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On the generating function for consecutively weighted permutations

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ABSTRACT

We show that the analytic continuation of the exponential generating function associated to consecutive weighted pattern enumeration of permutations only has poles and no essential singularities. The proof uses the connection between permutation enumeration and functional analysis, and as well as the Laurent expansion of the associated resolvent. As a consequence, we give a partial answer to a question of Elizalde and Noy: when is the multiplicative inverse of the exponential generating function for the number permutations avoiding a single pattern an entire function? Our work implies that it is enough to verify that this function has no zeros to conclude that the inverse function is entire.

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1. Weighted enumeration

For a vector $x = (x_1, \ldots, x_k)$ of k distinct real numbers, define $\Pi(x)$ to be the standardization of the vector x, that is, the unique permutation $\sigma = (\sigma_1, \ldots, \sigma_k)$ in $\mathfrak{S}_k$ such that for all indices $1 \leq i < j \leq k$ the inequality $x_i < x_j$ is equivalent to $\sigma_i < \sigma_j$. Let S be a set of permutations in the symmetric group $\mathfrak{S}_{m+1}$. We say that a permutation π in $\mathfrak{S}_n$ avoids the set S consecutively if there is no index $1 \leq j \leq n - m$ such that $\Pi(\pi_j, \pi_{j+1}, \ldots, \pi_{j+m}) \in S$.

Following [2], we consider the extension of consecutive pattern avoidance to weighted enumeration. Let wt be a real-valued weight function on the symmetric group $\mathfrak{S}_{m+1}$. Similarly, let $\mathrm{wt}_1, \mathrm{wt}_2$ be two real-valued weight functions on the symmetric group $\mathfrak{S}_m$. We call wt_1 and wt_2 the initial and

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final weight function, respectively. We extend these three weight functions to the symmetric group $\mathfrak{S}_n$ for $n \geq m$ by defining

$$\mathrm{Wt}(\pi) = \mathrm{wt}_1(\Pi(\pi_1, \pi_2, \ldots, \pi_m)) \cdot \prod_{i=1}^{n-m} \mathrm{wt}(\Pi(\pi_i, \pi_{i+1}, \ldots, \pi_{i+m})) \cdot \mathrm{wt}_2(\Pi(\pi_{n-m+1}, \pi_{n-m+2}, \ldots, \pi_n)).$$

Let α_n be the sum of all the weights of permutations in $\mathfrak{S}_n$, that is,

$$\alpha_n = \sum_{\pi \in \mathfrak{S}_n} \mathrm{Wt}(\pi).$$

This framework can be used to study consecutive pattern avoidance by defining the weight

$$\mathrm{wt}(\sigma) = \begin{cases} 1 & \text{if } \sigma \notin S, \\ 0 & \text{if } \sigma \in S. \end{cases}$$

Furthermore, let both the initial and final weight functions be constant 1. Then for $n \geq m$, α_n is the number of permutations in $\mathfrak{S}_n$ that avoid the set S. In this case we extend α_n by setting $\alpha_n = n!$ for $n \leq m - 1$.

The classical combinatorial way to understand a sequence is by its generating function. Our main result is an analytic result about the associated generating function.

Theorem 1.1. *All the singularities of the analytic continuation of the exponential generating function $Q(z) = \sum_{n \geq m} \alpha_n \cdot z^n/n!$ for weighted permutations are either removable or poles.*

In other words, the function $Q(z)$ does not have any essential singularities.

To prove this result, we now outline the operator approach described in the two papers [2,3]. Define the linear operator T on the space $L^2([0, 1]^m)$ by

$$T(f)(x_1, \ldots, x_m) = \int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(t, x_1, \ldots, x_{m-1}) dt,$$

where the function χ is defined by $\chi(t, x_1, \ldots, x_m) = \mathrm{wt}(\Pi(t, x_1, \ldots, x_m))$. Similarly define the two functions κ and μ on the m-dimensional unit cube $[0, 1]^m$ by $\kappa(x) = \mathrm{wt}_1(\Pi(x))$ and $\mu(x) = \mathrm{wt}_2(\Pi(x))$. Then the quantity α_n is given by the inner product

$$\left(T^{n-m}(\kappa), \mu\right) = \alpha_n/n!.$$

In this paper the initial and final weight functions wt_1 and wt_2 are equal to 1, that is, κ and μ are the constant function $\mathbf{1}$ on the m-dimensional unit cube. For an application, where the weight functions wt_1 and wt_2 are different from 1; see [2, Section 6].

Now the exponential generating function $Q(z)$ can be written as

$$Q(z) = \sum_{n \geq m} \left(T^{n-m}(\kappa), \mu\right) \cdot z^n = z^m \cdot \left((I - z \cdot T)^{-1}(\kappa), \mu\right).$$

We observe that this expression is not defined when the operator $I - z \cdot T$ is singular, that is, when z^{-1} is an eigenvalue of the operator T. However the operator T^m is compact. Thus Theorem 6 in [1, Section VII.4] states that the set of eigenvalues has no point of accumulation but 0 and all the non-zero eigenvalues have a finite index. Recall that the index of an eigenvalue for an operator on a finite-dimensional vector space is the size of the largest Jordan block associated with that eigenvalue. Now Theorem 1.1 follows from this lemma:

Lemma 1.2. *If λ is a non-zero eigenvalue of the operator T of index k, then λ^{-1} is a pole of order at most k of the function $Q(z)$.*

Proof. Consider the limit

$$\begin{aligned}\lim_{z\longrightarrow\lambda^{-1}}(z-\lambda^{-1})^k\cdot(I-z\cdot T)^{-1} &= \lim_{\xi\longrightarrow\lambda}(\xi^{-1}-\lambda^{-1})^k\cdot(I-\xi^{-1}\cdot T)^{-1}\\ &= \lim_{\xi\longrightarrow\lambda}\left(\frac{\lambda-\xi}{\xi\cdot\lambda}\right)^k\cdot\xi\cdot(\xi\cdot I-T)^{-1}\\ &= \lambda^{1-2k}\cdot\lim_{\xi\longrightarrow\lambda}(\lambda-\xi)^k\cdot(\xi\cdot I-T)^{-1}, \qquad (1.1)\end{aligned}$$

where we used the substitution $\xi = z^{-1}$. The function $R(\xi; T) = (\xi\cdot I - T)^{-1}$ is known as the *resolvent function* of T, and is defined for ξ not in the spectrum of T. By the proof of Theorem 16 in [1, Section VII.3], see also the proof of Theorem 18, we have the Laurent expansion of $R(\xi; T)$ in the neighborhood $0 < |\xi-\lambda| < \epsilon$, is given by

$$R(\xi;T)=\sum_{j=-k}^{\infty}A_j\cdot(\lambda-\xi)^j,$$

where the operators A_j do not depend on the variable ξ. Hence the limit in Eq. (1.1) exists and is given by

$$\lim_{z\longrightarrow\lambda^{-1}}(z-\lambda^{-1})^k\cdot(I-z\cdot T)^{-1}=\lambda^{1-2k}\cdot A_{-k}.$$

That is, the limit

$$\lim_{z\longrightarrow\lambda^{-1}}(z-\lambda^{-1})^k\cdot\left((I-z\cdot T)^{-1}(\kappa),\mu\right)=\lambda^{1-2k}\cdot(A_{-k}(\kappa),\mu)$$

exists. Since this limit could be zero, we conclude that the analytic continuation has a pole at λ^{-1} of order at most k. □

2. Pattern avoidance

Elizalde and Noy [4,5] studied a refined version of consecutive pattern avoidance, by considering how many times patterns from the set S occur in the permutation π. Let $c_S(\pi)$ denote the number of times a pattern from the set S occurs consecutively in the permutation π, that is,

$$c_S(\pi)=|\{i:\Pi(\pi_i,\ldots,\pi_{i+m})\in S\}|.$$

Define the generating function $P(u,z)$ by

$$P(u,z)=\sum_{n\geq 0}\sum_{\pi\in\mathfrak{S}_n}u^{c_S(\pi)}\cdot z^n/n!.$$

Elizalde and Noy [4,5] asked if the function $\omega(u,z) = 1/P(u,z)$ is an entire function when the set S consists of a single permutation. They explicitly asked this question when $u = 0$; see [5, Question 7.2].

In order to give a partial answer to this question we introduce the weight function wt by

$$\mathrm{wt}(\sigma)=\begin{cases}1 & \text{if } \sigma\notin S,\\ u & \text{if } \sigma\in S,\end{cases}$$

and let wt_1 and wt_2 both be the constant 1.

Now the associated generating function is given by

$$\begin{aligned}P(u,z) &= \sum_{n\geq 0}\alpha_n\cdot z^n/n!\\ &= \sum_{n=0}^{m-1}z^n+Q(z),\end{aligned}$$

where we suppress the dependency of $Q(z)$ on the set S and the parameter u.

Proposition 2.1. *For a fixed parameter u, the only possible obstruction to* $\omega(u, z) = 1/P(u, z)$ *to be an entire function is if the function* $P(u, z)$ *has a zero.*

Proof. Note that the analytic continuations of the two generating functions $Q(z)$ and $P(u, z)$ share the same singularities. Furthermore, by Theorem 1.1 all the singularities of $Q(z)$ are poles, which correspond to zeros of $\omega(u, z)$. Hence the only possibility that $\omega(u, z)$ is not an entire function is when $P(u, z)$ has a zero. □

We also note that when the parameter u is real and positive the operator T is positivity improving. Kreĭn and Rutman [6, Theorem 6.3] showed that T has a positive real eigenvalue λ which is simple and λ is greater than all other eigenvalues in modulus. In this case, it follows that $\alpha_n = c \cdot \lambda^n + O(r^n)$, where c is a positive constant and r is bounded from above by λ and below by the modulus of the next largest eigenvalue. Hence $1/\lambda$ is a simple zero of $\omega(u, z) = 1/P(u, z)$ and it is the smallest zero in modulus. For more details; see [3].

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