Discrete Comput Geom OF1–OF12 (2004)
DOI: 10.1007/s00454-004-1135-1

Discrete & Computational
Geometry

Enumerative Properties of Ferrers Graphs*

Richard Ehrenborg[1] and Stephanie van Willigenburg[2]

[1]Department of Mathematics, University of Kentucky,
Lexington, KY 40506-0027, USA
jrge@ms.uky.edu

[2]Department of Mathematics, University of British Columbia,
1984 Mathematics Road, Vancouver, British Columbia, Canada V6T 1Z2
steph@math.ubc.ca

Also for the $3 \cdot 4 \cdot 5$th birthday of André Joyal

Abstract. We define a class of bipartite graphs that correspond naturally with Ferrers diagrams. We give expressions for the number of spanning trees, the number of Hamiltonian paths when applicable, the chromatic polynomial and the chromatic symmetric function. We show that the linear coefficient of the chromatic polynomial is given by the excedance set statistic.

1. Introduction and Preliminaries

Geometric and algebraic combinatorics span many areas, from the geometry of hyperplane arrangements [2], [6], through graph theory [3], [14], to the more algebraic permutation statistics [8], [11], [12]. An important aspect of all these areas is enumeration, which often illuminates the finer structure of the object under investigation, be it computing the faces of a polytope [1], [4], [5] or the distribution of permutations satisfying certain criteria [7].

In this paper we unite these facets of combinatorics via the study of Ferrers graphs, and in particular answer some of the more pertinent questions concerning enumeration. More precisely, we define a class of bipartite graphs that we call Ferrers graphs, so called since the edges in the graphs are in direct correspondence with the boxes in a Ferrers diagram. First, we calculate the number of spanning trees. The technique we use to prove this utilizes electrical networks. In fact, the first reference to spanning trees is in an article

* The first author was partially supported by National Science Foundation Grant 0200624. The second author was partially supported by the National Sciences and Engineering Research Council of Canada.

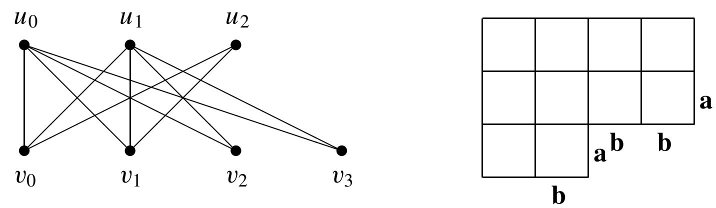

Fig. 1. The Ferrers graph and the Ferrers diagram associated with the partition $(4, 4, 2)$, the dual partition $(3, 3, 2, 2)$ and the **ab**-word **babba**.

by Kirchhoff [16], thus the study of trees and the study of electrical networks share their origin in the work of Kirchhoff. Second, when the two parts in the vertex partition have the same cardinality we determine the number of Hamiltonian paths in the Ferrers graph. This result is based upon the previous result and the proof is inspired by Joyal's proof of Cayley's formula. Third, and most mysterious, we prove that the linear coefficient of the chromatic polynomial of a Ferrers graph is given by the excedance set statistic of permutations. Lastly, we compute the chromatic symmetric function, thus generating a family of symmetric functions arising from Ferrers diagrams other than Schur functions. It should be noted that our Ferrers graphs are not those appearing in [13].

Definition 1.1. Define a *Ferrers graph* to be a bipartite graph on the vertex partition $U = \{u_0, \ldots, u_n\}$ and $V = \{v_0, \ldots, v_m\}$ such that

- if (u_i, v_j) is an edge, then so is (u_p, v_q) for $0 \le p \le i$ and $0 \le q \le j$,
- (u_0, v_m) and (u_n, v_0) are edges.

For a Ferrers graph G we have the associated partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, where λ_i is the degree of the vertex u_i. Similarly, we have the dual partition $\lambda' = (\lambda'_0, \lambda'_1, \ldots, \lambda'_m)$, where λ'_j is the degree of the vertex v_j. The associated Ferrers diagram is the diagram of boxes where we have a box in position (i, j) if and only if (u_i, v_j) is an edge in the Ferrers graph.

There is another natural way to index Ferrers graphs. Consider the Ferrers diagram associated with the graph. Walk along the path on the border of the Ferrers diagram starting at the lower right-hand corner of the box indexed by $(n, 0)$ and ending at the lower right-hand corner of the box indexed by $(0, m)$. Label a horizontal step by **b** and a vertical step by **a**. It is straightforward to see that the **ab**-words obtained this way are in one to one correspondence with Ferrers graphs. This is essentially the same encoding as in Exercise 7.59 in [19]. See Fig. 1 for an example of a Ferrers graph, its Ferrers diagram, partition, dual partition and **ab**-word.

2. The Number of Spanning Trees

For a spanning tree T of a Ferrers graph G define the weight $\sigma(T)$ to be

$$\sigma(T) = \prod_{p=0}^{n} x_p^{\deg_T(u_p)} \cdot \prod_{q=0}^{m} y_q^{\deg_T(v_q)}.$$

For a Ferrers graph G define $\Sigma(G)$ to be the sum $\Sigma(G) = \sum_T \sigma(T)$, where T ranges over all spanning trees T of the Ferrers graph G. Also let $\tau(G)$ denote the number of spanning trees of the graph G, that is, $\tau(G) = \Sigma(G)|_{x_0=\cdots=x_n=y_0=\cdots=y_m=1}$.

Theorem 2.1. *Let G be the Ferrers graph corresponding to the partition λ and the dual partition λ'. Then the sum of the weights of spanning trees T of the Ferrers graph G is given by*

$$\Sigma(G) = x_0 \cdots x_n \cdot y_0 \cdots y_m \cdot \prod_{p=1}^{n} (y_0 + \cdots + y_{\lambda_p - 1}) \cdot \prod_{q=1}^{m} (x_0 + \cdots + x_{\lambda'_q - 1}).$$

Hence the number of spanning trees of G is given by

$$\tau(G) = \prod_{p=1}^{n} \lambda_p \cdot \prod_{q=1}^{m} \lambda'_q.$$

Using the theory of electrical networks, originating with Kirchhoff [16] (for a more accessible reference see [9]), we can deduce the following:

Proposition 2.2. *Let H be a Ferrers graph and let G be the Ferrers graph obtained from H by adding the edge (u_i, v_j), where $i, j \geq 1$. Then the ratio between $\Sigma(G)$ and $\Sigma(H)$ is given by*

$$\frac{\Sigma(G)}{\Sigma(H)} = \frac{x_0 + \cdots + x_{i-1} + x_i}{x_0 + \cdots + x_{i-1}} \cdot \frac{y_0 + \cdots + y_{j-1} + y_j}{y_0 + \cdots + y_{j-1}}.$$

Proof. Let N be given by $(x_0 + \cdots + x_i) \cdot (y_0 + \cdots + y_j)$. View the Ferrers graph as an electrical network where the edge (u_p, v_q) is a resistor with resistance $R(u_p, v_q) = (x_p y_q)^{-1}$. Assign to each edge in the Ferrers graph G a current $w(u_p, v_q)$ by the following rule:

$$w(u_p, v_q) = \begin{cases} -x_p y_q / N & \text{if} \quad p < i, \quad q < j, \\ y_q \sum_{p=0}^{i-1} x_p / N & \text{if} \quad p = i, \quad q < j, \\ x_p \sum_{q=0}^{j-1} y_q / N & \text{if} \quad p < i, \quad q = j, \\ \left(x_i y_j + y_j \sum_{p=0}^{i-1} x_p + x_i \sum_{q=0}^{j-1} y_q \right) / N & \text{if} \quad p = i, \quad q = j, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, by Ohm's law we have the potential difference $P(u_p, v_q) = R(u_p, v_p) \cdot w(u_p, v_q)$. It is then straightforward to verify that $w(u_p, v_q)$ and $P(u_p, v_q)$ satisfy Kirchhoff's two laws when a current of size 1 enters the vertex u_i and leaves at v_j. Also observe that the vertices $u_0, \ldots, u_{i-1}$ have the same potential and hence no current goes through vertices $v_{j+1}, \ldots, v_m$. Similarly, there is no current through the vertices $u_{i+1}, \ldots, u_n$. Hence the current through the edge (u_i, v_j) is given by

$$\begin{aligned} w(u_i, v_j) &= \frac{x_i y_j + y_j \sum_{p=0}^{i-1} x_p + x_i \sum_{q=0}^{j-1} y_q}{N} \\ &= \frac{N - \left(\sum_{p=0}^{i-1} x_p\right) \cdot \left(\sum_{q=0}^{j-1} y_q\right)}{N}. \end{aligned}$$

However, the current through the edge (u_i, v_j) can also be determined by the theory of electrical networks so

$$w(u_i, v_j) = \frac{\sum_{(u_i,v_j)\in T} \prod_{e\in T} R(e)^{-1}}{\sum_T \prod_{e\in T} R(e)^{-1}} = \frac{\Sigma(G) - \Sigma(H)}{\Sigma(G)},$$

where the sum in the denominator is over all spanning trees T of the Ferrers graph G and the sum in the numerator is over all spanning trees containing the edge (u_i, v_j). By combining the last two identities the result follows. □

Proof of Theorem 2.1. The proof is by induction on the number of edges. The smallest Ferrers graph is the tree with $n + m + 1$ edges where (u_i, v_j) is an edge if and only if $i \cdot j = 0$. This tree has weight $x_0 \cdots x_n \cdot y_0 \cdots y_m \cdot x_0^m \cdot y_0^n$. The induction step adds one edge at a time, and the result follows from Proposition 2.2. □

As a corollary of Theorem 2.1 we obtain the classical result for the complete bipartite graphs. For the history and different approaches of this corollary, see Exercise 5.30 in [19].

Corollary 2.3. *For the complete bipartite graph $K_{n+1,m+1}$ the sum of the weights of spanning trees T is given by*

$$\Sigma(K_{n+1,m+1}) = x_0 \cdots x_n \cdot y_0 \cdots y_m \cdot (y_0 + \cdots + y_m)^n \cdot (x_0 + \cdots + x_n)^m.$$

Thus the number of spanning trees of $K_{n+1,m+1}$ is given by $\tau(K_{n+1,m+1}) = (m+1)^n \cdot (n+1)^m$.

3. The Number of Hamiltonian Paths

We now turn our attention to enumerating the number of Hamiltonian (open) paths in a Ferrers graph in the case when $n = m$, that is, when the two parts in the vertex partition of the bipartite graph have the same cardinality. Observe that for convenience we identify a Hamiltonian path with its reversal.

There are two important structures to consider. The first one is vertebrates:

Definition 3.1. Define a *vertebrate* (T, h, t) of a Ferrers graph as a spanning tree T together with one vertex h from the set U called the head and one vertex t from the set V called the tail. Call the set of vertices on the unique path from the head h to the tail t the *joints* of the vertebrate.

Since there are λ'_0 ways to choose a head and λ_0 ways to choose a tail, we have as a direct corollary to Theorem 2.1:

Corollary 3.2. *Let G be the Ferrers graph corresponding to the partition λ and the dual partition λ'. Then the number of vertebrates of the Ferrers graph G is given by*

$$\prod_{p=0}^{n} \lambda_p \cdot \prod_{q=0}^{m} \lambda'_q.$$

The other important structure we work with is permissible functions on the set $U \cup V$. We call a function $f\colon U \cup V \longrightarrow U \cup V$ *permissible* if, for all $z \in U \cup V$, $(z, f(z))$ is an edge in the associated Ferrers graph. Observe that the product in Corollary 3.2 also enumerates the number of permissible functions on the Ferrers graph G.

For a function f let f^k denote the kth power of the function under composition, that is, $f^k = f \circ \cdots \circ f$. For a permissible function f call the set $E(f) = \bigcap_{k \geq 1} \mathrm{Im}(f^k)$ the essential set of the function f. Observe that f restricts to a permissible permutation on the set $E(f)$. Moreover, the essential set $E(f)$ intersects the sets U and V in equally large subsets.

Using similar ideas of Joyal [15] we are able to prove for Ferrers graphs:

Theorem 3.3. *Let G be a Ferrers graph with $n = m$, that is, each of the two parts in the vertex partition have the same cardinality. Then the number of Hamiltonian paths in G is equal to the square of the number of placements of $n+1$ rooks on the associated Ferrers board.*

Observe that the number of rook placements on a Ferrers board with $n + 1$ rooks is $\lambda_n \cdot (\lambda_{n-1} - 1) \cdots (\lambda_0 - n)$, where λ is the associated partition. Similarly, this is also equal to $\lambda'_n \cdot (\lambda'_{n-1} - 1) \cdots (\lambda'_0 - n)$, where λ' is the dual partition.

Proof of Theorem 3.3. First observe that the number of rook placements squared is equal to the number of permissible bijections π on the Ferrers graph G.

The proof of the statement is by induction on n. The induction basis is $n = 0$ which is straightforward. Now the induction step.

Let S be a proper subset of $U \cup V$ such that $S \cap U$ and $S \cap V$ have equal size. We claim that the number of vertebrates of the Ferrers graph G with the joints being the set S is equal to the number of permissible functions on G having essential set S. By the induction hypothesis we know that the number of Hamiltonian paths on G restricted to the set S is equal to the number of permissible permutations on the set S. Now, a vertebrate is a path such that each vertex in the path is the root of a tree. Similarly a function is a permutation such that each entry in the permutation is the "root" of a "tree". For instance, for a root s in the essential set S of a permissible function f the tree is the collection of vertices z such that $f^k(z) \in S$ implies there exists $i \leq k$ such that $f^i(z) = s$ but $f^j(z) \notin S$ for $j < i$. Hence the claim follows by changing a path on the set S to a permissible permutation on the set S.

Now by summing over all S strictly contained in $U \cup V$ we have that the number of vertebrates that are not paths is equal to the number of permissible functions that are not permutations. Since the cardinalities of vertebrates and permissible functions are the same we are done. □

4. The Chromatic Polynomial and the Linear Coefficient

Before we embark on deriving the chromatic polynomial we recall the excedance set statistic. It was first studied in [11] and [12]. We follow their notation and instead of speaking of the excedance set, we talk about the excedance word.

Define *the excedance word* of a permutation $\pi = \pi_1 \cdots \pi_{k+1}$ in S_{k+1} to be the word $w = w_1 \cdots w_k$ where $w_i = \mathbf{a}$ if $\pi_i \leq i$ and $w_i = \mathbf{b}$ if $\pi_i > i$. For an **ab**-word w of length k let $[w]$ denote the number of permutations in S_{k+1} with excedance word w.

Following [11] let $R_m = \{\mathbf{r} = (r_0, \ldots, r_m) \, : \, r_0 = 1, r_{i+1} - r_i \in \{0, 1\}\}$. Thus, each vector $\mathbf{r} = (r_0, \ldots, r_m)$ in R_m starts with $r_0 = 1$ and increases by at most one at each coordinate. Let $h(\mathbf{r})$ be the number of indices i such that $r_{i+1} = r_i$. We then have the following result; see Theorem 6.3 of [11].

Theorem 4.1. *Let w be an* **ab**-*word with exactly m* **b**'*s. That is, we can write* $w = \mathbf{a}^{n_0}\mathbf{b}\mathbf{a}^{n_1}\mathbf{b} \cdots \mathbf{b}\mathbf{a}^{n_m}$. *Then the excedance set statistic $[w]$ is given by*

$$[w] = \sum_{\mathbf{r} \in R_m} (-1)^{h(\mathbf{r})} \cdot r_0^{n_0+1} \cdot r_1^{n_1+1} \cdots r_m^{n_m+1}.$$

For an **ab**-word w, let $\chi(w)$ denote the chromatic polynomial in t of the Ferrers graph G associated with w. Moreover, let $|w|$ denote the length of the **ab**-word w. Now we can state the relationship between the linear coefficient of the chromatic polynomial and the excedance set statistic.

Theorem 4.2. *The linear coefficient of the chromatic polynomial $\chi(w)$ is given by* $(-1)^{|w|+1} \cdot [w]$.

It is straightforward to observe that $\chi(\mathbf{a}w) = \chi(w\mathbf{b}) = (t-1) \cdot \chi(w)$ and $\chi(1) = t \cdot (t-1)$, where the 1 in $\chi(1)$ denotes the empty word.

For a vector $\mathbf{r}$ in the set R_m and $1 \leq i \leq m$ define $f_i(\mathbf{r}) = f_i$ by $f_i = t - r_{i-1}$ if $r_i - r_{i-1} = 1$ and $f_i = r_{i-1}$ otherwise.

Theorem 4.3. *Let w be an* **ab**-*word with exactly m* **b**'*s, that is,* $w = \mathbf{a}^{n_0}\mathbf{b}\mathbf{a}^{n_1}\mathbf{b} \cdots \mathbf{b}\mathbf{a}^{n_m}$. *Then the chromatic polynomial $\chi(w)$ of the associated Ferrers graph G is given by*

$$\chi(w) = \sum_{\mathbf{r} \in R_m} t \cdot (t - r_0)^{n_0} \cdot f_1 \cdot (t - r_1)^{n_1} \cdot f_2 \cdots f_{m-1} \cdot (t - r_{m-1})^{n_{m-1}} \cdot f_m \cdot (t - r_m)^{n_m+1}.$$

Proof. For a proper coloring of the graph G let r_i be the number of distinct colors appearing on the $i+1$ nodes v_0 through v_i. Let us determine how many colorings there are of the graph with a given vector $\mathbf{r} = (r_0, \ldots, r_m)$.

The node v_0 can be colored in t ways. If $r_i - r_{i-1} = 1$, then the node v_i is colored with a color not used before, and there are $t - r_{i-1}$ such colors. If $r_{i+1} - r_i = 0$, then the node is colored with an "old" color, and there are r_{i-1} such colors. In both cases we have f_i possibilities.

For $i \leq m-1$ observe that there are n_i u-nodes that are connected exactly to the nodes $v_0, \ldots, v_i$. There are $(t - r_i)^{n_i}$ ways to color these n_i nodes, since they all have to avoid the r_i colors of the nodes $v_0, \ldots, v_i$. Finally, there are $n_m + 1$ u-nodes that are connected to all the v-nodes $v_0, \ldots, v_m$. Similarly, there are $(t - r_m)^{n_m+1}$ ways to color these nodes. Hence there are

$$t \cdot f_1 \cdot f_2 \cdots f_m \cdot (t - r_0)^{n_0} \cdots (t - r_{m-1})^{n_{m-1}} \cdot (t - r_m)^{n_m+1}$$

ways to color the graph G with a given $\mathbf{r}$-vector. Now summing over all possible $\mathbf{r}$-vectors the result follows. □

We now prove the main result:

Proof of Theorem 4.2. To obtain the linear coefficient in $\chi(w)$ divide by t and set $t = 0$. Observe that f_i evaluated at $t = 0$ is equal to r_{i-1} with a sign change if $r_i - r_{i-1} = 1$. The number of such sign changes is $m - h(\mathbf{r})$. Moreover, we also obtain $n_0 + n_1 + \cdots + n_m + 1$ sign changes from the other factors. Hence the total number of sign changes is $m - h(\mathbf{r}) + n_0 + n_1 + \cdots + n_m + 1 = |w| - h(\mathbf{r}) + 1$.

The remainder of the term corresponding to $\mathbf{r}$ can now be written as $r_0^{n_0+1} \cdot r_1^{n_1+1} \cdots r_m^{n_m+1}$, and the result follows by Theorem 4.1. □

There is one important special case of Theorem 4.3:

Proposition 4.4. *The chromatic polynomial of the complete bipartite graph $K_{n+1,m+1}$ is given by*

$$\chi(\mathbf{b}^m \mathbf{a}^n) = \sum_{k=1}^{m+1} S(m+1, k) \cdot t \cdot (t-1) \cdots (t-k+1) \cdot (t-k)^{n+1},$$

where $S(m, k)$ denotes the Stirling number of the second kind.

Proof. Begin to color the vertices $v_0, v_1, \ldots, v_m$ with exactly k colors where $1 \le k \le m + 1$. This can be done in $S(m + 1, k) \cdot t \cdot (t - 1) \cdots (t - k + 1)$ ways. There are $(t - k)^{n+1}$ ways to color the remaining vertices $u_0, u_1, \ldots, u_n$. □

The linear coefficient of the chromatic polynomial (up to a sign) also has the interpretation of being the number of acyclic orientations of the graph with a *unique* given sink [14]. Also observe that it is enough to note that there are no directed 4-cycles in an orientation of the edges in a Ferrers graph to guarantee that the orientation is acyclic. Expressing this in terms of the associated Ferrers diagram we have:

Corollary 4.5. *The excedance set statistic $[w]$ is the number colorings of the boxes in the Ferrers diagram associated to the* **ab**-*word w with colors red and blue such that*

(i) *there are no four boxes (p, r), (p, s), (q, r), (q, s) such that (p, r) and (q, s) are colored red and (p, s) and (q, r) are colored blue,*
(ii) *there is a unique given row where all the boxes are colored red, and*
(iii) *there is no column where all the boxes are colored blue.*

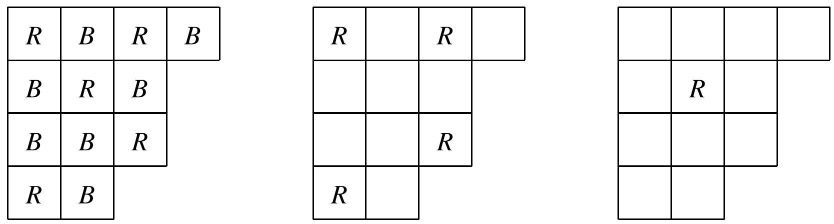

Fig. 2. A Ferrers diagram with colored boxes, and its constituents.

5. The Chromatic Symmetric Function

A natural generalization of the chromatic polynomial, known as the *chromatic symmetric function* was defined in [18], and it is natural to ask whether we can explicitly compute these for Ferrers graphs. This would give us a set of symmetric functions other than the Schur functions that can be computed from Ferrers diagrams.

Observe that unlike the Schur functions, the chromatic symmetric functions of Ferrers graphs will not form a basis for the symmetric functions as the chromatic symmetric function of the Ferrers graph corresponding to the partition λ and λ' will be identical.

Before we continue we need to define the constitution of a Ferrers diagram whose boxes have been colored red and blue. First choose a red box. Score through that row and column. For every red box with a score going through it in one direction score through it in the other direction. Repeat until all the red boxes either have two scores or no scores through them. Extract all the boxes with two scores in them. Choose another red box, and repeat until none remain. The list of extractions is the *constitution* and each extraction is called a *constituent* (see Fig. 2).

In addition, let RB_λ be the set of all red–blue colorings of the Ferrers diagram corresponding to the partition λ (without the restriction of Corollary 4.5). For $r \in RB_\lambda$ let $|r|$ be the number of constituents of r and let $|r|_{\text{red}}$ be the number of boxes in r colored red.

Theorem 5.1. *Let G be the Ferrers graph corresponding to the partition λ. Then the chromatic symmetric function X_G in terms of the power sum symmetric functions p_μ is given by*

$$X_G = \sum_{r \in RB_\lambda} (-1)^{|r|_{\text{red}}} \cdot p_{r_1} \cdot p_{r_2} \cdots p_{r_{|r|}} \cdot p_1^b,$$

*where r_i is the number of rows plus the number of columns in the i*th *constituent of r, $1 \le i \le |r|$, and b is the number of rows plus the number of columns of r that contain no red boxes.*

Proof. Recall that for a graph G with a set of edges E the definition of the chromatic symmetric function in terms of the power sum basis is [18, Theorem 2.5]

$$X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{|C_0|} \cdots p_{|C_m|},$$

where $|C_i|$ is the number of vertices in each connected component C_i, $0 \leq i \leq m$, of G with the edges not in S removed.

Now observe that for a Ferrers graph G with edge set E there is an natural bijection between $S \subseteq E$ and red–blue colorings $r \in RB_\lambda$ of the Ferrers diagram associated with λ, given by

$$(u_i, v_j) \in S \Leftrightarrow (i, j) \text{ is colored red in } r.$$

This gives us the index of summation and the exponent of -1 in our formula. To complete the proof note the constituents of r yield precisely the connected components of G containing more than one vertex, and if the ith row (column) of r contains only blue boxes, then u_i (v_i) is not connected to any other vertex in G. □

A more specific formula can be found for the two extreme cases of Ferrers graphs. First the case when the Ferrers graph is a tree.

Corollary 5.2. *Let G be the Ferrers graph corresponding to the partition $(m+1)1^n$. Then the chromatic symmetric function X_G in terms of the power sum symmetric functions p_μ is given by*

$$X_G = \sum_{i=0}^{m+n} (-1)^i \left(\left(\sum_{j+k=i} \binom{m}{j} \binom{n}{k} p_{j+1} p_{k+1} p_1^{m+n-i} \right) - \binom{m+n}{i} p_{i+2} p_1^{m+n-i} \right).$$

Proof. Observe that in the case where the Ferrers diagram associated with λ is a hook, for $r \in RB_\lambda$ if (0, 0) is blue, then we obtain the function

$$\sum_{i=0}^{m+n} (-1)^i \sum_{j+k=i} \binom{m}{j} \binom{n}{k} p_{j+1} p_{k+1} p_1^{m+n-i},$$

whereas if it is red, then we obtain the function

$$\sum_{i=1}^{m+n+1} (-1)^i \binom{m+n}{i-1} p_{i+1} p_1^{m+n+1-i}.$$ □

The other extreme case is the complete bipartite graph $K_{n,m}$, which is the Ferrers graph associated with the partition m^n. A change of basis is required for the simplest description of the chromatic symmetric function.

Corollary 5.3. *The chromatic symmetric function $X_{K_{n,m}}$ in terms of the monomial symmetric functions m_μ is given by*

$$X_{K_{n,m}} = \sum_{\sigma \in \Pi_n} \sum_{\tau \in \Pi_m} (r_1! \, r_2! \, \cdots) \cdot m_{\mu(\sigma,\tau)},$$

where Π_n is the collection of all set partitions of $\{1, \ldots, n\}$, $\mu(\sigma, \tau)$ is the partition determined by the block sizes of σ and τ, and r_i is the multiplicity of i in $\mu(\sigma, \tau)$.

Proof. Recall that a stable partition of the vertices of a graph G is a partition of the vertices such that each block is totally disconnected. Then Proposition 2.4 in [18] states

$$X_G = \sum_{\pi} (r_1!\, r_2! \cdots) m_{\mu(\pi)},$$

where the sum ranges over all stable partitions π of the graph G. The result follows by noting that in the complete bipartite graph $K_{n,m}$, every block in a stable partition either lies entirely in the n vertices $\{u_0, \ldots, u_{n-1}\}$ or lies entirely in the m vertices $\{v_0, \ldots, v_{m-1}\}$. □

The symmetric functions appearing in Corollary 5.3 have the following explicit exponential generating function, generalizing Exercise 5.6 in [19]:

$$\sum_{n,m \geq 0} X_{K_{n,m}} \frac{s^n}{n!} \frac{t^m}{m!} = \prod_{i \geq 1} (e^{s x_i} + e^{t x_i} - 1),$$

where we view the symmetric functions in terms of the variables $\{x_i\}_{i \geq 1}$.

Lastly, note that to recover the earlier chromatic polynomial we set $x_1 = \cdots = x_t = 1$ and all other $x_i = 0$.

6. Concluding Remarks

Is it possible to obtain an expression for the Tutte polynomial of a Ferrers graph, that would both encode the number of spanning trees in Theorem 2.1 and the chromatic polynomial in Theorem 4.3? For the enumerative results in this paper it is natural to ask for combinatorial proofs. From a bijection given in [17], a bijective proof for Theorem 2.1 can be obtained via some modifications. In [10] bijective proofs for Theorems 2.1 and 3.3 have been derived using box labeling. However, it would also be desirable to have a bijective proof for Corollary 4.5.

The excedance set statistic $[w]$ satisfies the recursion $[u\mathbf{ba}v] = [u\mathbf{ab}v] + [u\mathbf{a}v] + [u\mathbf{b}v]$ where u and v are two **ab**-words. Is there a similar recursion for the chromatic polynomial? A partial answer to this question is the following proposition, whose proof we omit.

Proposition 6.1. *The chromatic polynomial* $\chi(w)$ *of the associated Ferrers graph satisfies the recursion*:

$$\chi(w\mathbf{ba}^{k-1}) = t \cdot \chi(w\mathbf{a}^{k-1}) + \sum_{0 \leq i \leq k-1} (-1)^{k-i} \cdot \binom{k}{i} \cdot \chi(w\mathbf{a}^i).$$

On the excedance statistic level this recursion corresponds to $[w\mathbf{ba}^{k-1}] = \sum_{0 \leq i \leq k-1} \binom{k}{i} \cdot [w\mathbf{a}^i]$; see Proposition 2.5 of [11]. Moreover, can this proposition be extended to the chromatic symmetric function?

Another question related to the chromatic polynomial arises from the following observation. A Ferrers graph G can be equivalently viewed as an $(n+m+2)$-dimensional hyperplane arrangement given by

$$x_i = y_j \qquad \text{if and only if } (u_i, v_j) \text{ is an edge in } G.$$

Thus the chromatic polynomial of the Ferrers graph G is also the characteristic polynomial of the associated hyperplane arrangement, see [14]. Hence, can a combinatorial expression be found for the number of acyclic orientations of the Ferrers graph, or equivalently for the number of regions of the associated hyperplane arrangement?

Finally, one can define the Ferrers graph associated with a skew partition λ/μ. Do any of the results in this paper extend naturally to skew partitions?

Acknowledgments

The authors thank Margaret Readdy, Tom Zaslavsky and the two referees for their comments on earlier drafts of this paper.

References

1. M. M. Bayer and L. J. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.
2. L. J. Billera, K. S. Brown and P. Diaconis, Random walks and plane arrangements in three dimensions, *Amer. Math. Monthly* **6** (1999), 502–524.
3. L. J. Billera, C. S. Chan and N. Liu, Flag complexes, labelled rooted trees, and star shellings, *Contemp. Math.* **223** (1999), 91–102.
4. L. J. Billera and R. Ehrenborg, Monotonicity of the **cd**-index for polytopes, *Math. Z.* **233** (2000), 421–441.
5. L. J. Billera, R. Ehrenborg and M. Readdy, The **c**-2**d**-index of oriented matroids, *J. Combin. Theory Ser. A* **80** (1997), 79–105.
6. L. J. Billera, P. Filliman and B. Sturmfels, Constructions and complexity of secondary polytopes, *Adv. Math.* **83** (1990), 155–179.
7. L. J. Billera, S. K. Hsiao and S. van Willigenburg, Peak quasisymmetric functions and Eulerian enumeration, *Adv. Math.* **176** (2003), 248–276.
8. L. J. Billera and A. Sarangarajan, The combinatorics of permutation polytopes, *Discrete Math. Theoret. Comput. Sci.* **24** (1996), 1–23.
9. B. Bollobás, *Modern Graph Theory*, Springer-Verlag, New York, 1998.
10. J. Burns, Bijective proofs for "Enumerative properties of Ferrers graphs", Preprint 2003, 6pp, `http://arXiv.org/abs/math.CO/0312282`.
11. R. Ehrenborg and E. Steingrímsson, The excedance set of a permutation, *Adv. in Appl. Math.* **24** (2000), 284–299.
12. R. Ehrenborg and E. Steingrímsson, Yet another triangle for the Genocchi numbers, *European J. Combin.* **21** (2000), 593–600.
13. A. Frumkin, G. James and Y. Roichman, On trees and characters, *J. Algebraic Combin.* **17** (2003), 323–334.
14. C. Greene and T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, *Trans. Amer. Math. Soc.* **280** (1983), 97–126.
15. A. Joyal, Une théorie combinatoire des séries formelles, *Adv. Math.* **42** (1981), 1–82.
16. G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, *Ann. Phys. Chem.* **72** (1847), 497–508.

17. J. B. Remmel and S. G. Williamson, Spanning trees and function classes, *Electron. J. Combin.* **9** (2002), R34, 24pp.
18. R. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* **111** (1995), 166–194.
19. R. Stanley, *Enumerative Combinatorics*, Vol. II, Cambridge University Press, Cambridge, 1999.

Received May 12, 2003, *and in revised form February* 24, 2004. *Online publication August* 24, 2004.