign}((\alpha_e, x)))_{e\in E}$, where $\mathrm{sign} : \mathbb{R} \longrightarrow \{+,-,0\}$ is the map sending positive numbers to $+$, negative numbers to $-$ and zero to 0.

Remark 2.1.2. The image of the map $s : V \longrightarrow \{+,-,0\}^E$ is the set of covectors of an oriented matroid (see for example [BLVS+99, Definition 4.1.1]), which is the oriented matroid corresponding to the hyperplane arrangement. In fact, some of our results extend to general oriented matroids. In this article we have chosen to concentrate on hyperplane arrangements to keep the exposition more concrete. In particular, we do not assume that the reader knows what an oriented matroid is.

We denote by $\mathscr{L}$ the set of nonempty subsets of V of the form $C = s^{-1}(X)$, for a sign vector $X \in \{+,-,0\}^E$. The elements of $\mathscr{L}$ are called *faces* of the arrangement. The set $\mathscr{L}$ has a natural partial order given by $C \le D$ if and only if $C \subseteq \overline{D}$. The relation $C \le D$ is equivalent to the fact that for every $e \in E$ we have $s(C)_e = 0$ or $s(C)_e = s(D)_e$. The set $\mathscr{L}$ with this partial order is called the *face poset* of the arrangement. Note that V_0 is the minimal element of $\mathscr{L}$. When we adjoin a maximal element $\widehat{1}$ to the poset $\mathscr{L}$, we obtain a lattice $\mathscr{L} \cup \{\widehat{1}\}$ known as the *face lattice*. Note that, under our convention, faces other than V_0 are not closed subsets of V: for every $C \in \mathscr{L}$, the closure $\overline{C}$ is a closed convex polyhedral cone in V, and it is an intersection of closed half-spaces $\overline{H^{\pm}_{\alpha_e}}$. The poset $\mathscr{L}$ is graded with the rank of a face $C \in \mathscr{L}$ given by $\rho(C) = \dim(C) - \dim(V_0)$, where we write $\dim(C)$ for $\dim(\mathrm{Span}(C))$.

We denote by $\mathscr{T}$ the set of maximal faces of $\mathscr{L}$. These elements are often called *chambers*, *regions* or *topes*, and are the connected components of $V - \bigcup_{e\in E} H_{\alpha_e}$. If $T \in \mathscr{T}$ then T is an open subset of V, and its closure is a closed convex polyhedral cone of dimension $\dim(V)$.

If $X, Y \in \{+,-,0\}^E$, their *composition* $X \circ Y$ is the sign vector defined by

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise.} \end{cases}$$

If $C, D \in \mathscr{L}$ then $s(C) \circ s(D)$ is also the image of a face of $\mathscr{L}$, and we denote this face by $C \circ D$. This is the unique face of $\mathscr{L}$ that contains all vectors of V of the form $x + \varepsilon y$, with $x \in C$, $y \in D$ and $\varepsilon > 0$ sufficiently small (relative to x and y). Define the *separation set* of C and D to be the set

$$S(C, D) = \{e \in E : s(C)_e = -s(D)_e \neq 0\}.$$

This is the set of $e \in E$ such that C and D are on different sides of the hyperplane H_{α_e}.

Fix a chamber $B \in \mathscr{T}$. We can then define a partial order $\preceq_B$ on $\mathscr{T}$ by declaring that $T \preceq_B T'$ if and only if $S(B, T) \subseteq S(B, T')$. The resulting poset is called the *chamber poset with base chamber* B. We will denote it by $\mathscr{T}_B$. It is a poset with minimal element B and maximal element $-B$. When all the hyperplanes are distinct, this poset is also graded with the poset rank function $\rho(T) = |S(B, T)|$; see [BLVS+99, Proposition 4.2.10].

We also consider the graph with vertex set $\mathscr{T}$, where two chambers $T, T' \in \mathscr{T}$ are connected by an edge if and only if $\overline{T} \cap \overline{T}'$ spans a hyperplane (necessarily one of the H_{α_e}). In this situation, we say that this hyperplane is a *wall* of the chambers T and T'. This graph is called the *chamber graph*. In the case when all the hyperplanes of the arrangement $\mathcal{H}$ are distinct, the distance between two chambers T and T' in this graph is $|S(T, T')|$; see [BLVS+99, Proposition 4.2.3].

Consider the sphere $\mathbb{S}$ of center 0 and radius 1 in V/V_0. The intersections $\overline{C} \cap \mathbb{S}$, for $C \in \mathscr{L}$, form a regular cell decomposition $\Sigma(\mathscr{L})$ of $\mathbb{S}$, and we will identify $\mathscr{L}$ with the face poset of this regular cell decomposition.

We will need the following notion and result. For instance, see [BLVS$^+$99, Definition 4.7.14].

Definition 2.1.3. A pure n-dimensional polytopal complex Δ is *shellable* if it is 0-dimensional (and hence a collection of a finite number of points), or if there is a linear order of the facets $F_1, F_2, \dots, F_k$ of Δ, called a *shelling order*, such that:

(i) The boundary complex of F_1 is shellable.

(ii) For $1 < j \leq k$ the intersection of $\overline{F_j}$ with the union of the closures of the previous facets is nonempty and is the beginning of a shelling of the $(n-1)$-dimensional boundary complex of F_j, that is,

$$\overline{F_j} \cap (\overline{F_1} \cup \overline{F_2} \cup \cdots \cup \overline{F_{j-1}}) = \overline{G_1} \cup \overline{G_2} \cup \cdots \cup \overline{G_r},$$

where $G_1, G_2, \dots, G_r, \dots, G_t$ is a shelling order of ∂F_j and $r \geq 1$.

Theorem 2.1.4. *([BLVS$^+$99, Theorem 4.3.3].) Let B be a chamber in $\mathscr{T}$. Then any linear extension of the chamber poset with base chamber B is a shelling order on the facets of $\Sigma(\mathscr{L})$.*

Finally, we recall the definition of the star of a face in $\mathscr{L}$.

Definition 2.1.5. Let $C \in \mathscr{L}$. The *star* of C in $\mathscr{L}$ is $\{D \in \mathscr{L} : C \leq D\}$. Geometrically it is the set of faces of $\mathscr{L}$ whose closure contains C. We will denote it by $\mathscr{L}_{\geq C}$.

Lemma 2.1.6. *Let $C \in \mathscr{L}$ and let $E(C) = \{e \in E : C \subset H_{\alpha_e}\}$. Consider the hyperplane arrangement $\mathcal{H}(C) = (H_{\alpha_e})_{e \in E(C)}$ and let $\mathscr{L}_{\mathcal{H}(C)}$ be its face poset. Then the following three statements hold:*

(i) Each face D of $\mathscr{L}$ is contained in a unique face D' of $\mathscr{L}_{\mathcal{H}(C)}$, and the map $D \longmapsto D'$ induces an isomorphism of posets $\iota_C : \mathscr{L}_{\geq C} \xrightarrow{\sim} \mathscr{L}_{\mathcal{H}(C)}$. In particular, it sends the chambers of $\mathscr{T} \cap \mathscr{L}_{\geq C}$ to the chambers of $\mathscr{L}_{\mathcal{H}(C)}$.

(ii) If $D_1, D_2 \in \mathscr{L}_{\geq C}$ then the inclusion $S(D_1, D_2) \subset E(C)$ holds. In particular, we have the equality $S(D_1, D_2) = S(\iota_C(D_1), \iota_C(D_2))$, where the isomorphism ι_C is as in (i).

(iii) The isomorphism $\iota_C : \mathscr{L}_{\geq C} \xrightarrow{\sim} \mathscr{L}_{\mathcal{H}(C)}$ preserves composition and dimension, that is, for all $D, D' \in \mathscr{L}_{\geq C}$, the identities $\iota_C(D \circ D') = \iota_C(D) \circ \iota_C(D')$ and $\dim(\iota_C(D)) = \dim(D)$ hold.

In particular, $\mathscr{L}_{\geq C}$ is also isomorphic to the face poset of a regular cell decomposition of the unit sphere in $V / \bigcap_{e \in E(C)} H_{\alpha_e}$ that we denote by $\Sigma(\mathscr{L}_{\geq C})$.

Proof of Lemma 2.1.6. Statement (i) is clear.

We prove (ii). Let $D_1, D_2 \in \mathscr{L}_{\geq C}$, and let $e \in S(D_1, D_2)$. Suppose for example that $s(D_1)_e = +$ and $s(D_2)_e = -$. (The other case is similar.) Then $\overline{D}_1 \subset \overline{H^+_{\alpha_e}}$ and $\overline{D}_2 \subset \overline{H^-_{\alpha_e}}$, so $C \subset \overline{D}_1 \cap \overline{D}_2 \subset \overline{H^+_{\alpha_e}} \cap \overline{H^-_{\alpha_e}} = H_{\alpha_e}$, which implies that $e \in E(C)$.

The first statement of (iii) follows easily from the definitions: the composition $D \circ D'$ is defined on the sign vectors of D and D', and the isomorphism ι_C just forgets the coordinates outside of $E(C)$ in these sign vectors.

We prove the second statement of (iii). Let $D \in \mathscr{L}_{\geq C}$, and let D' be the unique face of $\mathscr{L}_{\mathcal{H}(C)}$ containing D. We clearly have $\dim(D) \leq \dim(D')$. If $\dim(D') > \dim(D)$ then there exists $e \in E$ such that $D \subset H_{\alpha_e}$ and $D' \not\subset H_{\alpha_e}$. But $C \subset \overline{D}$, so this implies that $e \in E(C)$. As D' is not included in H_{α_e}, it must be contained in one of the open half-spaces $H^{\pm}_{\alpha_e}$, contradicting the fact that D' contains D.

Remark 2.1.7. Let $C \in \mathscr{L}$ and let $F' = \{e \in E : C \not\subset H_{\alpha_e}\}$. Then the set $\mathscr{T} \cap \mathscr{L}_{\geq C}$ is equal to $\{T \in \mathscr{T} : \forall e \in F',\ s(T)_e = s(C)_e\}$, so it is a T-convex subset of $\mathscr{T}$ in the sense of [BLVS+99, Definition 4.2.5]; see [BLVS+99, Proposition 4.2.6]. In other words, it contains every shortest path in the chamber graph between any two of its elements, so it is a lower order ideal in $\mathscr{T}_B$ for every choice of base chamber $B \in \mathscr{T} \cap \mathscr{L}_{\geq C}$.

2.2 The weighted complex

Definition 2.2.1. Let $\lambda \in V$. We consider the following subset of the face poset $\mathscr{L}$:

$$\mathscr{L}_\lambda = \{C \in \mathscr{L} : C \subseteq \overline{H_\lambda^+}\}.$$

More generally, if C_0 is a fixed face of $\mathscr{L}$, we also consider the intersection

$$\mathscr{L}_{\lambda,\geq C_0} = \mathscr{L}_\lambda \cap \mathscr{L}_{\geq C_0}.$$

Remark 2.2.2. (See [BLVS+99, Section 4.5] for definitions.) If $\lambda \neq 0$, then the hyperplane arrangement $\{H_\lambda\} \cup \{H_{\alpha_e} : e \in E\}$ defines an affine oriented matroid with distinguished hyperplane H_λ. If H_λ is in general position relative to the H_{α_e}, that is, if λ is not in the span of any family $(\alpha_e)_{e\in F}$ for $|F| \leq \dim(V) - 1$, then $\mathscr{L}_\lambda$ coincides with the bounded complex of this affine oriented matroid. In general, $\mathscr{L}_\lambda$ is larger.

The basic properties of the subsets $\mathscr{L}_\lambda$ and $\mathscr{L}_{\lambda,\geq C_0}$ are given in the following proposition.

Proposition 2.2.3. *The following two statements hold:*

(i) For a fixed face C_0 of $\mathscr{L}$ the set $\mathscr{L}_{\lambda,\geq C_0}$ is a lower order ideal in $\mathscr{L}_{\geq C_0}$.

(ii) Let $C \in \mathscr{L}_\lambda$. Then there exists $T \in \mathscr{T} \cap \mathscr{L}_\lambda$ such that $C \leq T$.

Proof. It suffices to prove (i) when C_0 is the minimal face of $\mathscr{L}$. Let $C, D \in \mathscr{L}$ such that $C \leq D$ and $D \in \mathscr{L}_\lambda$. The hypothesis implies that $C \subset \overline{D}$ and $D \subset \overline{H_\lambda^+}$. As $\overline{H_\lambda^+}$ is closed, this immediately gives $C \subset \overline{H_\lambda^+}$, hence $C \in \mathscr{L}_\lambda$.

To show (ii) let $D_1, D_2, \ldots, D_p$ be the chambers of $\mathscr{T}$ that are larger than C with respect to the partial order $\leq$. If one of them is contained in $\overline{H_\lambda^+}$, then we are done. Otherwise, for every $i \in \{1, 2, \ldots, p\}$, we can find a point $x_i \in D_i$ such that $(\lambda, x_i) < 0$. Let $\mathcal{H}'$ be the subarrangement of $\mathcal{H}$ where we remove all the hyperplanes of $\mathcal{H}$ that contain the cone C. In the arrangement $\mathcal{H}'$ all the points x_i are contained in the same chamber C'. In particular, the convex hull P of the points $x_1, x_2, \ldots, x_p$ is contained in C'. The convex hull P intersects the linear span of the cone C in a point x. Since all the points x_i are in the open half-space H_λ^-, so is the point x, that is, $(\lambda, x) < 0$. By inserting the hyperplanes of $\mathcal{H}$ that contain the cone C back in the arrangement $\mathcal{H}'$, we subdivide the region C' into regions $C_1, C_2, \ldots, C_p$. But the point x belongs to the closure of each region C_i, thus the point x belongs to the cone C. This is a contradiction since C is contained in the half-space $\overline{H_\lambda^+}$, so $(\lambda, x) \geq 0$.

Definition 2.2.4. The subcomplex of the cell decomposition $\Sigma(\mathscr{L})$, respectively $\Sigma(\mathscr{L}_{\geq C_0})$, whose face poset is the lower order ideal $\mathscr{L}_\lambda$, respectively $\mathscr{L}_{\lambda,\geq C_0}$, is called the *weighted complex* and denoted by $\Sigma(\mathscr{L}_\lambda)$, respectively $\Sigma(\mathscr{L}_{\lambda,\geq C_0})$. By Proposition 2.2.3 it is pure of the same dimension as $\Sigma(\mathscr{L}_\lambda)$, respectively $\Sigma(\mathscr{L}_{\lambda,\geq C_0})$.

We now introduce a geometric condition on the hyperplane arrangement $\mathcal{H}$ that will imply the shellability of the weighted complex.

Condition (A). *Denote by (A) the following condition on the family $(\alpha_e)_{e\in E}$ (or the corresponding arrangement): For every $T \in \mathscr{T}$ and for every $e \in E$ such that $S = \overline{T} \cap H_{\alpha_e}$ is of dimension* $\dim(V) - 1$*, that is, S is a facet of the convex cone $\overline{T}$, the following inclusions hold:*

$$T \subseteq \operatorname{relint}(S) + \mathbb{R}_{>0}\alpha_e \text{ if } T \subseteq H^+_{\alpha_e},$$
$$T \subseteq \operatorname{relint}(S) + \mathbb{R}_{<0}\alpha_e \text{ if } T \subseteq H^-_{\alpha_e},$$

where $\operatorname{relint} S$ *is the relative interior of the cone S, that is, the interior of S in* $\operatorname{Span}(S)$.

Geometrically, condition (A) means that if $T \in \mathscr{T}$ then the dihedral angle between any two adjacent facets (facets whose intersection is a face of dimension $\dim(V)-2$) of the convex polyhedral cone $\overline{T}$ is acute, that is, less than or equal to $\pi/2$.

Proposition 2.2.5. *Suppose that the arrangement $\mathcal{H}$ satisfies condition (A). Let $T, T' \in \mathscr{T}$ and $e \in E$ such that $S(T,T') = \{e\}$, the inner product (α_e, λ) is nonnegative and the inclusion $T' \subset H^-_{\alpha_e}$ holds. Then $T' \in \mathscr{L}_\lambda$ implies that $T \in \mathscr{L}_\lambda$.*

Proof. The hypothesis implies that $\overline{T} \cap \overline{T}' = \overline{T} \cap H_{\alpha_e} = \overline{T}' \cap H_{\alpha_e}$. We denote this intersection by S. It is a facet of both $\overline{T}$ and $\overline{T}'$. By condition (A), we have $T \subset \operatorname{relint} S + \mathbb{R}_{>0}\alpha_e$ and $T' \subset \operatorname{relint} S + \mathbb{R}_{<0}\alpha_e$. In particular, if $x \in T$ then there exists $c > 0$ such that $x - c \cdot \alpha_e \in T'$. Then we have $(x, \lambda) = (x - c\cdot\alpha_e, \lambda) + c \cdot (\alpha_e, \lambda) \geq 0$. This implies that $T \subset \overline{H^+_\lambda}$, that is, $T \in \mathscr{L}_\lambda$.

Corollary 2.2.6. *Suppose that the arrangement $\mathcal{H}$ satisfies condition (A). If $(\lambda, \alpha_e) \geq 0$ for every $e \in E$ and if there exists $B \in \mathscr{T}$ such that $B \subset H^+_{\alpha_e}$ for every $e \in E$, then $\mathscr{T} \cap \mathscr{L}_\lambda$ is a lower order ideal in $\mathscr{T}_B$. More generally, if $C \in \mathscr{L}$ and $E(C) = \{e \in E : C \subset H_{\alpha_e}\}$, if $(\lambda, \alpha_f) \geq 0$ for every $f \in E(C)$ and if there exists $B \in \mathscr{T} \cap \mathscr{L}_{\geq C}$ such that $B \subset H^+_{\alpha_f}$ for every $f \in E(C)$, then $\mathscr{T} \cap \mathscr{L}_{\lambda,\geq C}$ is a lower order ideal in $\mathscr{T}_B$.*

Proof. It suffices to prove the second statement. Let T, T' be such that $S(B,T) \subset S(B,T')$ and $T' \in \mathscr{L}_{\lambda,\geq C}$. We want to show that $T \in \mathscr{L}_{\lambda,\geq C}$. As $\mathscr{T}_B$ is a graded poset and the intersection $\mathscr{T} \cap \mathscr{L}_{\geq C}$ is a lower order ideal in $\mathscr{T}_B$ (see Remark 2.1.7), we know that $T \in \mathscr{T} \cap \mathscr{L}_{\geq C}$, and it suffices to treat the case where $S(T', B) - S(T, B)$ is a singleton. Let f be the single index of $S(B,T') - S(B,T)$. As $B, T' \in \mathscr{T} \cap \mathscr{L}_{\geq C}$, we have $f \in E(C)$ by Lemma 2.1.6(ii), so $B \subset H^+_{\alpha_f}$. As $f \in S(B,T') - S(B,T)$, we have $T' \subset H^-_{\alpha_f}$ and $T \subset H^+_{\alpha_f}$. Also, as $f \in E(C)$, we have $(\lambda, \alpha_f) \geq 0$. So we may apply Proposition 2.2.5, and we obtain that $T \in \mathscr{L}_\lambda$.

Corollary 2.2.7. *Suppose that the arrangement $\mathcal{H}$ satisfies condition (A). Then the complex $\Sigma(\mathscr{L}_\lambda)$ is shellable. Moreover, there exists a shelling order on its chambers which is an initial shelling of $\Sigma(\mathscr{L})$. In particular, if $\lambda \neq 0$ then $\Sigma(\mathscr{L}_\lambda)$ is a shellable PL ball of dimension* $\dim(V/V_0) - 1$.

Proof. If $\lambda = 0$, then $\mathscr{L}_\lambda = \mathscr{L}$, so $\Sigma(\mathscr{L}_\lambda) = \Sigma(\mathscr{L})$, and the corollary is just Theorem 2.1.4.

We now assume that $\lambda \neq 0$. By Theorem 2.1.4 and Corollary 2.2.6, it suffices to find a family of signs $(\varepsilon_e) \in \{\pm 1\}^E$ such that:

– for every $e \in E$, we have $(\lambda, \varepsilon_e \alpha_e) \geq 0$;

– there exists a chamber $B \in \mathscr{L}_\lambda$ with $B \subset H^+_{\varepsilon_e \alpha_e}$ for every $e \in E$.

Indeed, Corollary 2.2.6 will then imply that $\mathscr{T} \cap \mathscr{L}_\lambda$ is a lower order ideal in $\mathscr{T}_B$, so it will be an initial segment for at least one linear extension of $\preceq_B$.

Let $F = \{e \in E : (\lambda, \alpha_e) \neq 0\}$. For every $e \in F$, we choose $\varepsilon_e \in \{\pm 1\}$ such that $(\lambda, \varepsilon_e \alpha_e) > 0$. Let x_0 be a point in V not on any hyperplane of $\mathcal{H}$, that is, $x_0 \in V - \bigcup_{e \in E} H_{\alpha_e}$. Then for every $e \in F$, the inner product $(x_0 + c \cdot \lambda, \varepsilon_e \alpha_e) = (x_0, \varepsilon_e \alpha_e) + c \cdot (\lambda, \varepsilon_e \alpha_e)$ tends to $+\infty$ as c tends to $+\infty$, so it is positive for c large enough. Similarly, the inner product $(x_0 + c \cdot \lambda, \lambda) = (x_0, \lambda) + c \cdot (\lambda, \lambda)$ is positive for c large enough. On the other hand, if $e \in E - F$, then $(x_0 + c \cdot \lambda, \alpha_e) = (x_0, \alpha_e) \neq 0$ for every $c \in \mathbb{R}$. So, if $c \in \mathbb{R}$ is large enough, then $x = x_0 + c \cdot \lambda \in V - \bigcup_{e \in E} H_{\alpha_e}$, and x is in $H^+_{\varepsilon_e \alpha_e}$ for every $e \in F$ and in H^+_λ. In particular, there exists a chamber $B \in \mathscr{T}$ such that $x \in B$, and B is included in $H^+_{\varepsilon_e \alpha_e}$ for every $e \in F$ and in H^+_λ. Now, if $e \in E - F$, we choose $\varepsilon_e \in \{\pm 1\}$ such that $B \subset H^+_{\varepsilon_e \alpha_e}$. As $(\lambda, \alpha_e) = 0$, we clearly have $(\lambda, \varepsilon_e \alpha_e) \geq 0$.

2.3 Coxeter arrangements

Let (W, S) be a Coxeter system, that is, W is the group generated by the set S and the relations between the generators are of the form $(st)^{m_{s,t}} = 1$ where $m_{s,s} = 1$ and $m_{s,t} \geq 2$ for $s \neq t$; see [BB05, Section 1.1]. The corresponding Coxeter graph has vertex set S, and two generators s and t are connected with an edge if $m_{s,t} \geq 3$. If $m_{s,t} \geq 4$ it is customary to label the edge by the integer $m_{s,t}$.

There are three natural partial orders on the elements of the Coxeter group W. First the *strong Bruhat order* is defined by the following cover relation: $z \prec w$ if there exists $s \in S$ and $u \in W$ such that $(usu^{-1})z = w$ and $\ell(z) + 1 = \ell(w)$ where ℓ is the length function on W; see for example [BB05, Definition 2.1.1]. Next, we have the *right* (respectively *left*) *weak Bruhat order*, where the cover relation is $z \prec w$ if there exists $s \in S$ such that $z \cdot s = w$ (respectively $s \cdot z = w$) and $\ell(z) + 1 = \ell(w)$. The strong Bruhat order refines both the left and right weak Bruhat orders.

Let $V = \bigoplus_{s \in S} \mathbb{R} e_s$, with the symmetric bilinear form $(\cdot, \cdot)$ defined by

$$(e_s, e_t) = -\cos(\pi / m_{s,t}).$$

In particular, $(e_s, e_s) = 1$. The *canonical representation* of (W, S) is the representation of W on V given by

$$s(v) = v - 2 \cdot (e_s, v) \cdot e_s, \tag{2.1}$$

for every $s \in S$ and every $v \in V$. Note that this formula defines an orthogonal isomorphism of V for the symmetric bilinear form $(\cdot, \cdot)$. We refer the reader to [Bou68, Chapitre V, § 4, № 8, Théorème 2 p. 98] for the next result.

Theorem 2.3.1. *Equation* (2.1) *defines a faithful representation of W on V, and the form $(\cdot, \cdot)$ is positive definite if and only if W is finite.*

From now on, we assume that W is finite, and we write $\Phi = \{w(e_s) : w \in W,\ s \in S\}$ and $\Phi^+ = \Phi \cap \sum_{s \in S} \mathbb{R}_{\geq 0} e_s$. The set Φ is a *pseudo-root system*, its subset Φ^+ is the set of *positive pseudo-roots*, and the set $\Phi^- = -\Phi^+ = \Phi - \Phi^+$ is the set of *negative pseudo-roots*; see Definitions B.1.1 and B.1.4. Then $\mathcal{H} = (H_\alpha)_{\alpha \in \Phi^+}$ is an essential hyperplane arrangement on V. The set of chambers $\mathscr{T}$ of this arrangement is in canonical bijection with W: the unit element $1 \in W$ corresponds to the chamber $B = \bigcap_{\alpha\in\Phi^+} H_\alpha^+ = \bigcap_{s\in S} H_{e_s}^+$, and an arbitrary element w of W corresponds to the chamber $w(B)$.

More generally, a *parabolic subgroup* of W is a subgroup W_I generated by a subset I of S, and the left cosets of parabolic subgroups of W are called *standard cosets*. The *Coxeter complex* $\Sigma(W)$ of W is the set of standard cosets of W ordered by reverse inclusion. It is a simplicial complex, and we have an isomorphism of posets from $\Sigma(W)$ to the face poset $\mathscr{L}$ of $\mathcal{H}$ sending a standard coset wW_I to the cone $\{x \in V : \forall s \in I,\ (x, w(e_s)) = 0 \text{ and } \forall s \in S - I,\ (x, w(e_s)) > 0\}$. The fact that this is an isomorphism is proved in [Bou68, Chapitre V § 4 № 6 pp. 96–97], since the representation of W on $V^\vee$ is isomorphic to its canonical representation on V by Theorem 2.3.1. The fact that $\Sigma(W)$ is a simplicial complex then follows from [Bou68, Chapitre V § 3 № 3 Proposition 7 p. 85].

The definitions of B and of the isomorphism $\mathscr{T} \simeq W$ imply that, if $w, w' \in W$ and $T_w, T_{w'} \in \mathscr{T}$ are the corresponding chambers, then

$$\begin{aligned} S(T_w, T_{w'}) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^+ \text{ and } {w'}^{-1}(\alpha) \in \Phi^-\} \\ \cup \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^- \text{ and } {w'}^{-1}(\alpha) \in \Phi^+\}, \end{aligned}$$

and in particular

$$S(B, T_w) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^-\}.$$

By [BB05, Propositions 3.1.3 and 4.4.6] this implies that the isomorphism $\mathscr{T} \simeq W$ sends the partial order $\preceq_B$ to the right weak Bruhat order on W.

Definition 2.3.2. Let $\mathcal{H} = (H_{\alpha_e})_{e\in E}$ be a finite hyperplane arrangement on a finite-dimensional inner product space V, whose inner product is denoted by $(\cdot,\cdot)$. We say that $\mathcal{H}$ is a *Coxeter arrangement* if $\alpha_e \notin \mathbb{R}\alpha_f$ for distinct $e, f \in E$ and if for every $e \in E$ the family of hyperplanes $\mathcal{H}$ is stable by the (orthogonal) reflection s_{α_e} across H_{α_e}.

Theorem 2.3.3. *The hyperplane arrangement associated to a Coxeter system with finite Coxeter group is a Coxeter arrangement. Conversely, suppose that $\mathcal{H}$ is a Coxeter arrangement on an inner product space V, and that there exists a chamber B of $\mathcal{H}$ that is on the positive side of each hyperplane in $\mathcal{H}$. Let W be the subgroup of $\mathbf{GL}(V)$ generated by the set $\{s_{\alpha_e} : e \in E\}$, let F be the set of $e \in E$ such that $\overline{B} \cap H_{\alpha_e}$ is a facet of $\overline{B}$ and let $S = \{s_{\alpha_f} : f \in F\}$. Then (W, S) is a Coxeter system, the group W is finite, and the hyperplane arrangement induced by $\mathcal{H}$ on $V/\bigcap_{e\in E} H_{\alpha_e}$ is isomorphic to the arrangement associated to the Coxeter system (W, S).*

Proof. The first statement is an immediate consequence of the definition of the arrangement associated to a Coxeter system. The second and fourth statements follow from [Bou68, Chapitre V § 3 № 2 Théorème 1 p. 74]. The fact that W is finite follows from [Bou68, Chapitre V § 3 № 7 Proposition 4 p. 80] and from the fact that the arrangement $\mathcal{H}$ is central.

Lemma 2.3.4. *Every Coxeter arrangement $\mathcal{H}$ satisfies Condition (A).*

Proof. In a Coxeter arrangement, the dihedral angle between any two adjacent facets is π/n, with $n \geq 2$.

In particular, Corollaries 2.2.6 and 2.2.7 apply to Coxeter arrangements. But we can actually prove a stronger result in this case.

We fix a Coxeter arrangement $\mathcal{H}$ on an inner product space V, and we use the notation introduced above. We say that a vector $\lambda \in V$ is *dominant* if $(\lambda, \alpha) \geq 0$ for every $\alpha \in \Phi^+$.

Lemma 2.3.5. *Suppose that $\lambda \in V$ is dominant. Denote by B the chamber of $\mathcal{H}$ corresponding to $1 \in W$. Let $z, w \in W$ such that $z \leq w$ in the strong Bruhat order of W. Then for every $x \in B$ the following inequality holds:*

$$(z^{-1}(\lambda), x) \geq (w^{-1}(\lambda), x).$$

Proof. We may assume that w covers z, so that there exists $s \in S$ and $u \in W$ such that $w = (usu^{-1})z$. Let α be the unique pseudo-root of Φ^+ such that $u(e_s)$ is a multiple (positive or negative) of α. If s_α is the reflection across H_α, we have $usu^{-1} = s_\alpha$, and so $w = s_\alpha z$ and $s_\alpha w = z$. Since the elements of Φ^+ are unit vectors, s_α is given by the following formula: $s_\alpha(\mu) = \mu - 2 \cdot (\mu, \alpha) \cdot \alpha$ for $\mu \in V$. Hence

$$(s_\alpha w)^{-1}(\lambda) = (w^{-1} s_\alpha)(\lambda) = w^{-1}(\lambda) - 2 \cdot (\lambda, \alpha) \cdot w^{-1}(\alpha),$$

and so, if $x \in B$,

$$((s_\alpha w)^{-1}\lambda, x) = (w^{-1}(\lambda), x) - 2 \cdot (\lambda, \alpha) \cdot (w^{-1}(\alpha), x).$$

As λ is dominant, we have $(\lambda, \alpha) \geq 0$. By [BB05, Equation (4.25)], we have the equivalence $w^{-1}(\alpha) \in \Phi^+ \iff \ell(w^{-1}s_\alpha) > \ell(w^{-1})$, and [BB05, Proposition 1.4.2(iv)] states that $\ell(v^{-1}) = \ell(v)$ for every $v \in W$. Using these two facts and the condition $\ell(s_\alpha w) < \ell(w)$, we see that $w^{-1}(\alpha) \in \Phi^-$. Thus $(w^{-1}(\alpha), x) < 0$ by definition of B. Hence the term $-2 \cdot (\lambda, \alpha) \cdot (w^{-1}(\alpha), x)$ is nonnegative, that is, $(z^{-1}(\lambda), x) = ((s_\alpha w)^{-1}(\lambda), x) \geq (w^{-1}(\lambda), x)$. 🐾

Proposition 2.3.6. *Let (W, S) be a Coxeter system, and let $\mathcal{H} = (H_\alpha)_{\alpha \in \Phi^+}$ be the associated hyperplane arrangement on the space V of the canonical representation of (W, S). Let $\lambda \in V$ be a dominant vector. Then the set W_λ of $w \in W$ such that the corresponding chamber of $\mathcal{H}$ is in $\mathscr{T} \cap \mathscr{L}_\lambda$ is a lower order ideal with respect to the* strong *Bruhat order on W.*

Proof. We denote by B the chamber of $\mathcal{H}$ corresponding to $1 \in W$. By definition of W_λ, an element w of W is in W_λ if and only if for every $x \in B$ we have $(\lambda, w(x)) = (w^{-1}(\lambda), x) \geq 0$. By Lemma 2.3.5, if $z, w \in W$ and w is greater than z in the strong Bruhat order then for every $x \in B$, we have $(z^{-1}(\lambda), x) \geq (w^{-1}(\lambda), x)$. If moreover $w \in W_\lambda$, this immediately implies that $z \in W_\lambda$. 🐾

3 The weighted sum

3.1 Definition and first properties

We use the notation of Subsection 2.2.

Definition 3.1.1. Given a chamber $B \in \mathscr{T}$, we define a function f_B from the face poset $\mathscr{L}$ to the set of chambers $\mathscr{T}$ by $f_B(C) = C \circ B$.

The next proposition gives some basic properties of the function f_B.

Proposition 3.1.2. *The following two statements hold:*

(i) Fix a face $C \in \mathscr{L}$, and consider the poset isomorphism $\iota_C : \mathscr{L}_{\geq C} \xrightarrow{\sim} \mathscr{L}_{\mathcal{H}(C)}$ of Lemma 2.1.6. If $B \in \mathscr{T} \cap \mathscr{L}_{\geq C}$ then for every $D \in \mathscr{L}_{\geq C}$ we have $\iota_C(f_B(D)) = f_{\iota_C(B)}(\iota_C(D))$.

(ii) *Suppose that $\mathcal{H}$ is a Coxeter arrangement with a chamber B that is on the positive side of every hyperplane, and let (W, S) be the associated Coxeter system. Identify $\mathscr{L}$ with the Coxeter complex $\Sigma(W)$ as in Section 2.3. If the face $C \in \mathscr{L}$ corresponds to a standard coset $c \subset W$, then the element $w \in W$ corresponding to the chamber $f_B(C)$ is the shortest element of c which is also the minimal element of the coset c in the right weak Bruhat order.*

In particular, part (ii) implies that, for the type A Coxeter complex, the function f_B defined here (for B the chamber corresponding to $1 \in W$) is equal to the function f defined at the beginning of [EMR19, Section 4]. Note that the existence of a minimal element in every standard coset is proved in [BB05, Proposition 2.4.4].

Proof of Proposition 3.1.2. Statement (i) follows immediately from Lemma 2.1.6.

We now prove (ii). By definition of the composition $\circ$, the chamber $f_B(C) = C \circ B$ is the element of $\mathscr{T} \cap \mathscr{L}_{\geq C}$ closest to B in the chamber graph; in other words, it is the minimal element of $\mathscr{T} \cap \mathscr{L}_{\geq C}$ for the order $\preceq_B$; see Subsection 2.1. As we know that $\preceq_B$ corresponds to the right weak Bruhat order on W (see the discussion after Theorem 2.3.1), and as the elements of W corresponding to the chambers of $\mathscr{T} \cap \mathscr{L}_{\geq C}$ are the elements of the coset c, the result follows.

The link between the function f_B and the shellings of Theorem 2.1.4 is established in the following proposition. For the type A Coxeter complex, this result appeared implicitly in the proof of [EMR19, Proposition 4.1].

Proposition 3.1.3. *Let $B \in \mathscr{T}$, and let $B = T_1, T_2, \ldots, T_r = -B$ be a linear order of $\mathscr{T}$ refining the partial order $\preceq_B$. Then for every index $1 \leq i \leq r$ the fiber of f_B over T_i is given by*

$$f_B^{-1}(T_i) = \{C \in \mathscr{L} : C \subset \overline{T}_i - (\overline{T}_1 \cup \overline{T}_2 \cup \cdots \cup \overline{T}_{i-1})\} \tag{3.1}$$

$$= \{C \in \mathscr{L} : C \leq T_i \text{ and } C \not\leq T_j \text{ for } 1 \leq j < i\}. \tag{3.2}$$

By Theorem 2.1.4 the linear order $T_1, T_2, \ldots, T_r$ is a shelling order of the chambers of $\Sigma(\mathscr{L})$. In particular, the shelling order defines a partition of the faces of $\Sigma(\mathscr{L})$:

$$\mathscr{L} = \coprod_{i=1}^{r} \{C \in \mathscr{L} : C \leq T_i \text{ and } C \not\leq T_j \text{ for } 1 \leq j < i\}.$$

Proposition 3.1.3 says that this partition is independent of the linear refinement $T_1, T_2, \ldots, T_r$ of $\preceq_B$, and that its blocks are the fibers of the map $f_B : \mathscr{L} \longrightarrow \mathscr{T}$.

Proof of Proposition 3.1.3. The equivalence between equalities (3.1) and (3.2) is an immediate consequence of the definition of the order $\leq$ on $\mathscr{L}$. Let us prove equality (3.2).

Let $C \in f_B^{-1}(T_i)$, that is, $C \circ B = T_i$. In particular, we have $C \leq C \circ B = T_i$. Suppose that $T \in \mathscr{T}$ is another chamber such that $C \leq T$. Then for every $e \in S(B, T_i)$ we have $s(T_i)_e \neq s(B)_e$, but $s(C \circ B)_e = s(T_i)_e$, so $0 \neq s(C)_e = s(T_i)_e$. As $C \leq T$, this implies that $s(T)_e = s(C)_e = s(T_i)_e \neq s(B)_e$, hence that $e \in S(B, T)$. So we have proved that $S(B, T_i) \subseteq S(B, T)$, which means that $T_i \preceq_B T$. In particular, if $1 \leq j \leq i - 1$ then $C \not\leq T_j$.

Conversely, let $C \in \mathscr{L}$ be such that $C \leq T_i$ and $C \not\leq T_j$ for $1 \leq j < i$, and let $T = C \circ B$. If $e \in S(B, T)$ then $0 \neq s(C)_e = s(T)_e$. As $C \leq T_i$, this implies that $s(C)_e = s(T_i)_e$, so $s(T_i)_e =$

$s(C)_e = s(T)_e \neq s(B)_e$, that is, $e \in S(B, T_i)$. So we have proved that $S(B,T) \subseteq S(B,T_i)$, which means that $T \preceq_B T_i$. Hence there exists an index $1 \leq i' \leq i$ such that $T = T_{i'}$. As $C \leq T$ and $C \not\leq T_j$ for $1 \leq j < i$, we must have $i' = i$, that is, $f_B(C) = C \circ B = T = T_i$. 🐾

We are interested in the following quantity.

Definition 3.1.4. Let $\mathcal{H}$ be a hyperplane arrangement in an inner product space V. Let λ be a vector in V and B a chamber of $\mathcal{H}$, that is, $B \in \mathscr{T}$. Recall that $\mathscr{L}_\lambda$ is the collection of faces on the nonnegative side of the hyperplane H_λ. The *weighted sum* is defined to be

$$\psi_{\mathcal{H}}(B, \lambda) = \sum_{D \in \mathscr{L}_\lambda} (-1)^{\dim(D)} \cdot (-1)^{|S(B, D \circ B)|}. \tag{3.3}$$

More generally, if C is a face of the arrangement $\mathcal{H}$, that is, $C \in \mathscr{L}$, and if B is a chamber whose closure contains the face C, that is, $B \in \mathscr{T} \cap \mathscr{L}_{\geq C}$, we define the weighted sum to be

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{D \in \mathscr{L}_{\lambda, \geq C}} (-1)^{\dim(D)} \cdot (-1)^{|S(B, D \circ B)|}. \tag{3.4}$$

Remark 3.1.5. In our previous paper, we used the notation $S(\lambda)$ (see [EMR19, Equation (6.1)]) to denote what turns out to be a particular case of the sum in equation (3.4) in the type B Coxeter case; see equation (5.1) in Subsection 5.2 for the precise relation between the two. In this paper, we decided to follow the notation of [GKM97] in order to avoid overusing the letter S.

Proposition 3.1.6. *Let $C \in \mathscr{L}$ and $B \in \mathscr{T} \cap \mathscr{L}_{\geq C}$. Suppose that $\mathscr{T} \cap \mathscr{L}_{\lambda, \geq C}$ is a lower order ideal in the chamber poset $\mathscr{T}_B$. Then the following identity holds:*

$$\psi_{\mathcal{H}/C}(B, \lambda) = \begin{cases} (-1)^{\dim(V) + |E(C)|} & \text{if } \mathscr{L}_{\lambda, \geq C} = \mathscr{L}_{\geq C}, \\ 0 & \text{if } \mathscr{L}_{\lambda, \geq C} \subsetneq \mathscr{L}_{\geq C}. \end{cases}$$

In particular, if C is the minimal face V_0 in $\mathscr{L}$, then we have $\mathscr{L}_{\geq V_0} = \mathscr{L}$. Hence the condition $\mathscr{L}_{\lambda, \geq V_0} = \mathscr{L}_{\geq V_0}$ is equivalent to $\lambda = 0$.

Remark 3.1.7. If the hyperplane arrangement satisfies Condition (A), then we can use Corollary 2.2.6 to verify the hypothesis of Proposition 3.1.6. However, Proposition 3.1.6 holds in greater generality than in the case where the somewhat unnatural angle requirement of Condition (A) is satisfied.

Proof of Proposition 3.1.6. We use the notation of Lemma 2.1.6. In particular, we get a subarrangement $\mathcal{H}(C)$ of $\mathcal{H}$ whose hyperplanes are the hyperplanes of $\mathcal{H}$ containing C and an isomorphism of posets $\iota_C : \mathscr{L}_{\geq C} \xrightarrow{\sim} \mathscr{L}_{\mathcal{H}(C)}$. Let $B' = \iota_C(B)$. By part (ii) of Lemma 2.1.6 this isomorphism induces an isomorphism of posets $\mathscr{T}_B \cap \mathscr{L}_{\geq C} \xrightarrow{\sim} \mathscr{T}_{\mathcal{H}(C), B'}$, where $\mathscr{T}_{\mathcal{H}(C)}$ is the set of chambers of $\mathcal{H}(C)$.

Choose a linear order $T_1, T_2, \ldots, T_r$ of $\mathscr{T}_{\mathcal{H}(C), B'}$ refining $\preceq_{B'}$. As $\iota_C(\mathscr{T} \cap \mathscr{L}_{\lambda, \geq C})$ is a lower order ideal in $\mathscr{T}_{\mathcal{H}(C), B'}$ by assumption and by the previous paragraph, we may choose this linear order so that $\iota_C(\mathscr{T} \cap \mathscr{L}_{\lambda, \geq C}) = \{T_1, T_2, \ldots, T_s\}$, for some $1 \leq s \leq r$.

By Theorem 2.1.4 the chosen linear order on $\mathscr{T}_{\mathcal{H}(C),B'}$ is a shelling order, and by Proposition 3.1.3 the corresponding partition of $\mathscr{L}_{\mathcal{H}(C)}$ is

$$\mathscr{L}_{\mathcal{H}(C)} = \coprod_{i=1}^{r} f_{B'}^{-1}(T_i).$$

In particular, $\coprod_{i=1}^{s} f_{B'}^{-1}(T_i)$ is the set of faces of $\overline{T_1} \cup \overline{T_2} \cup \cdots \cup \overline{T_s}$, which is also equal to $\iota_C(\mathscr{L}_{\lambda,\geq C})$ by Proposition 2.2.3. As the function $D \longmapsto (-1)^{|S(B',\iota_C(D)\circ B)'|}$ is constant and equal to $(-1)^{|S(B',T)|}$ on the fiber $f_{B'}^{-1}(T)$ for every $T \in \mathscr{T}_{\mathcal{H}(C)}$ and as the isomorphism $\iota_C : \mathscr{L}_{\geq C} \xrightarrow{\sim} \mathscr{L}_{\mathcal{H}(C)}$ preserves dimensions (by part (iii) of Lemma 2.1.6), we obtain

$$\psi_{\mathcal{H}/C}(B,\lambda) = \sum_{i=1}^{s} (-1)^{|f_{B'}(T_i)|} \cdot \sum_{D \in f_{B'}^{-1}(T_i)} (-1)^{\dim(D)}. \tag{3.5}$$

Let a_i be the inner sum of (3.5), that is, $a_i = \sum_{D \in f_{B'}^{-1}(T_i)} (-1)^{\dim(D)}$ for $1 \leq i \leq r$. Our tool to evaluate a_i is the *Euler characteristic with compact support*, denoted by χ_c. This is defined for example in van den Dries's book [vdD98, Section 2 of Chapter 4], where it is simply called "Euler characteristic". The connection with cohomology with compact support is explained in Schürmann's book [Sch03, Section 2.0.1, page 84]. The properties that will be most relevant to us are the facts that $\chi_c(A) = (-1)^d$ if A is an open ball of dimension d, and that χ_c is a valuation, that is, $\chi_c(A) = \chi_c(A) + \chi_c(A \setminus A')$ for A a nice enough topological space and A' a nice enough subspace of A. Here "nice enough" means that A is locally compact and has a stratification by locally closed subsets called cells which are all homeomorphic to open balls, and A' is a union of some of the cells of A.

First consider the case $i = 1$. Then all the cones D in $f_{B'}^{-1}(T_1)$ except for the minimal cone, which is $\mathrm{Span}(C)$, intersect the unit sphere $\mathbb{S}$ in V in an open ball of dimension $\dim(D) - 1$. In particular, we have $(-1)^{\dim(D)} = -\chi_c(D \cap \mathbb{S})$. As for $\mathrm{Span}(C)$, its intersection with $\mathbb{S}$ is a sphere of dimension $\dim(C) - 1$. Note that if $C = \{0\}$ then this intersection is empty so it is a sphere of dimension -1. We have $(-1)^{\dim(C)} = 1 - \chi_c(\mathrm{Span}(C) \cap \mathbb{S})$. Hence we obtain

$$a_1 = 1 - \sum_{D \in f_{B'}^{-1}(T_1)} \chi_c(D \cap \mathbb{S}) = 1 - \chi_c(\overline{T_1} \cap \mathbb{S}).$$

But $\overline{T_1} \cap \mathbb{S}$ is a closed ball of dimension $\dim(V) - 1$, hence $\chi_c(\overline{T_1} \cap \mathbb{S}) = 1$ and so $a_1 = 0$.

Suppose now that $i \geq 2$. Then we have

$$a_i = -\sum_{D \in f_{B'}^{-1}(T_i)} \chi_c(D \cap \mathbb{S}) = -\chi_c(\overline{T_i} \cap \mathbb{S}) + \chi_c\left(\overline{T_i} \cap \left(\overline{T_1} \cup \overline{T_2} \cup \cdots \cup \overline{T_{i-1}}\right) \cap \mathbb{S}\right).$$

Again $\overline{T_i} \cap \mathbb{S}$ is a closed ball of dimension $\dim(V) - 1$, hence $\chi_c(\overline{T_i} \cap \mathbb{S}) = 1$. On the other hand, $\overline{T_i} \cap (\bigcup_{j=1}^{i-1} \overline{T_j}) \cap \mathbb{S}$ is a partial shelling of the boundary of $\overline{T_i} \cap \mathbb{S}$. As $\Sigma(\mathscr{L}_{\mathcal{H}(C)})$ is a sphere, T_r is the only homology facet in the shelling given by the order $T_1, T_2, \ldots, T_r$. So if $i \leq r - 1$ then $\overline{T_i} \cap (\bigcup_{j=1}^{i-1} \overline{T_j}) \cap \mathbb{S}$ is not equal to $\partial \overline{T_i} \cap \mathbb{S}$, so it is a closed ball of dimension $\dim(V) - 2$. In particular $\chi_c(\overline{T_i} \cap (\bigcup_{j=1}^{i-1} \overline{T_j}) \cap \mathbb{S}) = 1$ and thus $a_i = 0$. Finally, if $i = r$ then $\overline{T_r} \cap (\bigcup_{j=1}^{r-1} \overline{T_j}) \cap \mathbb{S}$ is equal to the boundary of $\overline{T_r} \cap \mathbb{S}$, so $a_r = -\chi_c(T_r \cap \mathbb{S}) = (-1)^{\dim(V)}$.

As $\psi_{\mathcal{H}/C}(B,\lambda) = \sum_{i=1}^{s}(-1)^{|S(B',T_i)|}a_i$, this shows that $\psi_{\mathcal{H}/C}(B,\lambda) = 0$ unless $s = r$. But $s = r$ if and only if $\mathscr{L}_{\geq C} = \mathscr{L}_{\lambda,\geq C}$. In this case, we have $\psi_{\mathcal{H},C}(B,\lambda) = (-1)^{|S(B',T_r)|} \cdot a_r = (-1)^{|S(B',T_r)|+\dim(V)}$. Finally, observe that the last facet T_r is given by $-B'$, so $S(B',-B')$ is the total index set of hyperplanes in $\mathcal{H}(C)$, that is, $E(C)$, and so $\psi_{\mathcal{H}/C}(B,\lambda) = (-1)^{\dim(V)+|E(C)|}$.

We state the following lemma, which reduces the calculation of $\psi_{\mathcal{H}/C}(B,\lambda)$ to the case of an essential arrangement.

Lemma 3.1.8. *Recall that V_0 is the intersection of all the hyperplanes of $\mathcal{H}$, that is, $V_0 = \bigcap_{e\in E} H_{\alpha_e}$. Let π denote the projection $V \longrightarrow V/V_0$. Let $\mathcal{H}/V_0$ be the hyperplane arrangement $(H_{\alpha_e}/V_0)_{e\in E}$ on V/V_0. Note that π induces an isomorphism between the posets of faces of $\mathcal{H}$ and of $\mathcal{H}/V_0$. Let $C \in \mathscr{L}$, let B be a chamber of $\mathcal{H}$ such that $B \in \mathscr{L}_{\geq C}$ and let $\lambda \in V$. Then the following identity holds:*

$$\psi_{\mathcal{H}/C}(B,\lambda) = \begin{cases} (-1)^{\dim(V_0)} \cdot \psi_{(\mathcal{H}/V_0)/\pi(C)}(\pi(B),\pi(\lambda)) & \text{if } \lambda \in V_0^{\perp}, \\ 0 & \text{if } \lambda \notin V_0^{\perp}. \end{cases}$$

Proof. Note that $\lambda \in V_0^{\perp}$ if and only if $V_0 \subset H_\lambda$. If $\lambda \notin V_0^{\perp}$ then the linear functional $(\lambda,\cdot)$ takes both positive and negative values on V_0. As $V_0 \subset \overline{D}$ for every $D \in \mathscr{L}$, this linear functional also takes both positive and negative values on D, so $D \notin \mathscr{L}_\lambda$. This shows that $\mathscr{L}_\lambda = \varnothing$ if $\lambda \notin V_0^{\perp}$ and gives the second case. Now suppose that $\lambda \in V_0^{\perp}$. Then $V_0 \subset H_\lambda$, and it is easy to see that $D \in \mathscr{L}_\lambda$, respectively $D \in \mathscr{L}_{\geq C}$, if and only if $\pi(D) \subset \overline{H^{+}_{\pi(\lambda)}}$, respectively $\pi(D) \geq \pi(C)$, and that $\dim(\pi(D)) = \dim(D) - \dim(V_0)$. This yields the first case.

Suppose that $V = V_1 \times \cdots \times V_r$ with the V_i mutually orthogonal subspaces of V and that $\mathcal{H}$ also decomposes as a product $\mathcal{H}_1 \times \cdots \times \mathcal{H}_r$. By this, we mean that there is a decomposition $E = E_1 \sqcup \cdots \sqcup E_r$ such that, for $i = 1,\ldots,r$ and every $e \in E_i$, we have $\alpha_e \in V_i$. The arrangement $\mathcal{H}_i = (V_i \cap H_{\alpha_e})_{e\in E_i}$ is a hyperplane arrangement on the subspace V_i, and each hyperplane of $\mathcal{H}$ is of the form $H \times \prod_{j\neq i} V_j$, where $i = 1,\ldots,r$ and H is one of the hyperplanes of $\mathcal{H}_i$.

Let $\mathscr{L}_i$ be the face poset of $\mathcal{H}_i$ for $i = 1,\ldots,r$. Then the faces of $\mathscr{L}$ are exactly the products $C_1 \times \cdots \times C_r$ where $C_i \in \mathscr{L}_i$, and the order on $\mathscr{L}$ is the product order. In particular, C is a chamber in $\mathscr{L}$ if and only if all the C_i are chambers in $\mathscr{L}_i$.

Lemma 3.1.9. *Assume that the arrangement $\mathcal{H}$ factors as $\mathcal{H}_1 \times \cdots \times \mathcal{H}_r$ as described in the two previous paragraphs. Let $C = C_1 \times \cdots \times C_r$ be a face in $\mathscr{L}$, and let $B = B_1 \times \cdots \times B_r$ be a chamber in $\mathscr{L}_{\geq C}$. Finally, let $\lambda \in V$. Then*

$$\psi_{\mathcal{H}/C}(B,\lambda) = \prod_{i=1}^{r} \psi_{\mathcal{H}_i/C_i}(B_i,\lambda_i),$$

where, for $i = 1,\ldots,r$, λ_i is the orthogonal projection of λ on V_i.

Proof. The expression for $\psi_{\mathcal{H}/C}(B,\lambda)$ follows from the fact that $\mathscr{L} = \mathscr{L}_1 \times \cdots \times \mathscr{L}_r$ as posets once we prove the following statement: Let $D = D_1 \times \cdots \times D_r \in \mathscr{L}$, with $D_i \in \mathscr{L}_i$. Then $D \in \mathscr{L}_\lambda$ if and only if $D_i \in \mathscr{L}_{i,\lambda_i}$ for $i = 1,\ldots,r$.

We prove this last fact. Note that $\lambda = (\lambda_1,\ldots,\lambda_r)$ in $V_1 \times \cdots \times V_r = V$ because the V_i are pairwise orthogonal. If $D_i \in \mathscr{L}_{i,\lambda_i}$ for every $i = 1,\ldots,r$ then for every $x = (x_1,\ldots,x_r) \in V$ we have $(\lambda, x) = \sum_{i=1}^r (\lambda_i, x_i) \geq 0$. Conversely, suppose that $D \in \mathscr{L}_\lambda$. Let $x_j \in D_j$ for $j = 1, \ldots, r$. As all the D_j are cones, for every $\varepsilon > 0$, the element εx_j is in D_j. Fix $i \in \{1, \ldots, r\}$, and consider the element $x_\varepsilon = (\varepsilon x_1, \ldots, \varepsilon x_{i-1}, x_i, \varepsilon x_{i+1}, \ldots, \varepsilon x_r)$. Then x_ε is in D. Thus we have the inequality $0 \leq (\lambda, x_\varepsilon) = (\lambda_i, x_i) + \varepsilon \sum_{j \neq i} (\lambda_j, x_j)$. Letting ε tend to 0, we obtain $(\lambda_i, x_i) \geq 0$. So $D_i \in \mathscr{L}_{i,\lambda_i}$.

3.2 A recursive expression

The goal of this subsection is to prove a recursive expression for $\psi_{\mathcal{H}/C}(B, \lambda)$ in terms of a lower-dimensional hyperplane arrangement.

Definition 3.2.1. Let $V' \subset V$ be a linear subspace of codimension 2. We define the *intersection multiplicity* of the arrangement $\mathcal{H}$ at V' to be the cardinality

$$|\{e \in E : H_{\alpha_e} \supseteq V'\}|.$$

Here is the main result of this subsection.

Proposition 3.2.2. *Let C be a face in $\mathscr{L}$, B a chamber in $\mathscr{T} \cap \mathscr{L}_{\geq C}$ and $\lambda \in V$. Fix $e_0 \in E$ such that $H_0 = H_{\alpha_{e_0}}$ contains C and is a wall of B, that is, the intersection $H_0 \cap \overline{B}$ is the closure of a facet A' of B. Let $B' \in \mathscr{T}$ be the chamber on the other side of A' from B, that is, $B' = A' \circ (-B)$. Assume that $\alpha_e \notin \mathbb{R}\alpha_{e_0}$ (which implies that $H_e \neq H_0$) for every $e \in E - \{e_0\}$.*

(i) The following identity holds:

$$\psi_{\mathcal{H}/C}(B, \lambda) + \psi_{\mathcal{H}/C}(B', \lambda) = 2 \cdot \psi_{\mathcal{H}'_0/C}(A', \mu),$$

where $\mathcal{H}'_0$ is the hyperplane arrangement in H_0 defined by the family of orthogonal projections on H_0 of the α_e, $e \in E - \{e_0\}$, and $\mu \in H_0$ is the orthogonal projection of $\lambda \in V$. Note that, by our hypothesis on H_0, the orthogonal projection of α_e on H_0 is nonzero for every $e \in E - \{e_0\}$, and that C is also a face of $\mathcal{H}'_0$.

(ii) Let $E(C) = \{e \in E : C \subset H_{\alpha_e}\}$. Note that $e_0 \in E(C)$ by assumption. Let F_1 be the set of $f \in E(C) - \{e_0\}$ such that the intersection multiplicity of $\mathcal{H}$ at $H_0 \cap H_{\alpha_f}$ is even. Choose a subset F of F_1 such that, for every $f_1 \in F_1$, there is exactly one $f \in F$ with $H_0 \cap H_{\alpha_{f_1}} = H_0 \cap H_{\alpha_f}$. Let $\mathcal{H}_0$ be the hyperplane arrangement in H_0 defined by the family of orthogonal projections on H_0 of the α_f, for $f \in F \sqcup (E - E(C))$. Since $\mathcal{H}_0$ is a subarrangement of $\mathcal{H}'_0$, there is a unique chamber of $\mathcal{H}_0$ containing A', respectively C. Denote this chamber by A, respectively C_0. Then the following identity holds:

$$\psi_{\mathcal{H}/C}(B, \lambda) + \psi_{\mathcal{H}/C}(B', \lambda) = 2 \cdot \psi_{\mathcal{H}_0/C_0}(A, \mu).$$

The proof of this proposition will occupy the rest of this subsection. The following definition will be useful.

Definition 3.2.3. Let $C \in \mathscr{L}$ and $\lambda \in V$. If $D, D' \in \mathscr{L}_{\geq C}$, we define $\psi_{D/C}(D', \lambda)$ by the sum

$$\psi_{D/C}(D', \lambda) = \sum_{\substack{C' \in \mathscr{L}_{\lambda, \geq C} \\ C' \circ D' \leq D}} (-1)^{\dim(C')}.$$

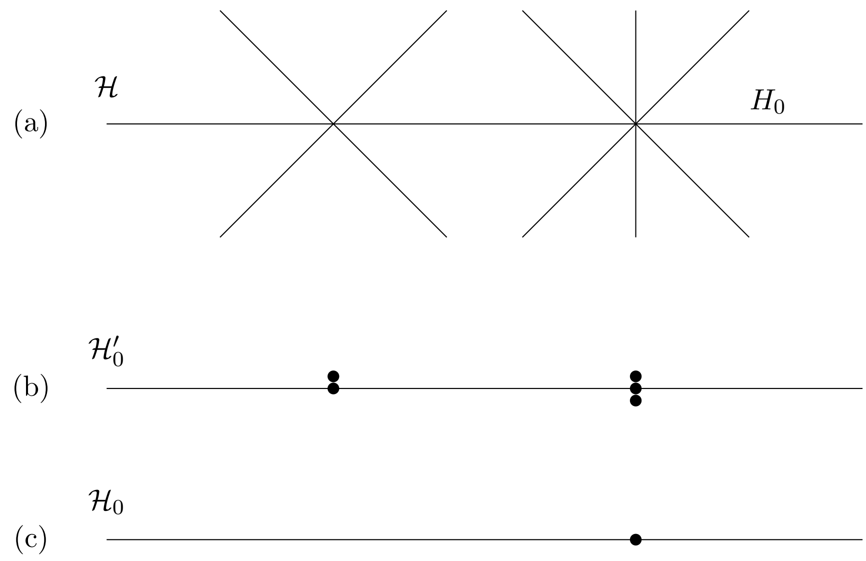

Figure 1: (a) A two-dimensional affine representation of a three-dimensional arrangement $\mathcal{H}$ with the hyperplane H_0. (b) The 1-dimensional representation of $\mathcal{H}'_0$ where the multiple dots indicate multiplicity. (c) The 1-dimensional representation of $\mathcal{H}_0$. (Here we assume that $C = \{0\}$, so that $E(C) = E$.)

Remark 3.2.4. Suppose that D' is a chamber. Then $C' \circ D'$ is a chamber for every $C' \in \mathscr{L}$, so $\psi_{D/C}(D', \lambda) = 0$ unless D is also a chamber. If D is a chamber, we have

$$\psi_{D/C}(D', \lambda) = \sum_{\substack{C' \in \mathscr{L}_{\lambda, \geq C} \\ C' \circ D' = D}} (-1)^{\dim(C')}.$$

The functions $\psi_{D/C}(D', \lambda)$ are related to $\psi_{\mathcal{H}/C}(B, \lambda)$ by the following lemma.

Lemma 3.2.5. *Let $C \in \mathscr{L}$ and $B \in \mathscr{T} \cap \mathscr{L}_{\geq C}$. Then for every $\lambda \in V$ the following identity holds:*

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{T \in \mathscr{T} \cap \mathscr{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \psi_{T/C}(B, \lambda).$$

Proof. Indeed, if $D \in \mathscr{L}_{\geq C}$ then the chamber $D \circ B$ is also in $\mathscr{L}_{\geq C}$. Hence, using (3.4) in Definition 3.1.4 and Remark 3.2.4, we obtain:

$$\psi_{\mathcal{H}/C}(B, \lambda) = \sum_{T \in \mathscr{T} \cap \mathscr{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \sum_{\substack{D \in \mathscr{L}_{\lambda, \geq C} \\ D \circ B = T}} (-1)^{\dim(D)} = \sum_{T \in \mathscr{T} \cap \mathscr{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \psi_{T/C}(B, \lambda).$$

Before Corollary A.1.6 of Appendix A, we define, for K a closed convex polyhedral cone in V, a function $\psi_K : V \times V^\vee \longrightarrow \mathbb{R}$. For fixed $(x, \ell) \in V \times V^\vee$, the function $K \longmapsto \psi_K(x, \ell)$ is a valuation on the set of closed convex polyhedral cones in V (see Definition A.1.1). This function is related to the functions $\psi_{D/C}(D', \lambda)$ in the following way.

Lemma 3.2.6. *Let $C \in \mathscr{L}$, let $D \in \mathscr{L}_{\geq C}$ and let $\lambda \in V$. Denote by $\ell \in V^\vee$ the linear functional $(\cdot, \lambda)$. Then for every $D' \in \mathscr{L}_{\geq C}$ the following identity holds:*

$$\psi_{D/C}(D', \lambda) = \psi_{\overline{D}}(x, \ell),$$

where x is any point in $D'_1 = (-C) \circ D'$.

Proof. As before we write $E(C) = \{e \in E : C \subset H_{\alpha_e}\}$. Note that $s(D'_1)_e = s(D')_e$ for $e \in E(C)$, and $s(D'_1) = -s(C)_e \neq 0$ for $e \in E - E(C)$. Also, by definition of $\mathscr{L}_{\geq C}$, if e is any index of $E - E(C)$ and $C' \in \mathscr{L}_{\geq C}$ then $s(C)_e = s(C')_e \neq 0$.

We claim that, for every $C' \in \mathscr{L}$, we have $C' \circ D'_1 \leq D$ if and only if $C' \in \mathscr{L}_{\geq C}$ and $C' \circ D' \leq D$. Indeed, suppose first that $C' \in \mathscr{L}_{\geq C}$ and $C' \circ D' \leq D$. Then for every $e \in E(C)$ we have $s(C' \circ D'_1)_e = s(C' \circ D')_e \leq s(D)_e$. Moreover, if $e \in E - E(C)$ then $s(C')_e = s(C)_e = s(D)_e \neq 0$, so $s(C' \circ D'_1)_e = s(C')_e = s(D)_e$. This shows that $C' \circ D'_1 \leq D$. Conversely, suppose that C' is a face of $\mathscr{L}$ such that $C' \circ D'_1 \leq D$. If $e \in E - E(C)$ then $0 \neq s(C)_e = s(D)_e = -s(D'_1)_e$, thus $s(C')_e \neq 0$, and so $s(C')_e = s(C' \circ D'_1)_e = s(D)_e$. This implies that $C' \in \mathscr{L}_{\geq C}$. Moreover, if $e \in E(C)$ we have $s(D'_1)_e = s(D')_e$, thus $s(C' \circ D')_e = s(C' \circ D'_1)_e \leq s(D)_e$. Hence we conclude that $C' \circ D' \leq D$.

By the claim, we obtain

$$\psi_{D/C}(D', \lambda) = \sum_{\substack{C' \in \mathscr{L}_\lambda \\ C' \circ D'_1 \leq D}} (-1)^{\dim(C')} = \psi_D(D'_1, \lambda).$$

We wish to show that this is equal to $\psi_{\overline{D}}(x, \ell)$, if $x \in D'_1$. As in Appendix A, we denote by $\mathcal{F}(\overline{D})$ the set of faces of the closed convex polyhedral cone $\overline{D}$. We have $\mathcal{F}(\overline{D}) = \{\overline{C'} \ : \ C' \in \mathscr{L},\ C' \leq D\}$, and the set $\{C' \in \mathscr{L} : C' \circ D'_1 \leq D\}$ is included in the set $\{C' \in \mathscr{L} : C' \leq D\}$. To prove the equality above, it suffices to show that the two following statements hold for $C' \in \mathscr{L}$ such that $C' \leq D$ (see Lemma A.1.3 for the definition of ψ_x and ψ_ℓ, and the beginning of Section A.1 for the notation $\overline{C'}^{\perp, \overline{D}}$):

(a) The face C' belongs to $\mathscr{L}_\lambda$ if and only $\psi_\ell(\overline{C'}) = 1$.

(b) The inequality $C' \circ D'_1 \leq D$ holds if and only if $\psi_x\left(\overline{C'}^{\perp, \overline{D}}\right) = 1$.

Statement (a) is just a direct translation of the definition of $\mathscr{L}_\lambda$. We prove (b). The assertion that $C' \circ D'_1 \leq D$ is equivalent to the fact that, for every $y \in C'$ and every sufficiently small $\varepsilon > 0$, we have $y + \varepsilon x \in \overline{D}$. On the other hand, the assertion that $\psi_x(\overline{C'}^{\perp, \overline{D}}) = 1$ is equivalent to the fact that $x \in (\overline{C'}^{\perp, \overline{D}})^* = \overline{D} + \mathrm{Span}(C')$. (Note that $\mathrm{Span}(\overline{C'}) = \mathrm{Span}(C')$ because $\mathrm{Span}(C')$ is a vector subspace of V, hence closed in V.) If $y + \varepsilon x \in \overline{D}$ for some $y \in C'$ and some $\varepsilon > 0$, then we clearly have $x \in \overline{D} + \mathrm{Span}(C')$. Conversely, suppose that $x \in \overline{D} + \mathrm{Span}(C')$. Write $x = a + b$, with $a \in \overline{D}$ and $b \in \mathrm{Span}(C')$. Let $y \in C'$. If $\varepsilon > 0$ is sufficiently small, then $y + \varepsilon b \in C'$, so $y + \varepsilon x \in \overline{D} + C' = \overline{D}$.

We are now ready to prove the main result of this section.

Proof of Proposition 3.2.2(i). Note that B' is also in $\mathscr{L}_{\geq C}$, because we chose e_0 such that $C \subseteq H_{\alpha_{e_0}}$. By Lemma 3.2.5, we have

$$\psi_{\mathcal{H}/C}(B,\lambda) = \sum_{T \in \mathscr{T} \cap \mathscr{L}_{\geq C}} (-1)^{|S(B,T)|} \cdot \psi_{T/C}(B,\lambda).$$

Similarly,

$$\psi_{\mathcal{H}/C}(B',\lambda) = \sum_{T \in \mathscr{T} \cap \mathscr{L}_{\geq C}} (-1)^{|S(B',T)|} \cdot \psi_{T/C}(B',\lambda)$$

also holds.

Fix $T \in \mathscr{T} \cap \mathscr{L}_{\geq C}$. We wish to evaluate the sum

$$a(T) = (-1)^{|S(B,T)|} \cdot \psi_{T/C}(B,\lambda) + (-1)^{|S(B',T)|} \cdot \psi_{T/C}(B',\lambda).$$

Suppose first that $e_0 \notin S(B,T)$. Then $S(B',T)$ is the disjoint union $S(B,T) \sqcup \{e_0\}$. Let $D \in \mathscr{L}$. Then $D \circ B = D \circ B'$ if and only if $s(D)_{e_0} \neq 0$. Moreover, if $s(D)_{e_0} = 0$ then $D \circ B'$ cannot be equal to T, because they are on opposite sides of the hyperplane H_0. This implies that

$$\begin{aligned} a(T) &= \sum_{\substack{D \in \mathscr{L}_{\lambda,\geq C} \\ D \circ B = T \\ D \circ B' = T}} \left((-1)^{|S(B,T)|} + (-1)^{|S(B,T')|} \right) \cdot (-1)^{\dim(D)} + \sum_{\substack{D \in \mathscr{L}_{\lambda,\geq C} \\ s(D)_{e_0} = 0 \\ D \circ B = T}} (-1)^{|S(B,T)|} \cdot (-1)^{\dim(D)} \\ &= \sum_{\substack{D \in \mathscr{L}_{\lambda,\geq C} \\ s(D)_{e_0} = 0 \\ D \circ B = T}} (-1)^{|S(B,T)|} \cdot (-1)^{\dim(D)}. \end{aligned}$$

Let $\mathscr{L}'_0$ be the face poset of $\mathcal{H}'_0$. Then the following equalities hold:

$$\mathscr{L}'_0 = \{D \in \mathscr{L} : D \subset H_0\} = \{D \in \mathscr{L} : s(D)_{e_0} = 0\}.$$

In particular, the cone C is also a face in $\mathscr{L}'_0$. Let $D \in \mathscr{L}$ such that $D \subset H_0$, and let us still denote by D the corresponding face of $\mathscr{L}'_0$. Then $D \in \mathscr{L}'_{0,\mu}$ if and only if $D \in \mathscr{L}_\lambda$, and $D \in \mathscr{L}'_{0,\geq C}$ if and only if $D \in \mathscr{L}_{0,\geq C}$. Also, $D \circ A'$ is a chamber of $\mathcal{H}'_0$, and it is on the same side of H_{α_e} as $D \circ B$ for every $e \in E - \{e_0\}$, so we can only have $D \circ B = T$ if H_0 is a wall of T, that is, if T has a facet T' such that $T' \subset H_0$. In this case, we have $D \circ B = T$ if and only if $D \circ A' = T'$. We conclude that $a(T) = 0$ unless H_0 is a wall of T. Moreover, if H_0 is a wall of T and T' is as before the unique facet of T such that $T' \subset H_0$, then we have $T' \in \mathscr{L}'_{0,\geq C}$ and $S(A',T') = S(B,T)$. In this case we have

$$a(T) = \sum_{\substack{D \in \mathscr{L}'_{0,\mu,\geq C} \\ D \circ A' = T'}} (-1)^{|S(A',T')|} \cdot (-1)^{\dim(D)} = (-1)^{|S(A',T')|} \cdot \psi_{T'/C}(A',\mu). \tag{3.6}$$

If $e_0 \in S(B,T)$ then $e_0 \notin S(B',T)$. Exchanging the roles of B and B' in the previous paragraph, and noting that A' is also the unique facet of B' contained in H_0, we see that $a(T) = 0$ unless H_0 is a wall of T. Moreover, if H_0 is a wall of T and T' is as before the unique facet of T such that $T' \subset H_0$, then we have $T' \in \mathscr{L}'_{0,\geq C}$ and $S(A',T') = S(B',T)$, and thus the expression for $a(T)$ is given by the same formula as in equation (3.6).

Finally, each chamber T' of $\mathcal{H}'_0$ appears as a facet of exactly two chambers of $\mathcal{H}$, one on either side of H_0. Hence it contributes twice the right-hand side of equation (3.6). Summing over T' in the set of chambers $\mathscr{T}'_0$ of $\mathcal{H}'_0$ and using Lemma 3.2.5 again, we obtain

$$\begin{aligned}\psi_{\mathcal{H}/C}(B,\lambda) + \psi_{\mathcal{H}/C}(B',\lambda) &= \sum_{T \in \mathscr{T} \cap \mathscr{L}_{\geq C}} a(T) \\ &= 2 \cdot \sum_{T' \in \mathscr{T}'_0 \cap \mathscr{L}'_{0,\geq C}} (-1)^{|S(A',T')|} \cdot \psi_{T'/C}(A',\mu) \\ &= 2 \cdot \psi_{\mathcal{H}'_0/C}(A',\mu).\end{aligned}$$

Proof of Proposition 3.2.2(ii). By (i), it suffices to prove that $\psi_{\mathcal{H}'_0/C}(A',\mu) = \psi_{\mathcal{H}_0/C_0}(A,\mu)$. As in the proof of (i), we denote by $\mathscr{L}'_0$, respectively $\mathscr{T}'_0$, the set of faces, respectively chambers, of $\mathcal{H}'_0$. We also denote by $\mathscr{L}_0$, respectively $\mathscr{T}_0$, the set of faces, respectively chambers, of $\mathcal{H}_0$. As $\mathcal{H}_0$ is a subarrangement of $\mathcal{H}'_0$, each chamber $T' \in \mathscr{T}'_0$ is contained in a unique chamber $T \in \mathscr{T}_0$. So, using Lemma 3.2.5, we can write

$$\psi_{\mathcal{H}'_0/C}(A',\mu) = \sum_{T \in \mathscr{T}_0} \sum_{\substack{T' \in \mathscr{T}'_0 \cap \mathscr{L}'_{0,\geq C} \\ T' \subseteq T}} (-1)^{|S(A',T')|} \cdot \psi_{T'/C}(A',\mu).$$

Fix $T \in \mathscr{T}_0$. We claim that, for every $T' \in \mathscr{T}'_0$ such that $T' \subset T$, the cardinalities $|S(A',T')|$ and $|S(A,T)|$ have the same parity. Let us prove this claim. Let $f \in S(A,T)$. Then every $f' \in F_1$ such that $H_{\alpha_{f'}} \cap H_0 = H_{\alpha_f} \cap H_0$ belongs to the separation set $S(A',T')$. Furthermore, there are an odd number of such elements $f' \in F_1$. Hence the parity of $|S(A,T)|$ is the same as the parity of $|S(A',T') \cap F_1|$. Next consider the set $S(A',T') \cap F_0$ where F_0 is the complement $(E - \{e_0\}) - F_1$. Assume that $f \in S(A',T') \cap F_0$. Then there are an even number of $f' \in F_0$ such that $H_{\alpha_{f'}} \cap H_0 = H_{\alpha_f} \cap H_0$, and they all belong to the set $S(A',T')$. Hence the cardinality of $S(A',T') \cap F_0$ is even, yielding $|S(A,T)| \equiv |S(A',T')| \bmod 2$. We then have

$$\psi_{\mathcal{H}'_0/C}(A',\mu) = \sum_{T \in \mathscr{T}_0} (-1)^{|S(A,T)|} \cdot \sum_{\substack{T' \in \mathscr{T}'_0 \cap \mathscr{L}'_{0,\geq C} \\ T' \subseteq T}} \psi_{T'/C}(A',\mu).$$

Next we note that if $T \in \mathscr{T}_0$ then $\{T' \in \mathscr{T}'_0 \cap \mathscr{L}'_{0,\geq C} : T' \subset T\}$ is empty if $T \notin \mathscr{L}_{0,\geq C_0}$. Indeed, if there exists $T' \in \mathscr{T}'_0 \cap \mathscr{L}'_{0,\geq C}$ such that $T' \subset T$, then $\overline{T} \supset \overline{T'} \supset C$, so $\overline{T} \supset C_0$. Moreover, if $T \in \mathscr{T}_0 \cap \mathscr{L}_{0,\geq C_0}$, then any $T' \in \mathscr{T}'_0$ such that $T' \subset T$ is in $\mathscr{L}'_{0,\geq C}$. Indeed, to go from $\mathcal{H}'_0$ to $\mathcal{H}_0$, we only removed hyperplanes that contain C. Thus if $T' \in \mathscr{T}'_0$ is such that $T' \subset T$ then for every $e \in E - E(C)$ we have $s(T')_e = s(T)_e = s(C_0)_e = s(C)_e$, which implies that $T' \geq C$. So we get that

$$\psi_{\mathcal{H}'_0/C}(A',\mu) = \sum_{T \in \mathscr{T}_0 \cap \mathscr{L}_{0,\geq C_0}} (-1)^{|S(A,T)|} \cdot \sum_{\substack{T' \in \mathscr{T}'_0 \\ T' \subseteq T}} \psi_{T'/C}(A',\mu).$$

It remains to show that for every $T \in \mathscr{T}_0 \cap \mathscr{L}_{0,\geq C_0}$ the following holds:

$$\psi_{T/C_0}(A,\mu) = \sum_{\substack{T' \in \mathscr{T}'_0 \\ T' \subseteq T}} \psi_{T'/C}(A',\mu).$$

Let $\ell \in H_0^\vee$ be the linear functional defined by $\ell = (\mu, \cdot)$, and let $x \in (-C) \circ A'$. By Lemma 3.2.6 there exists a valuation $K \longmapsto \psi_K(x, \ell)$ on the set of closed convex polyhedral cones in H_0 such that

- for every $D' \in \mathscr{L}'_{0,\geq C}$, we have $\psi_{D'/C}(A', \mu) = \psi_{\overline{D'}}(x, \ell)$;
- for every $D \in \mathscr{L}_{0,\geq C_0}$, we have $\psi_{D/C_0}(A, \mu) = \psi_{\overline{D}}(x, \ell)$ (recall that $C \subset C_0$ and $A' \subset A$, so $x \in (-C_0) \circ A$).

Let $T \in \mathscr{T}_0 \cap \mathscr{L}_{0,\geq C_0}$. Then we have

$$\overline{T} = \bigcup_{\substack{T' \in \mathscr{T}'_0 \\ T' \subseteq T}} \overline{T'}$$

and, if $T'_1, T'_2, \dots, T'_i$ are pairwise distinct elements of $\mathscr{T}'_0$ that are included in T and if $i \geq 2$, then $\overline{T'_1} \cap \overline{T'_2} \cap \dots \cap \overline{T'_i}$ is the closure of a face of $\mathscr{L}'_0$ that is not a chamber. As $\psi_{D'/C}(A', \mu) = 0$ for every $D' \in \mathscr{L}'_0$ that is not a chamber (see Remark 3.2.4), the general properties of valuations (see [Gro78, Theorem 1]) imply that

$$\psi_{T/C_0}(A, \mu) = \sum_{\substack{T' \in \mathscr{T}'_0 \\ T' \subseteq T}} \psi_{T'/C}(A', \mu),$$

as desired.

3.3 The case of Coxeter arrangements

In this section we make the recursive expression of Proposition 3.2.2(ii) more explicit in the case of Coxeter arrangements.

Suppose that $\mathcal{H}$ is a Coxeter arrangement, and fix a chamber B of $\mathcal{H}$. We suppose that B is on the positive side of every hyperplane of $\mathcal{H}$. As in Theorem 2.3.3, we obtain a Coxeter system (W, S) with W finite. Recall that W is the subgroup of $\mathbf{GL}(V)$ generated by the reflections across the hyperplanes of $\mathcal{H}$, and that $S \subset W$ is the set of reflections across the hyperplanes that are walls of the base chamber B.

We also consider the following action of W on E. If $e \in E$ and $w \in W$ then by assumption $w(H_{\alpha_e})$ is a hyperplane of $\mathcal{H}$, so it is of the form H_{α_f} for a unique $f \in E$, and we set $w(e) = f$. This is clearly a left action.

We have the following lemma.

Lemma 3.3.1. *Let $\lambda \in V$, $C, D \in \mathscr{L}$ and $w \in W$. Then the following statements hold:*

(i) $w(\mathscr{L}_\lambda) = \mathscr{L}_{w(\lambda)}$;

(ii) $w(C \circ D) = w(C) \circ w(D)$;

(iii) $S(w(C), w(D)) = w(S(C, D))$;

(iv) $\psi_C(D, \lambda) = \psi_{w(C)}(w(D), w(\lambda))$;

(v) for every $B \in \mathscr{T}$, *we have* $\psi_{\mathcal{H}}(B, \lambda) = \psi_{\mathcal{H}}(w(B), w(\lambda))$.

Proof. Statements (i), (ii) and (iii) are straightforward consequences of the fact that the action of W on V preserves the inner product $(\cdot, \cdot)$. Statement (iv) follows immediately from (i), (ii) and the definition of $\psi_C(D, \lambda)$, and (v) follows immediately from (iii) and (iv).

Remark 3.3.2. More generally, if $w \in \mathbf{GL}(V)$ is an isometry, the same proof shows that we have $\psi_{\mathcal{H}}(B, \lambda) = \psi_{w(\mathcal{H})}(w(B), w(\lambda))$.

We now consider the situation of Proposition 3.2.2(ii), with B the fixed base chamber of $\mathscr{T}$. For $e_0 \in E$, the hyperplane $H_0 = H_{\alpha_{e_0}}$ is a wall of B if and only the reflection s_0 across this hyperplane is an element of S (by definition of S). We fix such a e_0. We have the following essential fact.

Lemma 3.3.3. *Let $e \in E - \{e_0\}$. Then the intersection multiplicity of $\mathcal{H}$ at $H_0 \cap H_{\alpha_e}$ is even if and only if there exists $f \in E - \{e_0\}$ such that $H_0 \cap H_{\alpha_f} = H_0 \cap H_{\alpha_e}$ and that $(\alpha_{e_0}, \alpha_f) = 0$.*

Proof. Let $E' = \{f \in E - \{e_0\} : H_0 \cap H_{\alpha_f} = H_0 \cap H_{\alpha_e}\}$. Then $|E'| + 1$ is the intersection multiplicity of $\mathcal{H}$ at the intersection $H_0 \cap H_{\alpha_e}$, and we have $s_0(E') = E'$. As $s_0^2 = 1$, the set E' has odd cardinality if and only s_0 has a fixed point on E'. But $f \in E'$ is fixed by s_0 if and only if $s(H_{\alpha_f}) = H_{\alpha_f}$, which is equivalent to the condition that $(\alpha_{e_0}, \alpha_f) = 0$.

In particular, we can choose the set F of Proposition 3.2.2(ii) to be $F = \{f \in E : (\alpha_{e_0}, \alpha_f) = 0\}$, and we obtain a hyperplane arrangement $\mathcal{H}_0 = (H_0 \cap H_{\alpha_f})_{f \in F}$ on H_0. For every $f \in F$, if $s_f \in W$ is the reflection across H_{α_f} then $s_f(H_0) = H_0$ and $s_f(F) = F$. In particular, s_f preserves the arrangement $\mathcal{H}_0$. We conclude that $\mathcal{H}_0$ is a Coxeter arrangement on H_0. Finally, using Lemma 3.3.1(v), the conclusion of Proposition 3.2.2(ii) becomes that, for every $\lambda \in V$,

$$\psi_{\mathcal{H}}(B, \lambda) + \psi_{\mathcal{H}}(B, s(\lambda)) = 2 \cdot \psi_{\mathcal{H}_0}(A, \mu),$$

where A and μ are defined in Proposition 3.2.2(ii).

Example 3.3.4. Consider the A_3 arrangement in $\mathbb{R}^4$. This arrangement has 6 positive roots: $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq 4\}$. Let $B = \{x : x_1 > x_2 > x_3 > x_4\}$ be the base chamber. Choose α_0 to be the root $e_3 - e_4$, that is, the hyperplane H_0 is given by $\{x : x_3 = x_4\}$. The arrangement $\mathcal{H}_0'$ consists of the five hyperplanes $\{x \in H_0 : x_1 = x_3\} = \{x \in H_0 : x_1 = x_4\}$, $\{x \in H_0 : x_2 = x_3\} = \{x \in H_0 : x_2 = x_4\}$ and $\{x \in H_0 : x_1 = x_2\}$. But since the only hyperplane orthogonal to H_0 is $\{x : x_1 = x_2\}$, the arrangement $\mathcal{H}_0$ only consists of the hyperplane $\{x \in H_0 : x_1 = x_2\}$. Thus the arrangement $\mathcal{H}_0$ is the Coxeter arrangement of type A_1 embedded in a 3-dimensional space. Furthermore, the chamber A' in $\mathcal{H}_0'$ is given by $\{x : x_1 > x_2 > x_3 = x_4\}$, whereas the chamber A in $\mathcal{H}_0$ is given by $\{x : x_1 > x_2 \text{ and } x_3 = x_4\}$.

Example 3.3.5. Consider the B_3 arrangement in $\mathbb{R}^3$. This arrangement has 9 positive roots: $\Phi^+ = \{e_i \pm e_j : 1 \leq i < j \leq 3\} \cup \{e_i : 1 \leq i \leq 3\}$. Let $B = \{x : x_1 > x_2 > x_3 > 0\}$ be the base chamber. Then the simple roots are $\{e_1 - e_2, e_2 - e_3, e_3\}$. Now pick α_0 to be the root e_3, that is, the hyperplane H_0 is given by $\{x : x_3 = 0\}$. The arrangement $\mathcal{H}_0'$ consists of the eight hyperplanes $\{x \in H_0 : x_1 = x_2\}$, $\{x \in H_0 : x_1 = -x_2\}$ (both of multiplicity 1), and $\{x \in H_0 : x_1 = 0\}$, $\{x \in H_0 : x_2 = 0\}$ (both of multiplicity 3). Since all the multiplicities are odd, the two arrangements $\mathcal{H}_0$ and $\mathcal{H}_0'$ have the same set of hyperplanes. But all the hyperplanes of $\mathcal{H}_0$ have multiplicity 1, thus the arrangement $\mathcal{H}_0$ is a Coxeter arrangement of type B_2. Furthermore, the two chambers A and A' are equal and are given by $\{x : x_1 > x_2 > x_3 = 0\}$.

In the next section we consider a slightly more general situation. Suppose that $E = E^{(1)} \sqcup E^{(2)}$, with $\mathcal{H}^{(1)} = (H_{\alpha_e})_{e\in E^{(1)}}$ a Coxeter arrangement whose Coxeter group W stabilizes $\mathcal{H}^{(2)} = (H_{\alpha_e})_{e\in E^{(2)}}$, and that $C = (\bigcap_{e\in E^{(1)}} H_{\alpha_e}) \cap (\bigcap_{e\in E^{(2)}} H^+_{\alpha_e})$, so that $E(C) = E^{(1)}$. We apply Proposition 3.2.2 for some $e_0 \in E^{(1)}$. As in the case of a Coxeter arrangement, we can choose the subset F of $E^{(1)}$ to be $F = \{f \in E^{(1)} : (\alpha_{e_0}, \alpha_f) = 0\}$. The resulting hyperplane arrangement $\mathcal{H}_0$ on $H_0 = H_{\alpha_0}$ decomposes again as $\mathcal{H}_0 = \mathcal{H}_0^{(1)} \sqcup \mathcal{H}_0^{(2)}$, where $\mathcal{H}_0^{(1)} = (H_0 \cap H_{\alpha_f})_{f\in F}$ and $\mathcal{H}_0^{(2)} = (H_0 \cap H_{\alpha_e})_{e\in E^{(2)}}$, and we see as before that $\mathcal{H}_0^{(1)}$ is a Coxeter arrangement on H_0. Moreover, the unique face C_0 of $\mathcal{H}_0$ that contains C is clearly $C_0 = (\bigcap_{e\in F}(H_0 \cap H_{\alpha_e})) \cap (\bigcap_{e\in E^{(2)}}(H_0 \cap H^+_{\alpha_e}))$. If $\lambda \in V$ and if B is a chamber whose closure contains C and such that H_0 is a wall of B, we have

$$\psi_{\mathcal{H}/C}(B,\lambda) + \psi_{\mathcal{H}/C}(B, s_{\alpha_{e_0}}(\lambda)) = 2 \cdot \psi_{\mathcal{H}_0/C_0}(A,\mu),$$

where A and μ are defined in Proposition 3.2.2(ii).

Example 3.3.6. Let (W,S) be a Coxeter system, let V be the canonical representation of W, and let $\mathcal{H} = (H_\alpha)_{\alpha\in\Phi^+}$ be the associated hyperplane arrangement on V as in Section 2.3. Let I be a subset of S, set $\Phi^{(1)} = \Phi^+ \cap \big(\sum_{\alpha\in I} \mathbb{R}\alpha\big)$ and $\Phi^{(2)} = \Phi^+ - \Phi^{(1)}$. Then $\mathcal{H}^{(1)} = (H_\alpha)_{\alpha\in\Phi^{(1)}}$ is a Coxeter arrangement with associated Coxeter system (W_I, I), where W_I is the subgroup of W generated by I, and W_I preserves the arrangement $\mathcal{H}^{(2)} = (H_\alpha)_{\alpha\in\Phi^{(2)}}$. If $C = (\bigcap_{\alpha\in\Phi^{(1)}} H_\alpha) \cap (\bigcap_{\alpha\in\Phi^{(2)}} H^+_\alpha)$ as before, then the chamber B corresponding to $1 \in W$ is in $\mathscr{L}_{\geq C}$.

4 An identity for Coxeter arrangements

We now specialize to the case at the end of Subsection 3.3. That is, we assume that $E = E^{(1)} \sqcup E^{(2)}$, with $\mathcal{H}^{(1)} = (H_{\alpha_e})_{e\in E^{(1)}}$ a Coxeter arrangement whose Coxeter group stabilizes $\mathcal{H}^{(2)} = (H_{\alpha_e})_{e\in E^{(2)}}$, and that $C = (\bigcap_{e\in E^{(1)}} H_{\alpha_e}) \cap (\bigcap_{e\in E^{(2)}} H^+_{\alpha_e})$. To simplify some of the notation, without loss of generality we may assume that the vectors α_e, where $e \in E$, are all unit vectors. We assume that there exists a chamber B of $\mathcal{H}$ that is on the positive side of every hyperplane of $\mathcal{H}$ (not just $\mathcal{H}^{(1)}$). This also defines a chamber of the arrangement $\mathcal{H}^{(1)}$, and we denote by (W,S) the associated Coxeter system, as in Theorem 2.3.3.

The set $\Phi = \{\pm\alpha_e : e \in E^{(1)}\}$ is a normalized pseudo-root system (see Definition B.1.1), the subset $\Phi^+ = \{\alpha_e : e \in E^{(1)}\}$ is a system of positive pseudo-roots in Φ (see Definition B.1.4), and (W,S) is the corresponding Coxeter system. See Proposition B.1.6.

4.1 The main theorem

Recall the definition of 2-structures from Subsection B.2. A 2-*structure* for Φ is a subset $\varphi \subseteq \Phi$ such that:

(a) φ is a pseudo-root system whose irreducible components are all of type A_1, B_2 or $I_2(2^k)$ with $k \geq 3$;

(b) for every $w \in W$ such that $w(\varphi \cap \Phi^+) = \varphi \cap \Phi^+$, we have $\det(w) = 1$.

Recall that $\mathcal{T}(\Phi)$ is the set of 2-structures of Φ. By Proposition B.2.4, the group W acts transitively on $\mathcal{T}(\Phi)$. In Definition B.2.8 we define the sign $\epsilon(\varphi) = \epsilon(\varphi, \Phi^+)$ of any 2-structure $\varphi \in \mathcal{T}(\Phi)$. If $\varphi \in \mathcal{T}(\Phi)$, we also write $\varphi^+ = \varphi \cap \Phi^+$, and we denote by $\mathcal{H}_\varphi$ the hyperplane arrangement $(H_\alpha)_{\alpha\in\varphi^+\sqcup E^{(2)}}$ and by B_φ, respectively C_φ, the unique chamber of $\mathcal{H}_\varphi$ containing B, respectively C. By the choice of B, the chamber B_φ is also the unique chamber on the positive side of every hyperplane in $\mathcal{H}_\varphi$.

The main theorem of this article is the following.

Theorem 4.1.1. *Let $\mathcal{H}$, C and B be as above. For every $\lambda \in V$, we have*

$$\psi_{\mathcal{H}/C}(B,\lambda) = \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi)\cdot\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi,\lambda).$$

This theorem will be proved in Subsection 4.3. We note the following corollary, which extends [Her01, Theorem 5.3] to the case of Coxeter systems. Note we do not obtain a new proof of Herb's theorem, as Herb uses a similar induction argument to prove it.

As we are using the definition of the sign of a 2-structure from [Her83], our formula looks a bit different from the one of [Her01, Theorem 5.3]. This is explained in Remark 5.1 of [Her01], and we generalize the comparison between the two definitions of the sign in Corollary 4.1.3.

Corollary 4.1.2. *The sum of the signs of all 2-structures of a pseudo-root system is equal to 1, that is,*

$$\sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi) = 1.$$

Proof. Take $E^{(2)} = \varnothing$ and $\lambda = 0$ in the identity of Theorem 4.1.1. By Proposition 3.1.6, we have $\psi_{\mathcal{H}}(B,0) = (-1)^{\dim(V)+|E|} = (-1)^{\dim(V)+|\Phi^+|}$. By the same proposition, for every $\varphi\in\mathcal{T}(\Phi)$, we have $\psi_{\mathcal{H}_\varphi}(B_\varphi,0) = (-1)^{\dim(V)+|\varphi^+|}$. The equality then follows from the fact that $|\Phi^+ - \varphi^+|$ is even for every 2-structure φ, which is proved in Lemma B.2.12.

Corollary 4.1.3. *Let $\varphi\in\mathcal{T}(\Phi)$, and let $W(\varphi,\Phi^+) = \{w\in W : w(\varphi^+)\subset\Phi^+\}$ and $W_1(\varphi,\Phi^+) = \{w\in W : w(\varphi^+)\subset\varphi^+\}$. Then the sign $\epsilon(\varphi,\Phi^+)$ is given by*

$$\epsilon(\varphi,\Phi^+) = \frac{1}{|W_1(\varphi,\Phi^+)|}\sum_{w\in W(\varphi,\Phi^+)}\det(w).$$

Proof. By Corollary B.2.5, we have a bijection $W(\varphi,\Phi^+)/W_1(\varphi,\Phi^+)\longrightarrow\mathcal{T}(\Phi)$, $w\longmapsto w(\varphi)$. So, by Corollary 4.1.2 and and Lemma B.2.10, we get

$$1 = \frac{1}{|W_1(\varphi,\Phi^+)|}\sum_{w\in W(\varphi,\Phi^+)}\epsilon(w(\varphi),\Phi^+) = \epsilon(\varphi,\Phi^+)\frac{1}{|W_1(\varphi,\Phi^+)|}\sum_{w\in W(\varphi,\Phi^+)}\det(w).$$

4.2 Recursive formula for the right-hand side

In this subsection, we prove a recursive formula for the right-hand side of the identity in Theorem 4.1.1, mirroring the identity of Proposition 3.2.2(ii).

Let $\alpha_0 \in \Phi^+$ be a simple pseudo-root (see Definition B.1.4), and let $s_0 = s_{\alpha_0}$ be the corresponding element of the set of simple reflections S. As in Subsection 3.3, we set $H_0 = H_{\alpha_0}$ and $F = \{e \in E^{(1)} : (\alpha_e, \alpha_0) = 0\}$, and we consider the arrangement $\mathcal{H}_0 = \mathcal{H}_0^{(1)} \sqcup \mathcal{H}_0^{(2)}$ on H_0, where $\mathcal{H}_0^{(1)} = (H_0 \cap H_{\alpha_f})_{f \in F}$ and $\mathcal{H}_0^{(2)} = (H_0 \cap H_{\alpha_e})_{e \in E^{(2)}}$. Then $\mathcal{H}_0^{(1)}$ is a Coxeter arrangement, with normalized pseudo-root system $\Phi_0 = \{\pm\alpha_f : f \in F\}$ and system of positive pseudo-roots $\Phi_0^+ = \{\alpha_f : f \in F\}$. Its Coxeter group preserves $\mathcal{H}_0^{(2)}$, because it preserves H_0 and is a subgroup of W. We denote by A the unique chamber of $\mathcal{H}_0$ containing the interior (in H_0) of $\overline{B} \cap H_0$. This is also the unique chamber on the positive side of all the hyperplanes of $\mathcal{H}_0$. Finally, we fix $\lambda \in V$, and we denote by μ the orthogonal projection of λ on H_0.

The recursive formula is the following result:

Proposition 4.2.1. *We have:*

$$\begin{aligned}\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \lambda) + \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, s_0(\lambda)) \\ = 2 \cdot \sum_{\varphi_0 \in \mathcal{T}(\Phi_0)} \epsilon(\varphi_0) \cdot \psi_{\mathcal{H}_{0,\varphi_0} / C_{0,\varphi_0}}(A_{\varphi_0}, \mu). \qquad (4.1)\end{aligned}$$

Proof. As $s_0^2 = 1$, the left-hand side of (4.1) is equal to

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \lambda) + \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(s_0(\varphi)) \cdot \psi_{\mathcal{H}_{s_0(\varphi)} / C_{s_0(\varphi)}}(B_{s_0(\varphi)}, s_0(\lambda)).$$

Let $\mathcal{T}' = \{\varphi \in \mathcal{T} : s_0(\varphi) \neq \varphi\}$ and $\mathcal{T}'' = \{\varphi \in \mathcal{T} : s_0(\varphi) = \varphi\}$. Suppose that $\varphi \in \mathcal{T}'$. Then $\alpha_0 \notin \varphi$ holds, so $s_0(\varphi^+) \subset s_0(\Phi^+ - \{\alpha_0\}) \subset \Phi^+$ by [BB05, Lemma 4.4.3]. Hence $\epsilon(s_0(\varphi)) = -\epsilon(\varphi)$ by Lemma B.2.10. Moreover, again using the fact that $s_0(\varphi^+) \subset \Phi^+$, we see that $s_0(\varphi) \cap \Phi^+ = s_0(\varphi^+)$, so $\mathcal{H}_{s_0(\varphi)} = s_0(\mathcal{H}_\varphi)$, $C_{s_0(\varphi)} = C_\varphi = s_0(C_\varphi)$ and $B_{s_0(\varphi)} = s_0(B_\varphi)$. By Remark 3.3.2 we have $\psi_{\mathcal{H}_{s_0(\varphi)} / C_{s_0(\varphi)}}(B_{s_0(\varphi)}, s_0(\lambda)) = \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \lambda)$. This shows that

$$\sum_{\varphi \in \mathcal{T}'} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \lambda) + \sum_{\varphi \in \mathcal{T}'} \epsilon(s_0(\varphi)) \cdot \psi_{\mathcal{H}_{s_0(\varphi)} / C_{s_0(\varphi)}}(B_{s_0(\varphi)}, s_0(\lambda)) = 0,$$

so the left hand-side of (4.1) is equal to

$$\begin{aligned}\sum_{\varphi \in \mathcal{T}''} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \lambda) + \sum_{\varphi \in \mathcal{T}''} \epsilon(s_0(\varphi)) \cdot \psi_{\mathcal{H}_{s_0(\varphi)} / C_{s_0(\varphi)}}(B_{s_0(\varphi)}, s_0(\lambda)) \\ = \sum_{\varphi \in \mathcal{T}''} \epsilon(\varphi) \cdot (\psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \lambda) + \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, s_0(\lambda))).\end{aligned}$$

We wish to prove that the right-hand side of this last equality is equal to the right-hand side of (4.1). Note first that if $\varphi \in \mathcal{T}''$, we must have $\alpha_0 \in \varphi$. Indeed, we know that $s_{\alpha_0}(\Phi^+ - \{\alpha_0\}) \subset \Phi^+$ (see [BB05, Lemma 4.4.3]), so, if $\alpha_0 \notin \varphi$, then $s_0(\varphi^+) \subseteq \Phi^+ \cap \varphi = \varphi^+$, contradicting condition (b) in the definition of a 2-structure. By Proposition 3.2.2(ii) and Subsection 3.3, if $\varphi \in \mathcal{T}''$ then we have

$$\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda) + \psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, s_0(\lambda)) = 2 \cdot \psi_{\mathcal{H}_{0,\varphi\cap\Phi_0}/C_{0,\varphi_0}}(A_{\varphi\cap\Phi_0}, \mu),$$

where $\mathcal{H}_{0,\varphi\cap\Phi_0}$, C_{0,φ_0} and $A_{\varphi\cap\Phi_0}$ are defined in the natural way, that is, as in the paragraph before the statement of the proposition, even though $\varphi \cap \Phi_0$ is not necessarily a 2-structure for Φ_0.

Let $\mathcal{T}_1''$ be the set of $\varphi \in \mathcal{T}''$ such that $\varphi \cap \Phi_0$ is a 2-structure for Φ_0, and $\mathcal{T}_2'' = \mathcal{T}'' - \mathcal{T}_1''$. By statements (1) and (2) of Lemma B.2.11, we have

$$\sum_{\varphi\in\mathcal{T}_1''} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_{\varphi\cap\Phi_0}/C_{0,\varphi\cap\Phi_0}}(A_{\varphi\cap\Phi_0}, \mu) = \sum_{\varphi_0\in\mathcal{T}(\Phi_0)} \epsilon(\varphi_0) \cdot \psi_{\mathcal{H}_{0,\varphi_0}/C_{0,\varphi_0}}(A_{\varphi_0}, \mu). \tag{4.2}$$

Also, by statement (3) of Lemma B.2.11, there exists an involution ι of $\mathcal{T}_2''$ such that $\iota(\varphi) \cap \Phi_0 = \varphi \cap \Phi_0$ and $\epsilon(\iota(\varphi)) = -\epsilon(\varphi)$ for every $\varphi \in \mathcal{T}_2''$. This immediately implies that

$$\sum_{\varphi\in\mathcal{T}_2''} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_{\varphi\cap\Phi_0}/C_{0,\varphi\cap\Phi_0}}(A_{\varphi\cap\Phi_0}, \mu) = 0. \tag{4.3}$$

Putting equations (4.2) and (4.3) together, we get the desired result.

4.3 Proof of the main theorem

Proof of Theorem 4.1.1. We prove the result by induction on the rank of Φ. Suppose that Φ is empty. Then $\mathcal{H} = \mathcal{H}^{(2)}$, $C = B$ is a chamber of $\mathcal{H}$, and the only element of $\mathcal{T}(\Phi)$ is the empty pseudo-root system. So both sides of the identity of Theorem 4.1.1 are equal to $(-1)^{\dim(V)}$ if $B \in \mathscr{L}_\lambda$ and 0 otherwise.

Suppose now that Φ is not empty, and that we know the result for all pseudo-root systems of smaller rank.

We first treat the case where $\mathscr{L}_{\lambda,\geq C} = \mathscr{L}_{\geq C}$. This means that $\bigcup_{D\in\mathscr{L}_{\geq C}} D \subset \overline{H_\lambda^+}$. As $\psi_{\mathcal{H}/C}(B, \lambda)$ only depends on λ through $\mathscr{L}_{\lambda,\geq C}$, we have $\psi_{\mathcal{H}/C}(B, \lambda) = \psi_{\mathcal{H}/C}(B, 0)$. Similarly, for every $\varphi \in \mathcal{T}(\Phi)$, if $\mathscr{L}_\varphi$ is the face poset of $\mathcal{H}_\varphi$, then we have $\mathscr{L}_{\varphi,\lambda,\geq C_\varphi} = \mathscr{L}_{\varphi,C_\varphi}$ because $\bigcup_{D\in\mathscr{L}_{\varphi,\geq C_\varphi}} D = \bigcup_{D\in\mathscr{L}_{\geq C}} D = \bigcap_{e\in E^{(2)}} H_{\alpha_e}^+$, hence $\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda) = \psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, 0)$. So we may assume that $\lambda = 0$. Choose any simple pseudo-root α_0, and let $s_0 = s_{\alpha_0}$. As $s_0(0) = 0$, Proposition 3.2.2(ii) and Subsection 3.3 imply that $\psi_{\mathcal{H}/C}(B, 0) = \psi_{\mathcal{H}_0/C_0}(A, 0)$, and Proposition 4.2.1 implies that

$$\sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, 0) = \sum_{\varphi_0\in\mathcal{T}(\Phi_0)} \epsilon(\varphi_0) \cdot \psi_{\mathcal{H}_{0,\varphi_0}/C_{0,\varphi_0}}(A_{\varphi_0}, 0).$$

So the result follows from the induction hypothesis.

Finally, suppose that $\mathscr{L}_{\lambda,\geq C} \neq \mathscr{L}_{\geq C}$. Choose $w \in W$ such that $\lambda' = w^{-1}(\lambda)$ satisfies $(\lambda', \alpha) \geq 0$ for every $\alpha \in \Phi^+$. We do a second induction, now on the length of w. If w has length 0, that is, if $w = 1$, then B and λ satisfy the assumptions of Corollary 2.2.6, so by Proposition 3.1.6, we have $\psi_{\mathcal{H}/C}(B, \lambda) = 0$. Let $\varphi \in \mathcal{T}(\Phi)$. Then B_φ and λ also satisfy the assumptions of Corollary 2.2.6, so by Proposition 3.1.6 we have $\psi_{\mathcal{H}_\varphi/C_\varphi}(B_\varphi, \lambda) = 0$. So we get the identity in this case. Suppose that w has positive length, and choose a simple pseudo-root α_0 such that $w = s_{\alpha_0} w'$ with w' of length one less than w. By the induction hypothesis on w, we know that the identity holds for $s_{\alpha_0}(\lambda) = w'(\lambda')$. The result now follows from the recursive expressions for each side, that is, Proposition 3.2.2(ii) and Subsection 3.3 for the left-hand side, and Proposition 4.2.1 for the right-hand side, and from the induction hypothesis for Φ.

5 Applications

5.1 Coxeter systems

We now specialize Theorem 4.1.1 to the case where $\mathcal{H} = \mathcal{H}^{(1)}$ is a Coxeter arrangement. In particular, C is the minimal face of $\mathscr{L}$, so $\psi_{\mathcal{H}/C}(B,\lambda) = \psi_{\mathcal{H}}(B,\lambda)$ for every $\lambda \in V$.

Let $\varphi \in \mathcal{T}(\Phi)$, and let $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$ be the decomposition of φ into irreducible pseudo-root systems. Let $V_{i,\varphi} = \mathrm{Span}(\varphi_i)$ for $1 \leq i \leq r$. Then $V = V_{0,\varphi} \times V_{1,\varphi} \times \cdots \times V_{r,\varphi}$, where $V_{0,\varphi} = \varphi^{\perp}$. Note that the dimension of $V_{0,\varphi}$ is equal to $\dim(V) - \mathrm{rank}(\varphi)$, so it is independent of φ by Proposition B.2.4. Let $\mathcal{H}_{i,\varphi}$ be the hyperplane arrangement given by $\varphi_i \cap \Phi^+$ on $V_{i,\varphi}$ where $1 \leq i \leq r$. For a fixed index i let $B_{i,\varphi}$ be the chamber of the arrangement $\mathcal{H}_{i,\varphi}$ that is on the positive side of every hyperplane, and let $\lambda_{i,\varphi}$ be the orthogonal projection of λ on $V_{i,\varphi}$.

Combining Theorem 4.1.1 with Lemmas 3.1.8 and 3.1.9, we obtain:

Corollary 5.1.1. *For every $\lambda \in V$, we have*

$$\psi_{\mathcal{H}}(B,\lambda) = (-1)^{\dim(V)-R} \cdot \sum_{\substack{\varphi \in \mathcal{T}(\Phi) \\ \lambda \in \mathrm{Span}(\varphi)}} \epsilon(\varphi) \cdot \prod_{i=1}^{r} \psi_{\mathcal{H}_{i,\varphi}}(B_{i,\varphi}, \lambda_{i,\varphi}),$$

where R is the rank of any $\varphi \in \mathcal{T}(\Phi)$.

To finish the calculation of $\psi_{\mathcal{H}}(B,\lambda)$ in this case, we use the following proposition, whose proof is a straightforward calculation.

Proposition 5.1.2. *In types A_1, $B_2 = I_2(4)$ and $I_2(2^k)$ for $k \geq 3$, the function ψ is given by the following expressions:*

(1) Type A_1: Suppose that $V = \mathbb{R}e_1$ and that $\Phi^+ = \{e_1\}$. Then ψ is given by

$$\psi_{\mathcal{H}}(B, ce_1) = \begin{cases} 0 & \text{if } c > 0, \\ 1 & \text{if } c = 0, \\ 2 & \text{if } c < 0. \end{cases}$$

(2) Type $I_2(2^k)$, where $k \geq 2$: Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with the usual inner product. For every $v \in V - \{0\}$, let $\theta(v) \in [0, 2\pi)$ be the angle from e_1 to v. Suppose that Φ is the set of unit vectors that have an angle of $r\pi/2^k$ with e_1, where $r \in \mathbb{Z}$, and that B is the set of nonzero vectors $v \in V$ such that $0 < \theta(v) < \pi/2^k$. Then ψ is given by

$$\psi_{\mathcal{H}}(B,\lambda) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 2 & \text{if } \lambda \neq 0 \text{ and } \theta(\lambda) = r\pi/2^k \text{ with } 2^{k-1}+1 \leq r \leq 3 \cdot 2^{k-1}, \\ 4 & \text{if } \lambda \neq 0 \text{ and } r\pi/2^k < \theta(\lambda) < (r+1)\pi/2^k \\ & \text{with } r \text{ odd and } 2^{k-1}+1 \leq r \leq 3 \cdot 2^{k-1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

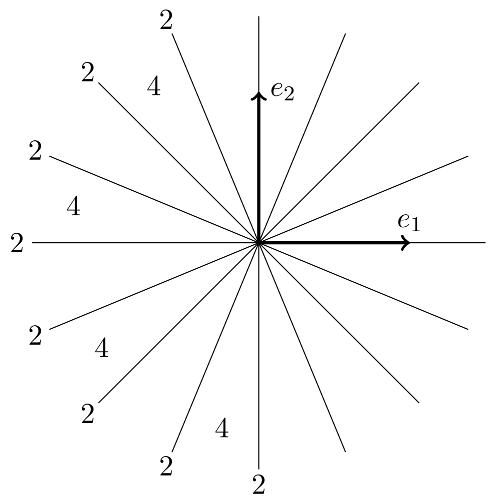

Figure 2: The function $\psi_{\mathcal{H}}(B, \lambda)$ in the dihedral pseudo-root system $I_2(8)$. The origin is assigned the value 1 and the unmarked faces are assigned the value 0.

Remark 5.1.3. If (W, S) arises from a root system Φ and -1 is an element of W (or, equivalently, the root system is generated by strongly orthogonal roots), then [GKM97, Theorem 3.1] and [Her00, Theorem 4.2] give two different expressions for the coefficients appearing in the formula for the averaged discrete series characters of a real reductive group with root system Φ. Corollary 5.1.1 asserts the equality of these two formulas. (Its proof is an induction similar to the ones used in the proofs of the two cited theorems.) In general, although there are no discrete series anymore, the formulas of Goresky–Kottwitz–MacPherson and Herb still make sense, and Corollary 5.1.1 says that they are still equal. Also, Corollary 5.1.1 implies that $\psi_{\mathcal{H}}(B, \lambda) = 0$ if λ is not in the span of any 2-structure for Φ, so it implies [GKM97, Theorem 5.3]. It is not clear whether this is an easier proof than the one given in [GKM97].

5.2 The type *A* identity involving ordered set partitions

We now show how to deduce [EMR19, Theorem 6.4] [1] from Theorem 4.1.1. We take $V = \mathbb{R}^n$ with the usual inner product, and we denote by $(e_1, \ldots, e_n)$ the standard basis of V. We consider the hyperplane arrangement $\mathcal{H}$ of type B_n on V, that is, $\mathcal{H} = (H_\alpha)_{\alpha \in \Phi_B^+}$, where $\Phi_B^+ = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_1, \ldots, e_n\}$. We write $\Phi_B^+ = \Phi^{(1)} \sqcup \Phi^{(2)}$, where $\Phi^{(1)} = \{e_i - e_j : 1 \leq i < j \leq n\}$, and we denote by $\mathcal{H} = \mathcal{H}^{(1)} \sqcup \mathcal{H}^{(2)}$ the corresponding decomposition of $\mathcal{H}$. The arrangement $\mathcal{H}^{(1)}$ is a Coxeter arrangement of type A_{n-1}, and we denote by $\Phi = \Phi^{(1)} \cup (-\Phi^{(1)})$ the associated root system. Let C be the intersection $(\bigcap_{\alpha \in \Phi^{(1)}} H_\alpha) \cap (\bigcap_{\alpha \in \Phi^{(2)}} H_\alpha^+)$. Then C is the open ray $\mathbb{R}_{>0} \cdot (e_1 + e_2 + \cdots + e_n)$.

Recall that $\mathscr{L}$ is the face poset of $\mathcal{H}$. We will now give a description of $\mathscr{L}$ in terms of signed ordered partitions. See also [ER99, Section 5] for this description. A *signed block* is a nonempty subset $\widetilde{B}$ of $\{\pm 1, \ldots, \pm n\}$ such that, for every $i \in \{1, \ldots, n\}$, at most one of $\pm i$ is in $\widetilde{B}$. We then denote by B the subset of $\{1, \ldots, n\}$ defined by $B = \{|i| : i \in \widetilde{B}\}$. A *signed ordered partition* of

[1] This is a reformulation of [Mor11, Proposition A.4].

a subset I of $\{1,\dots,n\}$ is a list $(\widetilde{B}_1,\dots,\widetilde{B}_r)$ of signed blocks such that $(B_1,\dots,B_r)$ is an ordered partition of I.

We consider the poset $\Pi_n^{\mathrm{ord},B}$ whose elements are pairs $\pi = (\widetilde{\pi}, Z)$, where $Z \subseteq \{1,\dots,n\}$ and $\widetilde{\pi}$ is a signed ordered partition of $\{1,\dots,n\} - Z$, and the cover relation is given by the following two rules:

$$((\widetilde{B}_1,\dots,\widetilde{B}_r), Z) \prec ((\widetilde{B}_1,\dots,\widetilde{B}_{r-1}), B_r \cup Z),$$
$$((\widetilde{B}_1,\dots,\widetilde{B}_r), Z) \prec ((\widetilde{B}_1,\dots,\widetilde{B}_{i-1}, \widetilde{B}_i \cup \widetilde{B}_{i+1}, \widetilde{B}_{i+2},\dots,\widetilde{B}_r), Z).$$

The set Z is usually called the *zero block* of π.

Let $\pi = (\widetilde{\pi}, Z)$ be an element of $\Pi_n^{\mathrm{ord},B}$, with $\widetilde{\pi} = (\widetilde{B}_1,\dots,\widetilde{B}_r)$. We define the cone C_π to be the set of $(x_1,\dots,x_n) \in V$ such that (with the convention that $x_{-i} = -x_i$ for $1 \le i \le n$):

(i) if $Z = \{i_1,\dots,i_m\}$, then the equalities $x_{i_1} = \cdots = x_{i_m} = 0$ hold;

(ii) for every block $\widetilde{B} = \{i_1,\dots,i_m\}$ in $\widetilde{\pi}$, the equalities and inequality $x_{i_1} = \cdots = x_{i_m} > 0$ hold;

(iii) for every two consecutive blocks $\widetilde{B}_s$ and $\widetilde{B}_{s+1}$ in $\widetilde{\pi}$ with $i \in \widetilde{B}_s$ and $j \in \widetilde{B}_{s+1}$, the inequality $|x_i| > |x_j|$ holds.

It is easy to see that the map $\varphi : \Pi_n^{\mathrm{ord},B} \longrightarrow \mathscr{L}$ sending π to C_π is a bijection, and that it induces an order-reversing isomorphism between the poset $\Pi_n^{\mathrm{ord},B}$ and the face poset $\mathscr{L}$. The inverse image of the ray $C = \mathbb{R}_{>0}\cdot(e_1+e_2+\cdots+e_n)$ by this bijection is the element $\pi_0 = ((\{1,\dots,n\}), \varnothing)$ of $\Pi_n^{\mathrm{ord},B}$, so the elements of $\mathscr{L}_{\ge C}$ correspond exactly to the (unsigned) ordered partitions of $\{1,\dots,n\}$. In other words, the bijection φ induces an order-reversing isomorphism between the poset Π_n^{ord} of ordered partitions of $\{1,\dots,n\}$ defined in [EMR19, Section 2] and the poset $\mathscr{L}_{\ge C}$.

Let $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$. For a signed block $\widetilde{B}$, we set $\lambda_{\widetilde{B}} = \sum_{i\in B} \lambda_i$, with the convention that $\lambda_{-i} = -\lambda_i$ for $1 \le i \le n$. Define the subset $\Pi_n^{\mathrm{ord},B}(\lambda)$ of $\Pi_n^{\mathrm{ord},B}$ by

$$\Pi_n^{\mathrm{ord},B}(\lambda) = \left\{ ((\widetilde{B}_1,\widetilde{B}_2,\dots,\widetilde{B}_r), Z) \in \Pi_n^{\mathrm{ord},B} \ : \ \sum_{i=1}^{s} \lambda_{\widetilde{B}_i} \ge 0 \text{ for } 1 \le s \le r \right\}.$$

Then an element π of $\Pi_n^{\mathrm{ord},B}$ is in $\Pi_n^{\mathrm{ord},B}(\lambda)$ if and only if C_π is in $\mathscr{L}_\lambda$. Moreover, the subset $\mathscr{L}_{\lambda,\ge C}$ corresponds to the set $\Pi_n^{\mathrm{ord}}(\lambda)$ of ordered partitions $(B_1,\dots,B_r)$ of $\{1,\dots,n\}$ such that, for every $1 \le s \le r$, we have $\sum_{i=1}^{s} \lambda_{B_i} \ge 0$. This is almost the set $\mathcal{P}(\lambda)$ of [EMR19, Section 3]; the only difference is that the inequalities defining $\mathcal{P}(\lambda)$ are strict. So we can give the following identity relating these two sets: For every $\varepsilon \in \mathbb{R}$, let $\lambda_\varepsilon = (\lambda_1 - \varepsilon,\dots,\lambda_n - \varepsilon)$. Then, if $\varepsilon > 0$ is sufficiently small, we have $\Pi_n^{\mathrm{ord}}(\lambda_\varepsilon) = \mathcal{P}(\lambda)$.

Let B be the unique chamber of $\mathscr{L}$ that is on the positive side of every hyperplane, that is, $B = \{x_1 > x_2 > \cdots > x_n > 0\}$. As we already observed, $\mathscr{L}_{\ge C}$ is isomorphic to the face poset of the arrangement $\mathcal{H}^{(1)}$, which is a Coxeter arrangement of type A_{n-1}, and the unique chamber of this arrangement containing B corresponds to the identity element in the symmetric group $\mathfrak{S}_n$. It then follows from Proposition 3.1.2 that the function $f_B : \mathscr{L}_{\ge C} \longrightarrow \mathscr{T} \cap \mathscr{L}_{\ge C}$ corresponds via $\varphi : \Pi_n^{\mathrm{ord}} \xrightarrow{\sim} \mathscr{L}_{\ge C}$ to the function $f : \Pi_n^{\mathrm{ord}} \longrightarrow \mathfrak{S}_n$ of [EMR19, Section 4]. We obtain the equality:

$$\psi_{\mathcal{H}/C}(B,\lambda) = \sum_{\pi \in \Pi_n^{\mathrm{ord}}(\lambda)} (-1)^{|\pi|} \cdot (-1)^{f(\pi)},$$

where $|\pi|$ denotes the number of blocks of the ordered partition $\pi = (B_1, \dots, B_r)$, in other words, $|\pi| = r$. Let $\overline{\lambda}$ denote the reverse of λ, that is, $\overline{\lambda} = (\lambda_n, \dots, \lambda_1)$. If $\varepsilon \in \mathbb{R}$, we write $\overline{\lambda}_\varepsilon$ for $(\lambda_n - \varepsilon, \dots, \lambda_1 - \varepsilon)$. By [EMR19, Lemma 7.1], we have

$$\psi_{\mathcal{H}/C}(B, \overline{\lambda}) = (-1)^{\binom{n}{2}} \cdot \sum_{\pi \in \Pi_n^{\mathrm{ord}}(\lambda)} (-1)^{|\pi|} \cdot (-1)^{g(\pi)},$$

where $g : \Pi_n^{\mathrm{ord}} \longrightarrow \mathfrak{S}_n$ is the function defined at the beginning of [EMR19, Section 6]. Finally, using the fact that $\Pi_n^{\mathrm{ord}}(\lambda_\varepsilon) = \mathcal{P}(\lambda)$ for sufficiently small $\varepsilon > 0$, then the sum $S(\lambda)$ of [EMR19, Section 6] is given by the expression:

$$S(\lambda) = (-1)^{\binom{n}{2}} \cdot \psi_{\mathcal{H}/C}(B, \overline{\lambda}_\varepsilon) \tag{5.1}$$

for any sufficiently small $\varepsilon > 0$.

We now find an expression for the sum $T(\lambda)$ of [EMR19, Section 6] in terms of 2-structures. As in [EMR19], we denote by M_n the set of maximal matchings on $\{1, 2, \dots, n\}$. Then we have a bijection $M_n \xrightarrow{\sim} \mathcal{T}(\Phi)$ sending a matching $p = \{p_1, \dots, p_m\}$, where $p_1 = \{i_1 < j_1\}, \dots, p_m = \{i_m < j_m\}$ are the edges of p, to the 2-structure $\varphi_p = \{\pm(e_{i_1} - e_{j_1}), \dots, \pm(e_{i_m} - e_{j_m})\}$. Moreover, we have $(-1)^p = \epsilon(\varphi_p)$. We can calculate $\psi_{\mathcal{H}_{\varphi_p}/C_{\varphi_p}}(B_{\varphi_p}, \lambda)$ using Lemma 3.1.9 for the decomposition $V = V_0 \times V_1 \times \cdots \times V_m$, where $V_k = \mathbb{R}e_{i_k} + \mathbb{R}e_{j_k}$ for $1 \leq k \leq m$, $V_0 = \{0\}$ if n is even, and $V_0 = \mathbb{R}e_i$ if n is odd and i is the unique unmatched element of $\{1, \dots, n\}$. By Lemma 3.1.9, we have

$$\psi_{\mathcal{H}_{\varphi_p}/C_{\varphi_p}}(B_{\varphi_p}, \lambda) = \prod_{k=1}^{m} d_2(\lambda_{i_k}, \lambda_{j_k}) \cdot \begin{cases} 1 & \text{if } n \text{ is even,} \\ d_1(\lambda_i) & \text{if } n \text{ is odd,} \end{cases}$$

where:

(a) The function $d_1 : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $d_1(a) = \psi_{\mathcal{H}_1/C_1}(B_1, a)$, where $\mathcal{H}_1$ is the hyperplane arrangement (H_e) on $\mathbb{R}e$ and $B_1 = C_1 = \mathbb{R}_{>0}e$.

(b) The function $d_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by $d_2(a, b) = \psi_{\mathcal{H}_2/C_2}(B_2, (a, b))$, where $\mathcal{H}_2$ is the hyperplane arrangement $(H_e, H_f, H_{e-f}, H_{e+f})$ on $\mathbb{R}e \oplus \mathbb{R}f$, $C_2 = \{\alpha e + \beta f : \alpha = \beta > 0\}$ and $B_2 = \{\alpha e + \beta f : \alpha > \beta > 0\}$.

In other words, the functions d_1 and d_2 are precisely the function $\psi_{\mathcal{H}/C}(B, \lambda)$ that we are trying to determine in the cases $n = 1$ and $n = 2$. A direct calculation yields:

$$d_1(a) = \begin{cases} -1 & \text{if } a \geq 0, \\ 0 & \text{if } a < 0, \end{cases} \qquad \text{and} \qquad d_2(a, b) = \begin{cases} -1 & \text{if } a, b \geq 0, \\ -2 & \text{if } b \geq -a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this with the formula defining $c(p, \lambda)$ in [EMR19, Section 6], we see that, for all $a, b \in \mathbb{R}$, if $\varepsilon > 0$ is sufficiently small relative to a and b, we have $d_1(a - \varepsilon) = -c_1(a)$ and $d_2(a - \varepsilon, b - \varepsilon) = -c_2(b, a)$, and hence

$$\begin{aligned} \psi_{\mathcal{H}_{\varphi_p}/C_{\varphi_p}}(B_{\varphi_p}, \overline{\lambda}_\varepsilon) &= c(p, \lambda) \cdot \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even,} \\ (-1)^{(n+1)/2} & \text{if } n \text{ is odd} \end{cases} \\ &= (-1)^n \cdot (-1)^{\binom{n}{2}} \cdot c(p, \lambda), \end{aligned}$$

if $\varepsilon > 0$ is sufficiently small relative to the λ_i. Combining all these calculations, we see that if $\varepsilon > 0$ is sufficiently small, then

$$\sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \cdot \psi_{\mathcal{H}_\varphi / C_\varphi}(B_\varphi, \overline{\lambda}_\varepsilon) = (-1)^n \cdot (-1)^{\binom{n}{2}} \cdot \sum_{p \in M_n} (-1)^p \cdot c(p, \lambda) = (-1)^n \cdot (-1)^{\binom{n}{2}} \cdot T(\lambda).$$

The identity $S(\lambda) = (-1)^n \cdot T(\lambda)$ in [EMR19, Theorem 6.4] now follows from Theorem 4.1.1, applied to $\overline{\lambda}_\varepsilon$ for $\varepsilon > 0$ sufficiently small.

6 Concluding remarks

As mentioned in the introduction, we are not aware of whether or not there is a representation-theoretic interpretation of the identity in Theorem 4.1.1. In other words, what is the meaning of the constants $\psi_{\mathcal{H}/C}(B, \lambda)$ for different values of λ?

In Appendix A we construct a signed convolution of two valuations that yields a new valuation. We wonder if there are other convolutions of valuations that also yield valuations. For instance, can the signed convolution of Definition A.1.4 be extended to valuations on (not necessarily polyhedral) cones in Euclidean space?

The main results in this paper are for Coxeter arrangements. However, to obtain the recursion formula in Section 3.3 we first prove it for general hyperplane arrangements in Proposition 3.2.2. Can the other side of the recursion also be viewed in a larger setting? That is, is there some analogue of 2-structures for more general hyperplane arrangements?

In Appendix B the proof of Proposition B.2.4 that the group W acts transitively on the set of 2-structures $\mathcal{T}$ consiste of verifying the result for all irreducible pseudo-root systems. Is there a general proof that does not use the classification of irreducible pseudo-root systems?

A Extending the construction of a valuation by Goresky, Kottwitz and MacPherson

We introduce a signed convolution of valuations on closed convex polyhedral cones in a finite-dimensional real vector space. As a special case we obtain in Corollary A.1.6 a valuation due to Goresky, Kottwitz and MacPherson; see [GKM97, Proposition A.4].

Subsection A.1 of this appendix contains definitions and statements of results. The proofs are relegated to Subsection A.2.

A.1 The signed convolution

Let V be a finite-dimensional real vector space and $V^\vee$ be its dual. A *closed convex polyhedral cone* in V is a subset of the form $\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \cdots + \mathbb{R}_{\geq 0}v_k$, where $v_1, v_2, \ldots, v_k \in V$ and $k \geq 0$.

For a subset X of the space V, define $X^\perp = \{\alpha \in V^\vee : \forall x \in X,\ \langle \alpha, x \rangle = 0\}$ and $X^* = \{\alpha \in V^\vee : \forall x \in X,\ \langle \alpha, x \rangle \geq 0\}$. Note that $X^\perp$ is a subspace of $V^\vee$ and depends only on the linear span of X, and that X^* is a convex cone in $V^\vee$ and depends only on the closed convex polyhedral cone generated by X.

For F a face[2] of a closed convex polyhedral cone K, define $F^{\perp,K} = F^\perp \cap K^*$. The map $F \longmapsto F^{\perp,K}$ is an order-reversing bijection from the set of faces of K to the set of faces of K^*.

[2]In this appendix, we take all faces to be closed faces, unlike in the rest of the article.

This statement and other basic properties of closed convex polyhedral cones are proved in [Ful93, Section 1.2].

Definition A.1.1. We denote by $\mathcal{C}(V)$ the set of closed convex polyhedral cones in V. A *valuation* on $\mathcal{C}(V)$ with values in an abelian group A is a function $f : \mathcal{C}(V) \longrightarrow A$ such that $f(\varnothing) = 0$ and that for any $K, K' \in \mathcal{C}(V)$ such that $K \cup K' \in \mathcal{C}(V)$, we have

$$f(K \cup K') + f(K \cap K') = f(K) + f(K'). \tag{A.1}$$

For $\lambda \in V^\vee$, we define the hyperplane H_λ and the two open half-spaces H_λ^+ and H_λ^- by

$$H_\lambda = \{x \in V : \langle \lambda, x\rangle = 0\}, \quad H_\lambda^+ = \{x \in V : \langle \lambda, x\rangle > 0\} \quad \text{and} \quad H_\lambda^- = \{x \in V : \langle \lambda, x\rangle < 0\}.$$

The closed half-spaces are given by $\overline{H_\lambda^+} = \{x \in V : \langle \lambda, x\rangle \geq 0\}$ and $\overline{H_\lambda^-} = \{x \in V : \langle \lambda, x\rangle \leq 0\}$.

We have the following criterion for recognizing valuations on closed convex polyhedral cones. This is known as *Groemer's first extension theorem* and is proved in [Gro78, Theorem 2].

Theorem A.1.2 (Groemer). *Let A be an abelian group and $f : \mathcal{C}(V) \longrightarrow A$ be a function such that $f(\varnothing) = 0$. Suppose that for every $K \in \mathcal{C}(V)$ and every $\mu \in V^\vee$ the following holds:*

$$f(K) + f(K \cap H_\mu) = f\big(K \cap \overline{H_\mu^+}\big) + f\big(K \cap \overline{H_\mu^-}\big). \tag{A.2}$$

Then the function f is a valuation.

Lemma A.1.3. *Let X be a subset of V such that the complement $V - X$ is convex. Then the function $\phi_X : \mathcal{C}(V) \longrightarrow \mathbb{Z}$ defined by*

$$\phi_X(K) = \begin{cases} 1 & \text{if } \emptyset \subsetneq K \subset X, \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation. In particular, if $\lambda \in V^\vee$ then the function $\psi_\lambda = \phi_{\overline{H_\lambda^+}}$ is a valuation.

Proof. Let $K \in \mathcal{C}(V)$ be nonempty and let $\mu \in V^\vee$. Let $K_0 = K \cap H_\mu$, $K_+ = K \cap \overline{H_\mu^+}$ and $K_- = K \cap \overline{H_\mu^-}$. We must check Criterion (A.2) in Theorem A.1.2, that is, $\phi_X(K) + \phi_X(K_0) = \phi_X(K_+) + \phi_X(K_-)$.

If $K \subset X$ then K_0, K_+ and K_- are also included in X, and the equality above is clear. If $K_+ \subset X$ but $K_- \not\subset X$, then $K_0 \subset X$ and $K \not\subset X$, so again the desired equality holds. The case where $K_- \subset X$ and $K_+ \not\subset X$ is symmetric. Finally, suppose that $K_+, K_- \not\subset X$. Then $K \not\subset X$, and so we must show that $K_0 \not\subset X$. Take $x \in K_+ - X$ and $y \in K_- - X$. Then the segment $[x, y]$ is contained in the convex set $V - X$. As this segment intersects K_0, this shows that $K_0 \not\subset X$.

Definition A.1.4. Let A be a ring, and let $f_1 : \mathcal{C}(V) \longrightarrow A$ and $f_2 : \mathcal{C}(V^\vee) \longrightarrow A$ be two functions. Define their *signed convolution* $f_1 * f_2 : \mathcal{C}(V) \longrightarrow A$ by

$$(f_1 * f_2)(K) = \sum_{F \in \mathcal{F}(K)} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K}),$$

where, for every $K \in \mathcal{C}(V)$, $\mathcal{F}(K)$ is the set of faces of the cone K.

The main result of this appendix is the following theorem whose proof is in Subsection A.2.

Theorem A.1.5. *The signed convolution of two valuations is a valuation.*

Given $x \in V$ and $\lambda \in V^\vee$, we have two valuations $\psi_\lambda : \mathcal{C}(V) \longrightarrow \mathbb{Z}$ and $\psi_x : \mathcal{C}(V^\vee) \longrightarrow \mathbb{Z}$ defined in Lemma A.1.3. Let $K \longmapsto \psi_K(x, \lambda)$ be their signed convolution, that is, for $K \in \mathcal{C}(V)$,

$$\psi_K(x, \lambda) = (\psi_\lambda * \psi_x)(K).$$

This function is defined in [GKM97, Appendix A] (at the top of page 540).

Corollary A.1.6. *For every $x \in V$ and every $\lambda \in V^\vee$, the function $K \longmapsto \psi_K(x, \lambda)$ from $\mathcal{C}(V)$ to $\mathbb{R}$ is a valuation.*

Since any valuation satisfies the additivity property, we obtain the next corollary, which is [GKM97, Proposition A.4].

Corollary A.1.7 (Goresky–Kottwitz–MacPherson)**.** *Let K be a closed convex polyhedral cone. Suppose that its relative interior K° is the disjoint union of the relative interiors $K_1^\circ, K_2^\circ, \ldots, K_r^\circ$ of r closed convex polyhedral cones $K_1, K_2, \ldots, K_r$. Then for every $x \in V$ and every $\lambda \in V^\vee$*

$$\psi_K(x, \lambda) = \sum_{i=1}^{r} (-1)^{\dim(K) - \dim(K_i)} \cdot \psi_{K_i}(x, \lambda).$$

Remark A.1.8. As a final note, valuations on $\mathcal{C}(V)$ can be extended to relatively open cones as well. Let G be a collection of sets that is closed under finite intersections. Define $B(G)$ to be the Boolean algebra generated by G, that is, the smallest collection of sets that contains G and is closed under finite unions, finite intersections and complements. Groemer's Integral Theorem states that a valuation on G can be extended to a valuation on the Boolean algebra $B(G)$; see [Gro78] and also [KR97, Chapter 2]. In the case where $G = \mathcal{C}(V)$, that is, the collection of closed convex polyhedral cones in E, the associated Boolean algebra $B(\mathcal{C}(V))$ contains all cones that are obtained by intersecting closed and open half-spaces.

A.2 Proofs

Before proving Theorem A.1.5, we state and prove the following lemma.

Lemma A.2.1. *Let $K \subset V$ be a closed convex polyhedral cone, let F be a face of the cone K and let $\mu \in V^\vee$. We write $K_0 = K \cap H_\mu$, $K_+ = K \cap \overline{H_\mu^+}$ and $K_- = K \cap \overline{H_\mu^-}$.*

(a) Assume that $F \subset \overline{H_\mu^+}$ but $F \not\subset H_\mu$, that is, F is a face of K_+ but not of K_0. Then the equality $F^\perp \cap K^* = F^\perp \cap K_+^*$ holds.

(b) Assume that $F \cap H_\mu^+ \neq \varnothing$ and $F \cap H_\mu^- \neq \varnothing$, in other words, the hyperplane H_μ cuts the face F in two. Then the equality $F^{\perp,K} = F^\perp \cap K_0^*$ holds.

(c) In the situation of (b), let $F_0 = F \cap H_\mu$. Then the equality $F_0^\perp \cap K^* = F^{\perp,K}$ holds.

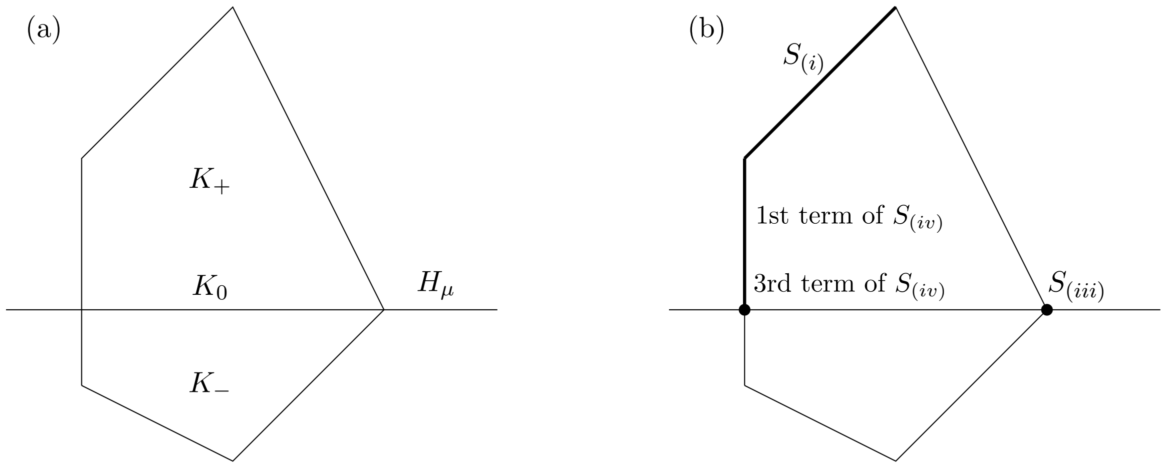

Figure 3: A two-dimensional representation of a five-sided three-dimensional cone K. In (b) the four different contributions to $g(K_+)$ are marked.

Proof. We first prove (a). The inclusion $F^\perp \cap K^* \subset F^\perp \cap K_+^*$ clearly holds, so we just need to show the reverse inclusion. Note that $K_+^* = K^* + \mathbb{R}_{\geq 0}\mu$. Let $\lambda \in F^\perp \cap K_+^*$, and write $\lambda = \lambda' + a\mu$, with $\lambda' \in K^*$ and $a \geq 0$. Choose $x^+ \in F \cap H_\mu^+$. Then we have $0 = \langle \lambda, x^+ \rangle = \langle \lambda', x^+ \rangle + a\langle \mu, x^+ \rangle$, with $\langle \lambda', x^+ \rangle \geq 0$ because $x^+ \in F \subset K$ and $\langle \mu, x^+ \rangle > 0$ because $x^+ \in H_\mu^+$, so $a \leq 0$. Hence $a = 0$, which implies that $\lambda \in K^*$.

We now prove (b). The relation $F^{\perp,K} = F^\perp \cap K^* \subset F^\perp \cap K_0^*$ clearly holds, so we just need to verify the reverse inclusion. Note that $K_0^* = K^* + \mathbb{R}\mu$. Let $\lambda \in F^\perp \cap K_0^*$, and write $\lambda = \lambda' + a\mu$, with $\lambda' \in K^*$ and $a \in \mathbb{R}$. Choose $x^+ \in F \cap H_\mu^+$ and $x^- \in F \cap H_\mu^-$. Then we have $0 = \langle \lambda, x^+ \rangle = \langle \lambda', x^+ \rangle + a\langle \mu, x^+ \rangle$, with $\langle \lambda', x^+ \rangle \geq 0$ because $x^+ \in F \subset K$, and $\langle \mu, x^+ \rangle > 0$ because $x^+ \in H_\mu^+$, so $a \leq 0$. On the other hand, we have $0 = \langle \lambda, x^- \rangle = \langle \lambda', x^- \rangle + a\langle \mu, x^- \rangle$, with $\langle \lambda', x^- \rangle \geq 0$ because $x^- \in F \subset K$, and $\langle \mu, x^- \rangle < 0$ because $x^- \in H_\mu^-$, so $a \geq 0$. This shows that $a = 0$, hence that $\lambda \in K^*$.

Finally, we prove (c). We have $F^{\perp,K} \subset F_0^\perp \cap K^*$, so we just need to show the reverse inclusion. Note that $F_0^\perp = F^\perp + \mathbb{R}\mu$. Let $\lambda \in F_0^\perp \cap K^*$, and write $\lambda = \lambda' + a\mu$, with $\lambda' \in F^\perp$ and $a \in \mathbb{R}$. As in the proof of (b), choose $x^+ \in F \cap H_\mu^+$ and $x^- \in F \cap H_\mu^-$. Then we have $0 \leq \langle \lambda, x^+ \rangle = \langle \lambda', x^+ \rangle + a\langle \mu, x^+ \rangle = a\langle \mu, x^+ \rangle$ and $\langle \mu, x^+ \rangle > 0$, so $a \geq 0$. On the other hand, we have $0 \leq \langle \lambda, x^- \rangle = \langle \lambda', x^- \rangle + a\langle \mu, x^- \rangle = a\langle \mu, x^- \rangle$ and $\langle \mu, x^- \rangle < 0$, so $a \leq 0$. Hence we must have $a = 0$, implying $\lambda \in F^\perp$.

Proof of Theorem A.1.5. Let $g = f_1 * f_2$ be the signed convolution of the valuations $f_1 : \mathcal{C}(V) \longrightarrow A$ and $f_2 : \mathcal{C}(V^\vee) \longrightarrow A$. We check the criterion of Theorem A.1.2. Let $K \in \mathcal{C}(V)$ and let $\mu \in V^\vee$. We define as before three closed convex polyhedral cones $K_+ = K \cap \overline{H_\mu^+}$, $K_- = K \cap \overline{H_\mu^-}$, $K_0 = K \cap H_\mu$. Note that $K_0^* = K_+^* \cup K_-^*$ and $K^* = K_+^* \cap K_-^*$. The faces F of the cone K come in four disjoint categories. For each category, we consider the contribution to the sum defining $g(K)$.

(i) F is a face of K_+, but not of K_0, that is, $F^\circ \subset H_\mu^+$. Then by Lemma A.2.1(a) we have

$F^\perp \cap K_+^* = F^\perp \cap K^*$, that is, $F^{\perp,K} = F^{\perp,K_+}$. Hence the contribution is

$$\begin{aligned} S_{(i)} &= \sum_{\substack{F\in\mathcal{F}(K)\cap\mathcal{F}(K_+)\\ F\notin\mathcal{F}(K_0)}} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K}) \\ &= \sum_{F\in(\mathcal{F}(K_+)\cap\mathcal{F}(K))-\mathcal{F}(K_0)} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K_+}). \end{aligned}$$

(ii) F is a face of K_-, but not of K_0, that is, $F^\circ \subset H_\mu^-$. As in case (i), we have $F^{\perp,K} = F^{\perp,K_-}$, and the contribution is

$$\begin{aligned} S_{(ii)} &= \sum_{\substack{F\in\mathcal{F}(K)\cap\mathcal{F}(K_-)\\ F\notin\mathcal{F}(K_0)}} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K}) \\ &= \sum_{F\in(\mathcal{F}(K_-)\cap\mathcal{F}(K))-\mathcal{F}(K_0)} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K_-}). \end{aligned}$$

(iii) F is a face of all three cones K_+, K_- and K_0, that is, we have $F \subset H_\mu$. Here the contribution is

$$\begin{aligned} S_{(iii)} &= \sum_{\substack{F\in\mathcal{F}(K)\\ F\subset H_\mu}} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K}) \\ &= \sum_{F\in\mathcal{F}(K_+)\cap\mathcal{F}(K_-)\cap\mathcal{F}(K_0)} (-1)^{\dim(F)} \cdot f_1(F) \cdot \big(f_2(F^{\perp,K_+}) + f_2(F^{\perp,K_-}) - f_2(F^{\perp,K_0})\big), \end{aligned}$$

since f_2 is a valuation and $F^{\perp,K_+} \cup F^{\perp,K_-} = F^{\perp,K_0}$ and $F^{\perp,K_+} \cap F^{\perp,K_-} = F^{\perp,K}$.

(iv) The face F gets cut into three faces: $F_+ = F \cap K_+$ in K_+, $F_- = F \cap K_-$ in K_- and $F_0 = F \cap K_0$ in K_0. Then we have $F^\perp = F_+^\perp = F_-^\perp$ because F, F_+ and F_- have the same span. By Lemma A.2.1(b), we have $F^\perp \cap K_0^* = F^\perp \cap K^*$, and so

$$F^{\perp,K} = F_+^{\perp,K_+} = F_-^{\perp,K_-} = F^\perp \cap K_0^*.$$

By Lemma A.2.1(c), we also have $F_0^\perp \cap K^* = F^{\perp,K}$. So the contribution is

$$\begin{aligned} S_{(iv)} &= \sum_{\substack{F\in\mathcal{F}(K)\\ F\text{ being cut}}} (-1)^{\dim(F)} \cdot f_1(F) \cdot f_2(F^{\perp,K}) \\ &= \sum_{\substack{F\in\mathcal{F}(K)\\ F\text{ being cut}}} (-1)^{\dim(F)} \cdot (f_1(F_+) + f_1(F_-) - f_1(F_0)) \cdot f_2(F^{\perp,K}) \\ &= \sum_{\substack{F\in\mathcal{F}(K)\\ F\text{ being cut}}} (-1)^{\dim(F)} \cdot (f_1(F_+) f_2(F_+^{\perp,K_+}) + f_1(F_-) f_2(F_-^{\perp,K_-}) - f_1(F_0) f_2(F^{\perp,K})) \end{aligned}$$

Using the fact that $F^{\perp,K} = F_0^{\perp} \cap K^*$, we get $f_2(F^{\perp,K}) = f_2(F_0^{\perp,K_+}) + f_2(F_0^{\perp,K_-}) - f_2(F_0^{\perp,K_0})$ and so

$$S_{(iv)} = \sum_{\substack{F\in\mathcal{F}(K)\\ F \text{ being cut}}} (-1)^{\dim(F)} \cdot \Big(f_1(F_+)f_2(F_+^{\perp,K_+}) + f_1(F_-)f_2(F_-^{\perp,K_-}) \\ - f_1(F_0)f_2(F_0^{\perp,K_+}) - f_1(F_0)f_2(F_0^{\perp,K_-}) + f_1(F_0)f_2(F_0^{\perp,K_0}) \Big).$$

Now expand $g(K)$ as $S_{(i)} + S_{(ii)} + S_{(iii)} + S_{(iv)}$. We use use the fact that $(-1)^{\dim(F)} = -(-1)^{\dim(F_0)}$ in the third, fourth and fifth terms of $S_{(iv)}$. The contributions to $g(K_+)$, respectively $g(K_-)$, are given by the sum $S_{(i)}$, respectively $S_{(ii)}$, the first term in the sum $S_{(iii)}$, respectively the second term, and the first and third terms in the sum $S_{(iv)}$, respectively the second and fourth terms. See Figure 3 (b). Finally, the third term of the sum $S_{(iii)}$ and the fifth term of the sum $S_{(iv)}$ yield the sum for $-g(K_0)$, which proves that $g(K) = g(K_+) + g(K_-) - g(K_0)$.

B Review of 2-structures

The concept of 2-structure for a root system was introduced by Herb to calculate discrete series characters on real reductive groups. See for example Section 5 of [Her01] or Section 4 of the review article [Her00]. In this section we review Herb's constructions and adapt them so that they work for an arbitrary Coxeter system having finite Coxeter group.

We fix a finite-dimensional $\mathbb{R}$-vector space V and an inner product $(\cdot,\cdot)$ on V. For every $v \in V - \{0\}$, we denote by s_v the (orthogonal) reflection across the hyperplane $v^{\perp}$.

Whenever we need to describe the irreducible root systems, we use the description given in the tables at the end of [Bou68], except that we write $(e_1,\dots,e_n)$ for the canonical basis of $\mathbb{R}^n$. When we need a system of positive roots in these root systems, we also use the ones given in these tables.

This appendix is organized as follows. Subsections B.1 and B.2 contain the definitions and results respectively. Subsections B.3 and B.4 contain the technical proofs. The verification that the Coxeter group W acts transitively on the set of 2-structures takes place in the fourth subsection.

B.1 Pseudo-root systems

Definition B.1.1. A finite subset Φ of $V - \{0\}$ is called a *pseudo-root system* if it satisfies the following conditions:

(a) for every $\alpha \in \Phi$, we have $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$;

(b) for every $\alpha, \beta \in \Phi$, the reflection s_α sends β to a vector of the form $c\gamma$, with $c \in \mathbb{R}_{>0}$ and $\gamma \in \Phi$.

If all the elements of Φ are unit vectors, we call Φ a *normalized pseudo-root system*. In that case, condition (b) become "$s_\alpha(\beta) \in \Phi$".

Remark B.1.2. We use this definition because it is convenient in the context of Coxeter systems. A root system (in the usual sense) is a pseudo-root system, which is not normalized in general. The converse is not true, even if we allow ourselves to replace the elements of Φ by scalar multiples, because of the existence of non-crystallographic Coxeter systems (see Proposition B.1.6).

Pseudo-root systems are called "root systems" in [Hum90, Section 1.2] and [BB05, Section 4.4]. We avoid this terminology because it is not compatible with the established definition of root systems in representation theory.

Remark B.1.3. If Φ is normalized or an actual root system then the group W preserves Φ, so the action of W on V restricts to an action of W on Φ. In general, we can still make W act on Φ by declaring that if $w \in W$ and $\alpha \in \Phi$ then $w \cdot \alpha$ is the unique element β of Φ such that $w(\alpha) \in \mathbb{R}_{>0}\beta$. This reduces to the previous action if Φ is normalized or an actual root system. Whenever we write an element of W acting on an element of Φ, this is the action that we mean.

Definition B.1.4. Let $\Phi \subset V$ be a pseudo-root system. A subset Δ of Φ is called a *system of simple pseudo-roots* if

(a) The set Δ is a vector space basis for the linear span of Φ.

(b) For every $\alpha \in \Phi$, we can write $\alpha = \sum_{\beta\in\Delta} n_\beta \beta$, where the coefficients n_β are in $\mathbb{R}$ and they are either all nonnegative or all nonpositive.

The corresponding *system of positive pseudo-roots* is then

$$\Phi^+ = \Phi \cap \left\{ \sum_{\beta\in\Delta} n_\beta \beta \ : n_\beta \in \mathbb{R}_{\geq 0} \ \forall \beta \in \Delta \right\}.$$

We also write $\Phi^- = -\Phi^+$.

Definition B.1.5. Let $\Phi \subset V$ be a pseudo-root system. We say that Φ is *irreducible* if there is no partition $\Phi = \Phi_1 \sqcup \Phi_2$, with Φ_1 and Φ_2 nonempty pseudo-root systems such that $(\alpha_1, \alpha_2) = 0$ for every $\alpha_1 \in \Phi_1$ and every $\alpha_2 \in \Phi_2$.

Proposition B.1.6. *The following two statements hold:*

(i) ([Hum90, Section 1.9] and [Hum90, Section 1.4].) Let $\Phi \subset V$ be a pseudo-root system and $\Delta \subset \Phi$ be a system of simple pseudo-roots. Let $W = W(\Phi)$ be the subgroup of $\mathbf{GL}(V)$ generated by the reflections s_α for $\alpha \in \Phi$, and let $S = \{s_\alpha : \alpha \in \Delta\}$. Then (W, S) is a Coxeter system where W is finite, and the Coxeter graph of (W, S) is connected if and only if Φ is irreducible.

Moreover, W acts transitively on the set of systems of positive pseudo-roots if we use the action of Remark B.1.3.

(ii) ([Hum90, Section 5.4].) Conversely, let (W, S) be a Coxeter system with W finite, and let $\rho : W \longrightarrow \mathbf{GL}(V)$ be its canonical representation on $V = \bigoplus_{s\in S} \mathbb{R}e_s$ (see the beginning of Subsection 2.3). Then $\Phi = \{\rho(w)(e_s) : w \in W,\ s \in S\}$ is a normalized pseudo-root system and $\Delta = \{e_s : s \in S\}$ is a system of simple pseudo-roots in Φ.

Definition B.1.7. Let $\Phi \subset V$ be an irreducible pseudo-root system. We say that Φ is of type A_n, respectively B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, $I_2(m)$ with $m \geq 3$, if the corresponding Coxeter system is of that type. Here we use the classification of simple finite Coxeter systems proved in [GB85, Chapter 5]. See Table 1 in [BB05, Appendix A].

Remark B.1.8. The Coxeter group of type $I_2(m)$ is the dihedral group of order $2m$. Note that types $I_2(3)$ and A_2 are isomorphic, types $I_2(4)$ and B_2 are isomorphic, and types $I_2(6)$ and G_2 are isomorphic. We did not include $I_2(2)$ in the list of irreducible types, because the corresponding Coxeter system is not irreducible, as it is isomorphic to $A_1 \times A_1$.

We will use the following lemma when introducing the sign associated to a 2-structure in Proposition B.2.7. Recall that, if $r \geq 1$, then the *lexicographic order* on $\mathbb{R}^r$ is defined by $(x_1, \ldots, x_r) < (y_1, \ldots, y_r)$ if there exists $1 \leq i \leq r$ such that $x_i < y_i$ and that $x_j = y_j$ for $1 \leq j \leq i-1$. It is a total order. Furthermore we say that a vector x is *positive* if $x > (0, 0, \ldots, 0)$.

Lemma B.1.9. *Let $\Phi \subset V$ be a pseudo-root system. Let $v_1, v_2, \ldots, v_r$ be linearly independent elements of V such that no element of Φ is orthogonal to every v_i. Define Φ^+ to be the set of $\alpha \in \Phi$ such that the element $((\alpha, v_1), (\alpha, v_2), \ldots, (\alpha, v_r))$ of $\mathbb{R}^r$ is positive with respect to the lexicographic order on $\mathbb{R}^r$. Then Φ^+ is a system of positive pseudo-roots.*

Proof. We complete $(v_1, \ldots, v_r)$ to a basis $(v_1, \ldots, v_n)$ of V, where n is the dimension of V. If $v, w \in V$, we say that $v < w$ if $((v, v_1), \ldots, (v, v_n)) < ((w, v_1), \ldots, (w, v_n))$ in the lexicographic order on $\mathbb{R}^n$. This defines a total order on V in the sense of [Hum90, Section 1.3], and Φ^+ is the corresponding positive system in Φ. By the theorem in [Hum90, Section 1.3], Φ^+ is a system of positive pseudo-roots in the sense of Definition B.1.4.

Definition B.1.10. If $\theta = (v_1, \ldots, v_r)$ is a sequence of linearly independent elements of V such that $\theta^\perp \cap \Phi = \varnothing$, we denote the system of positive pseudo-roots of Lemma B.1.9 by Φ_θ^+.

B.2 2-structures

We define 2-structures, generalizing a notion introduced by Herb for root systems; see for example the beginning of [Her00, Section 4]. We also generalize some of the results of [Her01, Section 5] to Coxeter systems with finite Coxeter groups.

We fix a pseudo-root system Φ in V and a system of positive pseudo-roots $\Phi^+ \subset \Phi$. We denote by (W, S) the corresponding Coxeter system (see Proposition B.1.6).

Definition B.2.1. A 2-*structure* for Φ is a subset φ of Φ satisfying the following properties:

(a) The subset φ is a disjoint union $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_i are pairwise orthogonal subsets of φ and each of them is an irreducible pseudo-root system of type A_1, B_2 or $I_2(2^n)$, for $n \geq 3$.

(b) Let $\varphi^+ = \varphi \cap \Phi^+$. If $w \in W$ is such that $w(\varphi^+) = \varphi^+$ then $\det(w) = 1$.

Remark B.2.2. Although condition (b) involves the set of positive pseudo-roots φ^+ in φ, it does not actually depend on the choice of φ^+, because the Coxeter group of φ acts transitively on sets of positive pseudo-roots in φ.

Remark B.2.3. If $\varphi \subseteq \Phi$ is a 2-structure then there is no $\alpha \in \Phi$ that is orthogonal to every element of φ. Indeed, if such an α existed then the associated reflection s_α would fix every element of φ, and in particular send φ^+ to itself, which would contradict condition (b) of Definition B.2.1.

Let $\mathcal{T}(\Phi) \subseteq 2^{\Phi}$ be the set of all 2-structures for the pseudo-root system Φ. The following proposition is proved in Subsection B.4, where we also show that each irreducible pseudo-root system contains a 2-structure and give the type of this 2-structure. This introduces no circularity in the arguments: the only results in this appendix that depend on Proposition B.2.4 are Lemmas B.2.11 and B.2.12, and these lemmas are not used in Subsections B.3 and B.4.

Proposition B.2.4. *The group W acts transitively on the collection of 2-structures $\mathcal{T}(\Phi)$.*

Let $\varphi \in \mathcal{T}(\Phi)$. We write $\varphi^+ = \varphi \cap \Phi^+$ and $\varphi^- = \varphi \cap \Phi^-$, and we define

$$W(\varphi, \Phi^+) = \{w \in W : w(\varphi^+) \subset \Phi^+\},$$
$$W_1(\varphi, \Phi^+) = \{w \in W : w(\varphi^+) \subset \varphi^+\} = \{w \in W : w(\varphi^+) = \varphi^+\}.$$

Note that $W_1(\varphi, \Phi^+)$ is a subgroup of W, and that the subset $W(\varphi, \Phi^+)$ of W is stable by right translations by elements of $W_1(\varphi, \Phi^+)$.

Corollary B.2.5. *Let $\varphi \in \mathcal{T}(\Phi)$. Then the map $W \longrightarrow \mathcal{T}(\Phi)$, $w \longmapsto w(\varphi)$ induces a bijection*

$$W(\varphi, \Phi^+)/W_1(\varphi, \Phi^+) \xrightarrow{\sim} \mathcal{T}(\Phi).$$

Proof. We denote by $f : W \longrightarrow \mathcal{T}(\Phi)$ the map defined by $f(w) = w(\varphi)$.

If $u \in W_1(\varphi, \Phi^+)$, then $u(\varphi) = \varphi$, so $f(wu) = f(w)$ for every $w \in W$. So the map f does induce a map from $W(\varphi, \Phi^+)/W_1(\varphi, \Phi^+)$ to $\mathcal{T}(\Phi)$, that we denote by $\overline{f}$.

We show that $\overline{f}$ is surjective. Let $\varphi' \in \mathcal{T}(\Phi)$. By Proposition B.2.4, there exists $w \in W$ such that $w(\varphi) = \varphi'$. By the theorem in [Hum90, Section 1.3], the set $w^{-1}(\Phi^+) \cap \varphi$ is a system of positive pseudo-roots in φ, so, by Proposition B.1.6, there exists $v \in W(\varphi)$, where $W(\varphi)$ is the Coxeter group of φ, such that $v(\varphi^+) = w^{-1}(\Phi^+) \cap \varphi$. Then $wv(\varphi^+) = \Phi^+ \cap w(\varphi) \subset \Phi^+$, so $wv \in W(\varphi, \Phi^+)$, and $wv(\varphi) = w(\varphi) = \varphi'$, that is, $f(wv) = \varphi'$.

We show that $\overline{f}$ is injective. Let $w, w' \in W(\varphi, \Phi^+)$ such that $w(\varphi) = w'(\varphi)$. Then we have $w^{-1}w'(\varphi) = \varphi$, and, again by the theorem in [Hum90, Section 1.3], the set $w^{-1}w'(\varphi^+)$ is a system of positive pseudo-roots in φ, so there exists $v \in W(\varphi)$ such that $v^{-1}w^{-1}w'(\varphi^+) = \varphi^+$. This means that we have $w' = wvu$ with $u \in W_1(\varphi, \Phi^+)$. So we will be done if we show that $v = 1$. Note that $wv(\varphi^+) = wvu(\varphi^+) = w'(\varphi^+) \subset \Phi^+$. Suppose that $v \neq 1$; then there exists $\alpha \in \varphi^+$ such that $v(\alpha) \in \varphi^-$, and then $wv(\alpha) = -w(-v(\alpha)) \in \Phi^-$ (because $w \in W(\varphi, \Phi^+)$), contradicting the fact that $wv(\varphi^+) \subset \Phi^+$. So $v = 1$. ∎

The following proposition, which follows immediately from Lemma 5.6 of [Her01] for root systems, can be deduced from Lemmas B.3.1 and B.3.4 for irreducible root systems not of type G_2, and proved via a direct calculation for the remaining irreducible types. We will not need this result, so we do not go into details.

Proposition B.2.6. *Let $\mathcal{T}_{(a)}(\Phi)$ be the set of $\varphi \subseteq \Phi$ that satisfy condition (a) of Definition B.2.1. Then $\mathcal{T}(\Phi)$ is exactly the set of elements of $\mathcal{T}_{(a)}(\Phi)$ that are maximal with respect to inclusion.*

Proposition B.2.7. *Let $\varphi \subseteq \Phi$ be a 2-structure. Define an ordered subset θ of φ as follows. Select a linear order of the irreducible components $\varphi_1, \varphi_2, \ldots, \varphi_r$ of φ. If φ_i is a pseudo-root system of type A_1, let θ_i be the singleton $\varphi_i \cap \varphi^+$. If φ_i is a pseudo-root system of type B_2 or $I_2(2^k)$ for $k \geq 3$,*

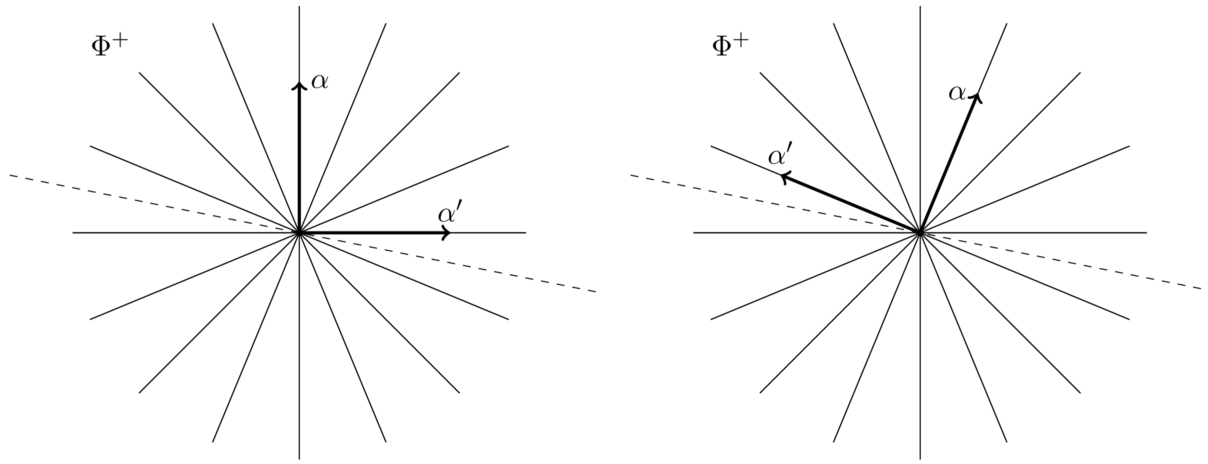

Figure 4: The dihedral pseudo-root system $I_2(8)$ with the two choices of $\theta = (\alpha, \alpha')$.

pick two orthogonal elements α and α' from $\varphi_i \cap \varphi^+$ such that $\varphi_i \cap \varphi^+ = \varphi^+_{i,(\alpha,\alpha')}$, that is, such that an element β of φ_i is in φ^+ if and only if either $(\beta, \alpha) > 0$, or $(\beta, \alpha) = 0$ and $(\beta, \alpha') > 0$. Let θ_i be the sequence (α, α'). Finally let θ be the concatenation of the sequences $\theta_1, \theta_2, \dots, \theta_r$.

Let Φ^+_θ be the system of positive pseudo-roots defined by the sequence θ as in Lemma B.1.9, and let w_θ be the unique element of W such that $w_\theta \cdot \Phi^+ = \Phi^+_\theta$. Then the sign $\det(w_\theta)$ *depends only on φ and not on the choices made to form θ.*

Note that there are several choices when producing the ordered set θ. First we have to select an order $\varphi_1, \varphi_2, \dots, \varphi_r$. There are $r!$ ways to do this. Second, if φ_i is of type B_2 or of type $I_2(2^k)$, there are two possible choices for the pseudo-roots α and α'; see Figure 4. These selections do not influence the sign of w_θ, although they do of course affect the set Φ^+_θ.

Proof of Proposition B.2.7. Let θ and θ' be the results of two possible sequences of choices. For an element w in W, recall that its length $\ell(w)$ [3] is also given by the cardinality of the intersection $w \cdot \Phi^+ \cap \Phi^-$; see [BB05, Proposition 4.4.4]. Note that $\Phi^+_\theta \cap \Phi^-_{\theta'} = w_\theta \cdot \Phi^+ \cap w_{\theta'} \cdot \Phi^- = w_{\theta'} \cdot (w_{\theta'}^{-1} w_\theta \cdot \Phi^+ \cap \Phi^-)$ which has cardinality $\ell(w_{\theta'}^{-1} w_\theta)$. Hence to prove that the signs agree, that is, that $\det(w_\theta) = \det(w_{\theta'})$, it suffices to show that the set $\Phi^+_\theta \cap \Phi^-_{\theta'}$ has an even number of elements.

We can reduce to the following two cases:

(a) there exists $1 \le i \le r$ such that θ and θ' differ only by the choice of the two pseudo-roots in the factor φ_i;

(b) there exists $1 \le i \le r-1$ such that $\theta_i = \theta'_{i+1}$, $\theta_{i+1} = \theta'_i$ and $\theta_j = \theta'_j$ if $j \ne i, i+1$.

We begin by treating case (a). We write $\theta_i = (\alpha, \alpha')$ and $\theta'_i = (\beta, \beta')$. Let $\gamma \in \Phi^+_\theta \cap \Phi^-_{\theta'}$. Then γ is orthogonal to $\varphi_1, \dots, \varphi_{i-1}$, and it is not orthogonal to φ_i. Also, as the sets of positive pseudo-roots in φ_i defined by θ_i and θ'_i are equal by assumption, we cannot have $\gamma \in \varphi_i$. Write $\gamma = c\alpha + c'\alpha' + \lambda$, with $\lambda \in \varphi_1^\perp \cap \cdots \cap \varphi_i^\perp$. By the previous sentence, we have $\lambda \ne 0$. The vector $\iota(\gamma) = -(s_\alpha s_{\alpha'})(\gamma) = c\alpha + c'\alpha' - \lambda$ is also in Φ. It is not equal to γ because $\lambda \ne 0$, and it is in

[3] By definition, this is the minimal number of factors in an expression of w as a product of reflections corresponding to simple pseudo-roots.

$\Phi_\theta^+ \cap \Phi_{\theta'}^-$ because γ and $\iota(\gamma)$ have the same inner product with any element of the set $\{\alpha, \alpha', \beta, \beta'\}$. Note that we clearly have $\iota(\iota(\gamma)) = \gamma$. We have constructed a fixed-point free involution ι on the set $\Phi_\theta^+ \cap \Phi_{\theta'}^-$, which proves that this set has even cardinality.

We treat case (b). Suppose first that φ_i and φ_{i+1} are both of type A_1, so we can write $\theta_i = (\alpha_i)$ and $\theta_{i+1} = (\alpha_{i+1})$. Let Φ' be the pseudo-root system $\Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$. If Φ' is of type $I_2(m)$ with $m \geq 3$, then m must be even because Φ' contains two orthogonal pseudo-roots. But then Φ' contains a multiple β of $\alpha_i - \alpha_{i+1}$, and the reflection s_β sends φ^+ to φ^+ because it fixes every element of φ_j for $j = i, i+1$ and exchanges α_i and α_{i+1}, contradicting the definition of a 2-structure. Hence Φ' is of type $A_1 \times A_1$, and then the fact that $|\Phi_\theta^+ \cap \Phi_{\theta'}^-|$ is even follows from Lemma B.3.2.

Suppose that φ_i is of type A_1 and φ_{i+1} is of type $I_2(2^m)$ with $m \geq 2$. Then we can write $\theta_i = (\alpha_i)$ and $\theta_{i+1} = (\alpha_{i+1}, \alpha'_{i+1})$. Let Φ', respectively Φ'', be the pseudo-root system $\Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$, respectively $\Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha'_{i+1})$, and let θ'' be the sequence that we obtain from θ by switching α_i and α_{i+1}. As φ_{i+1} is of type $I_2(2^m)$, it (and hence Φ) contains a pseudo-root β proportional to $\alpha_{i+1} - \alpha'_{i+1}$, and then $s_\beta(\Phi') = \Phi''$, so Φ' and Φ'' are of the same type. By Lemma B.3.2, the cardinalities of the sets $\Phi_\theta^+ \cap \Phi_{\theta''}^-$ and $\Phi_{\theta''}^+ \cap \Phi_{\theta'}^-$ have the same parity, and so $|\Phi_\theta^+ \cap \Phi_{\theta'}^-|$ is even. The case where φ_i is of rank 2 and φ_{i+1} of rank 1 follows from the previous case by switching the roles of φ_i and φ_{i+1}.

Finally, suppose that both φ_i and φ_{i+1} are of rank 2. Then we can write $\theta_i = (\alpha_i, \alpha'_i)$ and $\theta_{i+1} = (\alpha_{i+1}, \alpha'_{i+1})$. We move from θ to θ' by the following sequence of operations:

(1) We switch α'_i and α_{i+1}. By Lemma B.3.2 and Remark B.3.3, this changes the sign of w_θ by $(-1)^{m_1/2-1}$, where the pseudo-root system $\Phi_1 = \Phi \cap (\mathbb{R}\alpha'_i + \mathbb{R}\alpha_{i+1})$ is of type $I_2(m_1)$.

(2) We switch α'_i and α'_{i+1}. By the same lemma and remark, this changes the sign by $(-1)^{m_2/2-1}$, where the pseudo-root system $\Phi_2 = \Phi \cap (\mathbb{R}\alpha'_i + \mathbb{R}\alpha'_{i+1})$ is of type $I_2(m_2)$.

(3) We switch α_i and α_{i+1}. By the same lemma and remark, this changes the sign by $(-1)^{m_3/2-1}$, where the pseudo-root system $\Phi_3 = \Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$ is of type $I_2(m_3)$.

(4) We switch α_i and α'_{i+1}. By the same lemma and remark, this changes the sign by $(-1)^{m_4/2-1}$, where the pseudo-root system $\Phi_4 = \Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha'_{i+1})$ is of type $I_2(m_4)$.

The reflections $s_i = s_{\alpha_i - \alpha'_i}$ and $s_{i+1} = s_{\alpha_{i+1} - \alpha'_{i+1}}$ are both in W because φ_i contains a multiple of $\alpha_i - \alpha'_i$ and φ_{i+1} contains a multiple of $\alpha_{i+1} - \alpha'_{i+1}$. Observe now that $s_i(\Phi_1) = \Phi_3$, $s_i(\Phi_2) = \Phi_4$, $s_{i+1}(\Phi_1) = \Phi_2$ and $s_{i+1}(\Phi_3) = \Phi_4$. Thus the four pseudo-root systems Φ_1, Φ_2, Φ_3 and Φ_4 are isomorphic and hence $m_1 = m_2 = m_3 = m_4$. Hence performing operations (1) to (4) changes the sign by $((-1)^{m_1/2-1})^4 = 1$, that is, $\det(w_\theta) = \det(w_{\theta'})$.

Definition B.2.8. Let $\varphi \subseteq \Phi$ be a 2-structure, and let w_θ be as in Proposition B.2.7. Then the sign $(-1)^{r+r'} \det(w_\theta)$, where r is the number of irreducible factors of Φ of type A_{2n} with n odd and r' is the number of irreducible factors of Φ of type $I_2(2n'+1)$ with $n' \geq 3$ odd, is called the *sign* of φ and denoted by $\epsilon(\varphi, \Phi^+)$, or by $\epsilon(\varphi)$ if the system of positive pseudo-roots Φ^+ is understood.

Remark B.2.9. For a root system, this coincides with the definition of the sign of φ from Herb's paper [Her83], and it differs from the definition in Section 5 of Herb's paper [Her01]; see Remark 5.1 of [Her01] and Corollary 4.1.3.

Lemma B.2.10. *Let $\varphi \subseteq \Phi$ be a 2-structure, that is, $\varphi \in \mathcal{T}(\Phi)$.*

(i) For every $w \in W$, the identity $\epsilon(w(\varphi), w(\Phi^+)) = \epsilon(\varphi, \Phi^+)$ holds.

(ii) Let $w \in W$ be such that $w(\varphi^+) \subseteq \Phi^+$. Then the identity $\epsilon(w(\varphi), \Phi^+) = \det(w) \cdot \epsilon(\varphi, \Phi^+)$ holds.

Proof. Both identities follow easily from the definition of $\epsilon(\varphi, \Phi^+)$. Indeed, let θ be a subset of φ chosen as in Proposition B.2.7. For every $w \in W$, $w(\varphi)$ is a 2-structure for Φ and its subset $w(\theta)$ satisfies the same conditions for the system of positive pseudo-roots $w(\Phi^+)$, and also for the system of positive pseudo-roots Φ^+ if $w(\varphi^+) \subset \Phi^+$. Also, we have $\Phi^+_{w\cdot\theta} = w \cdot \Phi^+_\theta$. This immediately yields (i) and (ii).

Lemma B.2.11. *Let $\alpha_0 \in \Phi$ be a simple pseudo-root, let s_0 be the simple reflection defined by α_0, let $\Phi_0 = \alpha_0^\perp \cap \Phi$ and $\Phi_0^+ = \Phi_0 \cap \Phi^+$. Let $\mathcal{T}''$ be the set of $\varphi \in \mathcal{T}(\Phi)$ such that $s_0(\varphi) = \varphi$; we also consider the subsets $\mathcal{T}_1'' = \{\varphi \in \mathcal{T}'' : \varphi \cap \Phi_0 \in \mathcal{T}(\Phi_0)\}$ and $\mathcal{T}_2'' = \mathcal{T}'' - \mathcal{T}_1''$. Then the following statements hold:*

(1) The map $\mathcal{T}_1'' \longrightarrow \mathcal{T}(\Phi_0)$, $\varphi \longmapsto \varphi \cap \Phi_0$ is bijective.

(2) For every $\varphi \in \mathcal{T}_1''$, we have $\epsilon(\varphi, \Phi^+) = \epsilon(\varphi \cap \Phi_0, \Phi_0^+)$.

(3) There exists an involution ι of $\mathcal{T}_2''$ such that, for every $\varphi \in \mathcal{T}_2''$, we have $\varphi \cap \Phi_0 = \iota(\varphi) \cap \Phi_0$ and $\epsilon(\iota(\varphi), \Phi^+) = -\epsilon(\varphi, \Phi^+)$.

Proof. Note that a 2-structure φ for Φ is in $\mathcal{T}''$ if and only if $\alpha_0 \in \varphi$. Indeed, if $\alpha_0 \in \varphi$, then s_0 is in the Coxeter group of φ, so $s_0(\varphi) = \varphi$; conversely, we have $s_{\alpha_0}(\Phi^+ - \{\alpha_0\}) \subset \Phi^+$ by [BB05, Lemma 4.4.3], so, if $\varphi \in \mathcal{T}''$ and $\alpha_0 \notin \varphi$, then $s_0(\varphi^+) \subseteq \Phi^+ \cap \varphi = \varphi^+$, contradicting condition (b) in the definition of a 2-structure. Note also that the subset $\varphi \cap \Phi_0$ of Φ_0 always satisfies condition (a) in the definition of a 2-structure, but it does not always satisfy condition (b).

We prove (1). We may assume that Φ is irreducible, and we will freely use the explicit description of 2-structures given in Subsection B.4. If 2-structures for Φ are all of type A_1^s for some s, which happens in types A_n, D_n, E_6, E_7, E_8, H_3, H_4 and $I_2(m)$ for m odd, then $\varphi \cap \Phi_0 \in \mathcal{T}(\Phi_0)$ for every $\varphi \in \mathcal{T}(\Phi)$, that is, $\mathcal{T}_1'' = \mathcal{T}''$, and we see in the explicit description of 2-structures that the map of statement (1) is a bijection. It is easy to check that the same statement holds in type $I_2(m)$ for m even.

We now suppose that Φ is of type B_n or F_4. (Recall that from the point of view of Coxeter systems types B_n and C_n are isomorphic.) For convenience, in this case, we take Φ to be the actual root system, with possibly non-normalized roots; this does not affect any of the definitions that we made before. To study the map of (1), we may assume that $\alpha_0 = e_n$ or $\alpha_0 = e_1 - e_2$. Suppose first that $\alpha_0 = e_1 - e_2$. Then Φ_0 is reducible. Furthermore, it is of type $A_1 \times B_{n-2}$ if Φ is of type B_n, and of type $A_1 \times B_2$ if Φ is of type F_4, where the A_1 factor is $\{\pm(e_1 + e_2)\}$. In both cases, it is easy to see that $\mathcal{T}_1'' = \mathcal{T}''$ and that (1) holds. Suppose that $\alpha_0 = e_n$. Then Φ_0 is irreducible. Furthermore, it is of type B_{n-1} if Φ is of type B_n, and of type B_3 if Φ is of type F_4. If Φ is of type F_4 or B_n with n even then again it is easy to see that $\mathcal{T}_1'' = \mathcal{T}''$ and that (1) holds.

Finally, suppose that Φ is of type B_n with n odd and that $\alpha_0 = e_n$. If $\varphi \in \mathcal{T}''$ then we have $\varphi \in \mathcal{T}_1''$ if and only if $\{\pm e_n\}$ is an irreducible component of φ. The map sending $\varphi_0 \in \mathcal{T}(\Phi_0)$ to $\varphi_0 \sqcup \{\pm e_n\}$ is thus an inverse to the map of (1), so statement (1) holds.

We now prove (3). We have seen in the proof of (1) that $\mathcal{T}_2'' = \varnothing$ unless Φ is of type B_n with n odd and α_0 is the short simple root. Assume that we are in this case, which means that $\alpha_0 = e_n$. Let $\varphi \in \mathcal{T}_2''$. Then there exists $2 \leq i \leq n$ such that $\varphi_1 = \{\pm e_n, \pm e_i, \pm e_n \pm e_i\}$ is an irreducible component of φ. Write $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_k are irreducible and $\varphi_2 = \{\pm e_j\}$ is the unique rank 1 component of φ. Set $\iota(\varphi) = \{\pm e_n, \pm e_j, \pm e_n \pm e_j\} \sqcup \{\pm e_i\} \sqcup \varphi_3 \sqcup \cdots \sqcup \varphi_r$. This map switches the roles of e_i and e_j. Then $\iota(\varphi)$ is also in $\mathcal{T}_2''$, it is not equal to φ, we have $\iota(\varphi) \cap \Phi_0 = \varphi \cap \Phi_0$ and $\iota(\iota(\varphi)) = \varphi$. To finish the proof of (3), it suffices to show that $\epsilon(\iota(\varphi), \Phi^+) = -\epsilon(\varphi, \Phi^+)$ for every $\varphi \in \mathscr{T}_2''$. But this follows immediately from the definition of $\iota(\varphi)$ and from Lemma B.3.2.

We finally prove (2). Let $\varphi \in \mathcal{T}_1''$. Choose an ordered subset $\theta = \{\alpha_1, \ldots, \alpha_r\}$ of φ as in Proposition B.2.7. We may assume that $\alpha_0 \in \theta$. If α_0 is in an irreducible component of φ of type A_1, we may assume that $\alpha_0 = \alpha_1$. If α_0 is in an irreducible component of φ of rank 2, then, as it is a simple pseudo-root, it cannot be the first element of θ coming from this rank 2 factor of φ (see Figure 4 for an illustration in the case of $I_2(8)$, the general case is similar), so we may assume that $\alpha_0 = \alpha_r$.

Suppose first that α_0 is in an irreducible component of φ of rank 2 and that $\alpha_0 = \alpha_r$. By the description of 2-structures in Subsection B.4, this can only happen if Φ is of type B_n, F_4 or $I_2(m)$ with m even. The set $\{\alpha_1, \ldots, \alpha_{r-1}\}$ is an ordered subset of φ_0 satisfying the conditions of Proposition B.2.7, and $\Phi^+_{0,\theta_0} = \Phi^+_\theta \cap \Phi_0$. So the statement of (2) will follow if we can show that $X = (\Phi^+_\theta - \Phi^+_{0,\theta_0}) \cap \Phi^-$ has even cardinality. Let $s = s_{\alpha_r}$. We claim that $s(X) = X$ and that s has no fixed points in X, which implies that X has even cardinality because $s^2 = 1$. The fact that s has no fixed point in X follows from the facts that the fixed points of s are the elements of $\alpha_r^\perp$, that $\Phi \cap \alpha_r^\perp = \Phi_0$ and that $X \cap \Phi_0 = \varnothing$. As $\alpha_r \not\in \Phi^-$ and $-\alpha_r \not\in \Phi^+_\theta$, we have $X = (\Phi^- - \{-\alpha_r\}) \cap (\Phi^+_\theta - (\Phi^+_{0,\theta_0} \cup \{\alpha_r\}))$. As α_r is a simple pseudo-root, we have $s(\Phi^- - \{-\alpha_r\}) \subset \Phi^- - \{-\alpha_r\}$ by [BB05, Lemma 4.4.3]. So it suffices to prove that s preserves $\Phi^+_\theta - (\Phi^+_{0,\theta_0} \cup \{\alpha_r\})$. If $\beta \in \Phi^+_\theta - \Phi^+_{0,\theta_0}$ is such that $\beta \neq \alpha_r$, then we cannot have $(\beta, \alpha_i) = 0$ for every $i \in \{1, \ldots, r-1\}$; indeed, as Φ is of type B_n, F_4 or $I_2(m)$ with m even, the family $(\alpha_1, \ldots, \alpha_r)$ is an orthonormal basis of V, so the only element of Φ^+_θ that is orthogonal to $\alpha_1, \ldots, \alpha_{r-1}$ is α_r. So $\Phi^+_\theta - (\Phi^+_{0,\theta_0} \cup \{\alpha_r\})$ is the set pseudo-roots $\beta \in \Phi$ such that $((\beta, \alpha_1), \ldots, (\beta, \alpha_{r-1})) > 0$ (for the lexicographic order on $\mathbb{R}^{r-1}$) and that $(\beta, \alpha_r) \neq 0$. This set is stable by s, because, for every $\beta \in V$, we have $(s(\beta), \alpha_i) = (\beta, s(\alpha_i)) = (\beta, \alpha_i)$ if $1 \leq i \leq r-1$ and $(s(\beta), \alpha_r) = (\beta, s(\alpha_r)) = -(\beta, \alpha_r)$.

Now we suppose that α_0 is in an irreducible component of φ of rank 1 and that $\alpha_0 = \alpha_1$. Then $\theta_0 = \{\alpha_2, \ldots, \alpha_r\}$ is an ordered subset of φ_0 satisfying the conditions of Proposition B.2.7, and $\Phi^+_{0,\theta_0} = \Phi_0 \cap \Phi^+_\theta$, so $\Phi^+_\theta - \Phi^+_{0,\theta_0} = \{\beta \in \Phi : (\beta, \alpha_1) > 0\}$. Statement (2) will follow if we can show that

$$X = (\Phi^+_\theta - \Phi^+_{0,\theta_0}) \cap \Phi^- = \{\beta \in \Phi^- : (\beta, \alpha_1) > 0\}$$

has even cardinality if Φ is not of type A_{2n} or $I_2(2n'+1)$ with n' odd, and odd cardinality otherwise. As φ has an irreducible component of rank 1, we cannot be in type F_4. We can check that X has even cardinality by a computer calculation in the exceptional types E, G and H.

We now go through the remaining types one by one (in cases A, B and D, we use the description of the roots from the tables at the end of [Bou68], and not the normalized pseudo-root system):

- Type $I_2(m)$: If m is even, then φ is a rank 2 pseudo-root system; so m must be odd, and then φ_0 is empty and $\epsilon(\varphi_0) = 1$. There are exactly m pseudo-roots β such that $(\beta, \alpha_1) > 0$, and $(m-1)/2$ of these are in Φ^-. So $\epsilon(\varphi) = (-1)^{(m-1)/2}(-1)^{(m-1)/2} = 1$, which is what we wanted.

- Type A_n: We write $\alpha_0 = e_i - e_{i+1}$, with $1 \le i \le n$. Then

$$X = \{e_j - e_k : 1 \le k < j = i \text{ or } i+1 = k < j \le n+1\}$$

has cardinality $n-1$, that is, even if and only if n is odd.

- Type B_n: As φ has an irreducible component of rank 1, the integer n must be odd and α_0 is the short simple root, that is, $\alpha_0 = e_n$. Then

$$X = \{-e_i + e_n : 1 \le i \le n\}$$

has cardinality $n-1$, which is even.

- Type D_n: If $\alpha_0 = e_i - e_{i+1}$ with $1 \le i \le n-1$, then

$$\begin{aligned} X =& \{e_j - e_k : 1 \le k < j = i \text{ or } i+1 = k < j \le n\} \\ & \cup \{-(e_j + e_k) : i+1 = j < k \le n \text{ or } i \ne j < k = i+1\} \end{aligned}$$

has cardinality $2n-4$. If $\alpha_0 = e_{n-1} + e_n$, then

$$X = \{e_j - e_k : n = j > k \ne n-1 \text{ or } j = n-1 > k\}$$

also has cardinality $2n-4$.

Lemma B.2.12. *Let $\varphi \subseteq \Phi$ be a 2-structure. Then $|\Phi^+ - \varphi^+|$ is an even integer. More precisely, if Φ is irreducible, we have*

$$|\Phi^+ - \varphi^+| = \begin{cases} 2n \bmod 4 & \text{if } \Phi \text{ is of type } A_{2n}, \\ 0 \bmod 4 & \text{if } \Phi \text{ is of type } A_{2n+1}, B, D, E, F_4, G_2, \text{ or } H, \\ 2^r(m-1) & \text{if } \Phi \text{ is of type } I_2(2^r m) \text{ with } m \text{ odd.} \end{cases}$$

Proof. This follows from the explicit description of 2-structures for the irreducible types in Subsection B.4.

B.3 Orthogonal sets of pseudo-roots and 2-structures

For a pseudo-root system Φ let $\mathscr{O}(\Phi)$ be the set of all finite sequences $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of elements of Φ which are pairwise orthogonal and such that their entries all have the same length, that is, the following two conditions hold:

(a) $(\alpha_i, \alpha_j) = 0$ for all $1 \le i < j \le r$;

(b) $\|\alpha_1\| = \|\alpha_2\| = \cdots = \|\alpha_r\|$.

Recall that in the root systems of types A_n, D_n, E_6, E_7 and E_8, all the roots have the same length. Hence in part (ii) of the following lemma there is no need for an extra condition in these types.

Lemma B.3.1. *Suppose that Φ is a root system.*

(i) Let $\theta = (\alpha_1, \ldots, \alpha_r)$ and $\theta' = (\beta_1, \ldots, \beta_s)$ be elements of $\mathscr{O}(\Phi)$, and suppose that $\theta^\perp \cap \Phi = (\theta')^\perp \cap \Phi = \varnothing$ and that the elements of θ and θ' have the same length. Then there exists $w \in W$ such that $\{\alpha_1, \ldots, \alpha_r\} = \{w(\beta_1), \ldots, w(\beta_s)\}$. In particular, $r = s$ holds.

(ii) Suppose that Φ is irreducible and not of type G_2. Let $\theta \in \mathscr{O}(\Phi)$. If Φ is of type B_n and θ consists of long roots of the root system then assume that n is even. Similarly, if Φ is of type C_n and θ consists of short roots then again assume that n is even. Then there exists $\theta' \in \mathscr{O}(\Phi)$ such that θ is an initial segment of θ' and $(\theta')^\perp \cap \Phi = \varnothing$.

Proof. We start by proving (i). Let Φ^+_θ, respectively $\Phi^+_{\theta'}$, be the system of positive roots defined by θ, respectively θ', as in Definition B.1.10. As W acts transitively on the set of systems of positive roots, there exists $w \in W$ such that $w(\Phi^+_{\theta'}) = \Phi^+_\theta$. As $w(\Phi^+_{\theta'}) = \Phi^+_{w(\beta_1),\ldots,w(\beta_s)}$, we may assume that $\Phi^+_\theta = \Phi^+_{\theta'}$. We then wish to prove that θ and θ' are equal up to reordering their entries. We proceed by induction on the length of θ. If θ is empty then Φ is also empty because of the condition $\Phi \cap \theta^\perp = \varnothing$, so θ' is empty and we are done. Suppose that $r \geq 1$. Let j_0 be the smallest index j such that $(\alpha_1, \beta_j) \neq 0$. Since $\Phi \cap (\theta')^\perp = \varnothing$, this minimum exists. As $\beta_{j_0} \in \Phi^+_{\theta'} = \Phi^+_\theta$, we cannot have $(\alpha_1, \beta_{j_0}) < 0$, so $(\alpha_1, \beta_{j_0}) > 0$. As Φ is a root system and not just a pseudo-root system, the corollary after [Bou68, Chapitre VI, § 1, № 3, Théorème 1] implies that the difference $\gamma = \alpha_1 - \beta_{j_0}$ is an element of $\Phi \cup \{0\}$. Suppose that $\gamma \in \Phi^+_{\theta'}$. As $(\gamma, \beta_j) = 0$ for $1 \leq j < j_0$, we must then have $0 \leq (\gamma, \beta_{j_0}) = (\alpha_1, \beta_{j_0}) - (\beta_{j_0}, \beta_{j_0})$. The hypothesis states that $\|\alpha_1\| = \|\beta_{j_0}\|$ and hence we deduce that $(\alpha_1, \beta_{j_0}) \geq \|\beta_{j_0}\|^2 = \|\alpha_1\| \cdot \|\beta_{j_0}\|$. This inequality implies that $\alpha_1 = \beta_{j_0}$, contradicting the fact that γ is nonzero. Suppose that $\gamma \in \Phi^-_\theta$. Then $0 \leq (\alpha_1, -\gamma) = (\alpha_1, \beta_{j_0}) - (\alpha_1, \alpha_1)$, so $(\alpha_1, \beta_{j_0}) \geq \|\alpha_1\|^2$, and again this implies that $\alpha_1 = \beta_{j_0}$ and contradicts the assumption. Hence we conclude that $\gamma = 0$, that is, $\alpha_1 = \beta_{j_0}$. Let $\Phi_0 = \alpha_1^\perp \cap \Phi = \beta_{j_0}^\perp \cap \Phi$, $\theta_0 = (\alpha_2, \ldots, \alpha_r)$ and $\theta'_0 = (\beta_1, \ldots, \widehat{\beta_{j_0}}, \ldots, \beta_s)$. Then Φ_0 is a root system, θ_0 and θ'_0 are in $\mathscr{O}(\Phi_0)$, $\theta_0^\perp \cap \Phi_0 = (\theta'_0)^\perp \cap \Phi_0 = \emptyset$, and $\Phi^+_{0,\theta_0} = \Phi^+_\theta \cap \Phi_0 = \Phi^+_{\theta'} \cap \Phi_0 = \Phi^+_{0,\theta'_0}$. We can apply the induction hypothesis to conclude that $\{\alpha_2, \ldots, \alpha_r\} = \{\beta_1, \ldots, \widehat{\beta_{j_0}}, \ldots, \beta_s\}$, and this immediately implies that $\{\alpha_1, \ldots, \alpha_r\} = \{\beta_1, \ldots, \beta_s\}$.

We use the classification of irreducible root systems to prove (ii).

(a) Let $\Phi = A_n$. Then $\theta = (e_{i_1} - e_{j_1}, \ldots, e_{i_s} - e_{j_s})$ with $i_1, j_1, \ldots, i_s, j_s \in \{1, \ldots, n+1\}$ all distinct. After applying an element of $W = \mathfrak{S}_{n+1}$ to θ, we may assume that $(i_1, j_1, \ldots, i_s, j_s) = (1, 2, \ldots, 2s)$, and then we take $\theta' = (e_1 - e_2, \ldots, e_{2m-1} - e_{2m})$, where $m = \lfloor \frac{n+1}{2} \rfloor$.

(b) $\Phi = B_n$. Suppose that the elements of θ have the same length as the short roots in Φ. Then $\theta = (\pm e_{i_1}, \ldots, \pm e_{i_s})$, so after applying an element of W, we may assume that $\theta = (e_1, \ldots, e_s)$, and we can take $\theta' = (e_1, \ldots, e_n)$. Suppose that n is even and that the elements θ have the same length as the long roots in Φ. Then the elements of θ are of the form $\pm e_i \pm e_j$, with the following condition. If $\pm(e_i + e_j)$, respectively $\pm(e_i - e_j)$, appears, then $\mp(e_i + e_j)$, respectively $\mp(e_i - e_j)$, cannot appear and neither can any $\pm e_k \pm e_l$ with $|\{i,j\} \cap \{k,l\}| = 1$. After applying an element of W, we may assume that θ up to some reordering is given by the concatenation $(e_1 + e_2, e_1 - e_2, \ldots, e_{2s-1} + e_{2s}, e_{2s-1} - e_{2s}) \circ (e_{2s+1} + e_{2s+2}, \ldots, e_{2(s+t)-1} + e_{2(s+t)})$. Extend θ to a reordering θ' of $(e_1 + e_2, e_1 - e_2, \ldots, e_{2m-1} + e_{2m}, e_{2m-1} - e_{2m})$, where $m = n/2$.

(c) The cases of C_n and D_n are similar, except for some parity issues in the case of D_n.

Table 1: The number of orthogonal sets of roots or pseudo-roots of size k where the elements all have the same length in the exceptional/sporadic reflection arrangements. Note the double occurrence of 10! in the E_8 column. The equality of the columns in type F_4 comes from the fact that there is an automorphism of the underlying vector space that preserves angles, sends short roots to long roots, and sends long roots to doubles of short roots (for instance, the automorphism given by $e_1 \longmapsto e_1 + e_2$, $e_2 \longmapsto e_1 - e_2$, $e_3 \longmapsto e_3 + e_4$ and $e_4 \longmapsto e_3 - e_4$).

k	E_6	E_7	E_8	F_4 short	F_4 long	H_3	H_4
1	72	126	240	24	24	30	120
2	1080	3780	15120	72	72	60	1800
3	4320	32760	302400	96	96	40	2400
4	2160	75600	1965600	48	48		1200
5		90720	3628800				
6		60480	3628800				
7		17280	2073600				
8			518400				

(d) The cases of F_4, E_6, E_7 and E_8 can be checked with a computer. (Data from the calculations can be found in Table 1.)

Lemma B.3.2. *Let Φ be a normalized pseudo-root system, let $\theta = (\alpha_1, \dots, \alpha_r)$ be a sequence of pairwise orthogonal elements of Φ such that $\theta^\perp \cap \Phi = \varnothing$, and let θ' be the sequence obtained from θ by exchanging α_i and α_{i+1}. Consider the subroot system $\Phi' = \Phi \cap (\mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+1})$. Then Φ' is of type $A_1 \times A_1$ or $I_2(m)$ with $m \geq 4$ even, and the parity of the cardinality of $\Phi_\theta^+ \cap \Phi_{\theta'}^-$ is given by*

$$\begin{array}{lll} |\Phi_\theta^+ \cap \Phi_{\theta'}^-| & \equiv 0 \mod 2 & \textit{if } \Phi' = A_1 \times A_1, \\ |\Phi_\theta^+ \cap \Phi_{\theta'}^-| & \equiv m/2 - 1 \mod 2 & \textit{if } \Phi' = I_2(m). \end{array}$$

Proof. As Φ' is a pseudo-root system of rank 2 (because it is contained in a 2-dimensional vector space and contains the two linearly independent pseudo-roots α_i and α_{i+1}), it is of type $A_1 \times A_1$ or $I_2(m)$ with $m \geq 3$. Moreover, Φ' contains two orthogonal pseudo-roots, so it cannot be of type $I_2(m)$ with m odd.

We now set $C = \Phi_\theta^+ \cap \Phi_{\theta'}^-$ and calculate the parity of $|C|$. Let $\gamma \in C$. Then γ is orthogonal to $\alpha_1, \dots, \alpha_{i-1}$, so we can write $\gamma = c\alpha_i + d\alpha_{i+1} + \lambda$ with $\lambda \in \mathrm{Span}(\alpha_1, \dots, \alpha_{i+1})^\perp$ and $c\alpha_i + d\alpha_{i+1} \neq 0$. Set $\iota(\gamma) = -s_{\alpha_i} s_{\alpha_{i+1}}(\gamma)$. Then $\iota(\gamma) \in \Phi$ and $\iota(\gamma) = c\alpha_i + d\alpha_{i+1} - \lambda$, so $\iota(\gamma) \in C$. Also, we clearly have $\iota(\iota(\gamma)) = \gamma$, and $\iota(\gamma)$ is equal to γ if and only if $\lambda = 0$, that is, if and only if $\gamma \in \Phi'$. We have defined an involution ι of C, and we conclude that $|C| \equiv |C_0| \bmod 2$, where $C_0 = \Phi' \cap C$ is the set of fixed points of ι in C. If Φ' is of type $A_1 \times A_1$ then we easily see that C_0 is empty, so we are done. Suppose that Φ' is of type $I_2(m)$ with m even. Let $\gamma = c\alpha_i + d\alpha_{i+1} \in \Phi'$, with $c, d \in \mathbb{R}$. Then $\gamma \in C$ if and only if $c > 0$ and $d < 0$. The set C_0 contains exactly one quarter of the elements of $\Phi' - \{\pm\alpha_i, \pm\alpha_{i+1}\}$, that is, $|C_0| = (2m-4)/4 = m/2 - 1$.

Remark B.3.3. If we view the root system $A_1 \times A_1$ as the dihedral pseudo-root system $I_2(2)$ then the conclusion of Lemma B.3.2 is that $|\Phi_\theta^+ \cap \Phi_{\theta'}^-| \equiv m/2 - 1 \bmod 2$ if $\Phi' = I_2(m)$ with m even and $m \geq 2$.

Lemma B.3.4. *Suppose that Φ is an irreducible root system (not just a pseudo-root system) and not of type G_2. Let Φ^+ be a system of positive roots of Φ and let $\varphi \subseteq \Phi$ be a 2-structure. Define a subset θ of φ as in Proposition B.2.7. Then there is a choice of the sequences θ_i for which θ is an element of $\mathscr{O}(\Phi)$. Moreover, if Φ is of type B_n or F_4 we can choose θ to consist of short roots. Similarly, if Φ is of type C_n or F_4 we can choose θ to consist of long roots.*

Proof. By Remark B.2.3 we have $\theta^\perp \cap \Phi = \varnothing$. We use the notation of Proposition B.2.7. If all the roots of Φ have the same length (which is the case for A_n, D_n, E_6, E_7 and E_8), then there is nothing to prove. Note also that if φ_i is an actual root system of type B_2 (that is, with the correct root lengths), then the two possible choices for θ_i are the set of short positive roots and the set of long positive roots.

Suppose that Φ is of type B_n. If φ has no irreducible component of type A_1, then we choose the two short positive roots in each φ_i. Suppose that φ has a factor of type A_1. We show that this factor cannot contain long roots. Suppose on the contrary that this occurs. Without loss of generality, we may assume that $\varphi_1 = \{\pm(e_1 + e_2)\}$. The rank 2 factors of φ cannot contain $e_1 - e_2$, so they are all in $e_1^\perp \cap e_2^\perp$. All the rank 1 factors that do not contain $e_1 - e_2$ must also be in $e_1^\perp \cap e_2^\perp$. If $e_1 - e_2$ were not in φ then the reflection $s_{e_1 - e_2}$ would act as the identity on all the elements on φ, which contradicts the definition of a 2-structure. Hence $\{\pm(e_1 - e_2)\}$ is another rank 1 factor of φ. But then the reflection s_{e_1} preserves φ^+, which is impossible. Hence all the A_1 factors of φ contain only short roots, and we choose the θ_i in the B_2 factors to contain the two short positive roots.

The case of C_n is similar, with the roles of short and long roots uniformly exchanged.

Finally suppose that $\Phi = F_4$. In this case we can similarly show that the 2-structure φ has type B_2^2, allowing us to pick either short or long roots in each factor.

B.4 2-structures in the irreducible types

In this subsection we prove Proposition B.2.4, that is, the fact that the group W acts transitively on the collection of 2-structures $\mathcal{T}(\Phi)$. It is enough to prove this result for irreducible pseudo-root systems. We proceed by a case by case analysis.

Types A_n, D_n, E_6, E_7 and E_8

Suppose that Φ is a root system of type A_n, D_n or E_m with $m \in \{6, 7, 8\}$. As all the roots of Φ have the same length and as Φ contains no B_2 root system, the 2-structures for Φ are exactly the maximal sets $\varphi = \{\pm\alpha_1, \ldots, \pm\alpha_r\}$ such that $(\alpha_1, \ldots, \alpha_r) \in \mathscr{O}(\Phi)$. By Lemma B.3.1(i), for any $(\alpha_1, \ldots, \alpha_r)$ and $(\beta_1, \ldots, \beta_s)$ on $\mathscr{O}(\Phi)$, there exists $w \in W$ such that $\{\alpha_1, \ldots, \alpha_r\} = \{w(\beta_1), \ldots, w(\beta_s)\}$. Hence the group W acts transitively on $\mathcal{T}(\Phi)$. In particular, all the 2-structures for Φ are isomorphic, so we can determine their type. See Table 2.

Types B_n and C_n

Suppose that Φ is a root system of type B_n. This will also give the type C_n case, since B_n and C_n correspond to the same Coxeter system. We claim that W acts transitively on $\mathcal{T}(\Phi)$. In particular,

Table 2: The 2-structures in types A, D and E where $m = \lfloor (n+1)/2 \rfloor$ in type A and $m = \lfloor n/2 \rfloor$ in type D.

Type of root system Φ	2-structures are isomorphic to	Type of 2-structure
A_n	$\{\pm(e_1 - e_2),\ \ \pm(e_3 - e_4),\ \ \ldots,\ \ \pm(e_{2m-1} - e_{2m})\}$	A_1^m
D_n	$\{\pm e_1 \pm e_2,\ \ \pm e_3 \pm e_4,\ \ \ldots,\ \ \pm e_{2m-1} \pm e_{2m}\}$	A_1^{2m}
E_6	$\{\pm e_1 \pm e_2,\ \ \pm e_3 \pm e_4\}$	A_1^4
E_7	$\{\pm e_1 \pm e_2,\ \ \pm e_3 \pm e_4,\ \ \pm e_5 \pm e_6,\ \ \pm(e_7 - e_8)\}$	A_1^7
E_8	$\{\pm e_1 \pm e_2,\ \ \pm e_3 \pm e_4,\ \ \pm e_5 \pm e_6,\ \ \pm e_7 \pm e_8\}$	A_1^8

all the 2-structures for Φ are isomorphic to

$$\varphi_0 = \begin{cases} \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\} \sqcup \cdots \sqcup \{\pm e_{2m-1}, \pm e_{2m}, \pm e_{2m-1} \pm e_{2m}\} & \text{if } n = 2m, \\ \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\} \sqcup \cdots \sqcup \{\pm e_{2m-1}, \pm e_{2m}, \pm e_{2m-1} \pm e_{2m}\} \sqcup \{\pm e_{2m+1}\} & \text{if } n = 2m+1, \end{cases}$$

so they are of type B_2^m if $n = 2m$ is even, and of type $B_2^m \times A_1$ if $n = 2m+1$ is odd.

We prove the claim by induction on n. The case $n = 1$ is clear. Suppose that $n \geq 2$. Let $\varphi, \varphi' \in \Phi$. By Lemma B.3.4, we can choose sequences θ of φ and θ' of φ' as in Proposition B.2.7 such that $\theta, \theta' \in \mathscr{O}(\Phi)$ and that these subsets contain only short roots. By Lemma B.3.1(i), we may assume that θ and θ' coincide up to the order of their elements. Denote by $\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_s$ and $\varphi' = \varphi_1' \sqcup \cdots \sqcup \varphi_t'$ the decomposition into irreducible systems that gave rise to θ and θ'. We can always change the order on the φ_i and the φ_j'.

Suppose that φ_1 is of rank 1, so that $\varphi_1 = \{\pm\alpha_1\}$. We may assume that $\alpha_1 \in \varphi_1'$. If φ_1' is of rank 1 then $\varphi_1' = \varphi_1$. As $\Phi \cap \varphi_1^\perp$ is an irreducible root system of type B_{n-1}, the conclusion follows by the induction hypothesis.

If φ_1' is of rank 2 then φ_1' is a B_2 root system whose short positive roots are α_1 and some α_2, and we may assume that $\alpha_2 \in \varphi_2$. In particular, $\beta = \alpha_1 - \alpha_2 \in \Phi$. If $\varphi_2 = \{\pm\alpha_2\}$ then the reflection s_β preserves φ^+, which is not possible. So φ_2 is of rank 2 (in particular, $n \geq 3$), which means that it is a B_2 root system whose short roots are α_2 and some α_3. We may assume that $\alpha_3 \in \varphi_2'$. In particular, $\alpha_2 - \alpha_3 \in \Phi$, so $\gamma = s_\beta(\alpha_2 - \alpha_3) = \alpha_1 - \alpha_3$ is also a root. The irreducible components of $s_\gamma(\varphi)$ are $\varphi_1', \{\pm\alpha_3\}, \varphi_3, \ldots, \varphi_s$. As $\Phi \cap (\varphi_1')^\perp$ is a root system of type B_{n-2}, the induction hypothesis implies that there is a $w \in W$ such that $w(\varphi') = s_\gamma(\varphi)$, which finishes the proof in this case.

Suppose that φ_1 is of rank 2, and call its other short positive root α_2. We may assume that $\alpha_1 \in \varphi_1'$. If φ_1' is of rank 1 then $\varphi_1' = \{\pm\alpha_1\}$, and we can repeat the reasoning of the previous paragraph with the roles of φ and φ' exchanged. If $\varphi_1' = \varphi_1$ then the conclusion follows from the induction hypothesis applied to the B_{n-2} root system $\varphi_1^\perp \cap \Phi$. Finally, suppose that φ_1' is of rank 2 and $\varphi_1' \neq \varphi_1$. Let α_3 be the other short positive root of φ_1'. As α_2 and α_3 are both short roots, $\beta = \alpha_2 - \alpha_3 \in \Phi$. Note that the irreducible components of $s_\beta(\varphi)$ are $\varphi_1', s_\beta(\varphi_2), \ldots, s_\beta(\varphi_s)$, so again the induction hypothesis implies that there exists $w \in W$ such that $s_\beta(\varphi) = w(\varphi')$, and we are done.

Type F_4

Suppose that Φ is a root system of type F_4. Then we can show that W acts transitively on $\mathcal{T}(\Phi)$ exactly as in type B_n. In particular, any 2-structure is isomorphic to $\varphi_0 = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\} \sqcup \{\pm e_3, \pm e_4, \pm e_3 \pm e_4\}$, so it is of type B_2^2.

Dihedral types

Suppose that Φ is a pseudo-root system of type $I_2(m)$ with $m \geq 5$ (this includes the type G_2 root system). It is straightforward to see that W acts transitively on $\mathcal{T}(\Phi)$. If m is odd then all the 2-structures for Φ are isomorphic to $\varphi_0 = \{\pm e_1\}$, and in particular of type A_1. If m is even then all the 2-structures for Φ are of type $I_2(2^r)$, where 2^r is the largest power of 2 dividing m.

Types H_3 and H_4

Suppose that Φ is of type H_3 or H_4. We use the description of the pseudo-root systems H_3 and H_4 given in [GB85, Table 5.2] where they are called I_3 and I_4. In particular, we choose Φ to be normalized. We claim that W acts transitively on $\mathcal{T}(\Phi)$, and so every 2-structure for Φ is isomorphic to

$$\varphi_0 = \begin{cases} \{\pm e_1, \pm e_2, \pm e_3\} & \text{if } \Phi = H_3, \\ \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} & \text{if } \Phi = H_4, \end{cases} \tag{B.1}$$

and in particular it is of type A_1^3 if $\Phi = H_3$ and of type A_1^4 if $\Phi = H_4$.

It is clear by the chosen description of Φ that all of the inner products of elements of Φ are in $\mathbb{Q}[\sqrt{5}]$, and in particular $1/\sqrt{2}$ never appears. So there are no pseudo-roots in Φ with an angle of $\pi/4$ between them, which implies that Φ does not contain any pseudo-root system of type $I_2(m)$ with m a multiple of 4, and so 2-structures for Φ (if they exist) can only have irreducible components of type A_1.

We check easily that the set φ_0 given in equation (B.1) is a 2-structure, so it remains to show that all the maximal sets of pairwise orthogonal pseudo-roots are conjugate under W to ζ_0, where $\zeta_0 = \{e_1, e_2, e_3\}$ if $\Phi = H_3$ and $\zeta_0 = \{e_1, e_2, e_3, e_4\}$ if $\Phi = H_4$. Any element of the stabilizer W_0 of ζ_0 in W must act on $\mathrm{Span}(\Phi)$ by a permutation of the coordinates, and it must be an even permutation to be in W. This implies that the cardinality of W_0 is 3 for $\Phi = H_3$ and 12 for $\Phi = H_4$. Using a computer, it is not hard to count all the maximal sets of pairwise orthogonal pseudo-roots in H_3 and H_4 (see Table 1). We find that there are 40 such sets for H_3 and 1200 such sets for H_4. In both cases, this number is equal to $|W|/|W_0|$, so W does act transitively on the set of maximal sets of pairwise orthogonal pseudo-roots, and hence also on $\mathscr{T}(\Phi)$.

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