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A Bijective Proof of Infinite Variated Good's Inversion

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We use the theory of colored species to prove the plethystic Lagrange inversion formula and infinite variated Good's inversion formula. These inversion formulas are shown to be equivalent to transfer formulas in the infinite variated umbral calculus. © 1994 Academic Press, Inc.

1. INTRODUCTION

Recently, two similar generalizations of Joyal's theory of species in several variables were independently introduced. They both develop set-theoretical interpretations of operations with formal power series in an infinite number of variables. One of them is the theory of compositionals, introduced by Chen [C], which interprets the operations of formal power series in the variables $x_1, x_2, x_3, \ldots$, in terms of operations with compositions. A composition is a vector with an infinite number of entries, where each entry is a finite set. Polya's plethystic composition, and its set-theoretical interpretation, is obtained as particular instance of the operation of substitution of a summable family of power series into a power series. Chen developed a plethystic umbral calculas in this combinatorial setting.

The other theory, introduced by Mendez and Nava [M-N], is more general. It studies formal power series in the variables $\{x_i\}_{i \in I}$, where the elements of the index set I are called colors. The theory explains operations of these formal power series in terms of operations with colored sets.

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A colored set is a finite set, where each element has a color from the set I assigned to it. A general operation of substitution with power series is defined. Endowing the set I with a structure of a c-monoid, which is defined below, one can introduce a general notion of plethysm, and its combinatorial interpretation. Polya's plethysm is a special case of the general plethysm. To see this fact, let the c-monoid be positive integers with multiplication.

Using this framework we generalize Good's multivariated inversion formula [Go] to the infinite variated case, Theorem 3. Our proof is based on the construction of an involution over a set of colored functions enriched with a system of colored species. Previously Gessel presented a bijective proof of a Good's original multivariate formula [Ge].

A Lagrange inversion formula, Theorem 1, for the generalized plethysm can be obtained as a particular case of our generalization of Good's inversion formula, Theorem 3. A special case of this Lagrange inversion formula for Polya's plethysm was previously proved by Labelle [Lab2] using a procedure called lifting. However, we present a very simple combinatorial proof of the general plethystic Lagrange inversion formula, as it is of independent combinatorial interest. This proof is based on the enumeration of plethystic enriched forests using inclusion–exclusion over sets of plethystic enriched functions. The proof resembles Labelle's proof of the Lagrange inversion formula in one variable [Lab1].

We believe that Chen's plethystic umbral calculus can be completely restated in the more general context of colorations and the generalized plethysm. In particular, by restating Chen's plethystic transfer formula we prove in Theorem 7 the equivalence between the plethystic transfer formula and the plethystic Lagrange inversion formula. In a similar way we prove in Theorem 5 a transfer formula which is equivalent to our generalized version of Good's inversion formula.

Recently Gessel and Labelle have proved the plethystic Lagrange inversion formula for Polya's plethysm using ordinary species [Ge-Lab]. Their method uses the cycle indicator series of a species.

We are pleased to acknowledge Gian-Carlo Rota who brought us together and inspired us in the the pursuit of studying combinatorics.

2. Formal Power Series and Colored Sets

Let $\mathscr{T}$ be a set, possible infinite.

Definition 2.1. A *multi index* $\mathbf{n}$ is a vector whose entries are non-negative integers whose sum is finite and the entries are indexed by $\mathscr{T}$. That is $\mathbf{n}=(n_i)_{i\in\mathcal{T}}$, where $n_i\in\mathbb{N}$. Define the base multi-indices by $\mathbf{e}_j=(\delta_{j,i})_{i\in\mathcal{T}}$ for $j\in\mathcal{T}$.

Note that any multi-index $\mathbf{n}$ can be written as a linear combination of the base multi-indices

$$\mathbf{n}=\sum_{i\in\mathcal{T}} n_i\cdot\mathbf{e}_i,$$

where there are only a finite number of nonzero terms in the sum. For a finite subset $I\subseteq\mathcal{T}$, define

$$\mathbf{e}_I=\sum_{i\in I}\mathbf{e}_i.$$

Let $\mathcal{K}$ be a field of characteristic 0. We consider formal power series in the variables $(x_i)_{i\in\mathcal{T}}$. A power series is denoted with $f(\mathbf{x})$. To be able to write readable formulas we introduce the following notations:

$$\mathbf{n}=(n_i)_{i\in\mathcal{T}}$$

$$\mathbf{n}!=\prod_{i\in\mathcal{T}} n_i!$$

$$\mathbf{x}=(x_i)_{i\in\mathcal{T}}$$

$$\mathbf{x}^{\mathbf{n}}=\prod_{i\in\mathcal{T}} x_i^{n_i}$$

$$\binom{\mathbf{n}}{\mathbf{k}}=\prod_{i\in\mathcal{T}}\binom{n_i}{k_i}$$

$$(\mathbf{n})_{\mathbf{k}}=\prod_{i\in\mathcal{T}}(n_i)_{k_i}.$$

Thus a formal power series can be written as an exponential series

$$f(\mathbf{x})=\sum_{\mathbf{n}} a_{\mathbf{n}}\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}.$$

The sum of two power series, $f(\mathbf{x})=\sum_{\mathbf{n}} a_{\mathbf{n}}(\mathbf{x}^{\mathbf{n}}/\mathbf{n}!)$ and $g(\mathbf{x})=\sum_{\mathbf{n}} b_{\mathbf{n}}(\mathbf{x}^{\mathbf{n}}/\mathbf{n}!)$, is defined componentwise. That is

$$(f+g)(\mathbf{x})=\sum_{\mathbf{n}}(a_{\mathbf{n}}+b_{\mathbf{n}})\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}.$$

The product of two power series $f(\mathbf{x})$ and $g(\mathbf{x})$ is

$$(f \cdot g)(\mathbf{x}) = \sum_{\mathbf{n}} c_{\mathbf{n}} \frac{x^{\mathbf{n}}}{\mathbf{n}!},$$

where

$$c_{\mathbf{n}} = \sum_{0 \leqslant \mathbf{k} \leqslant \mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}} a_{\mathbf{k}} b_{\mathbf{n}-\mathbf{k}}.$$

DEFINITION 2.2. A collection power series $\mathbf{g}(\mathbf{x})$ is a set of formal power series indexed by the set $\mathscr{T}$. That is

$$\mathbf{g}(\mathbf{x}) = (g_i(\mathbf{x}))_{i \in \mathscr{T}}.$$

A summable collection $\mathbf{g}(\mathbf{x})$ is a collection such that for every multi-index $\mathbf{n}$ the coefficient $[\mathbf{x}^{\mathbf{n}}/\mathbf{n}!]\, g_i(\mathbf{x})$ is nonzero for only a finite number $i \in \mathscr{T}$.

DEFINITION 2.3. Let $f(\mathbf{x})$ be a formal power series and $\mathbf{g}(\mathbf{x})$ be a summable collection formal power series, such that $g_i(\mathbf{x})$ has no constant coefficient. Define the composition $f \circ \mathbf{g}$ as

$$(f \circ \mathbf{g})(\mathbf{x}) = f((g_i(\mathbf{x}))_{i \in \mathscr{T}}).$$

Observe that $\mathbf{x}^{\mathbf{n}} \circ \mathbf{g} = \prod_{i \in \mathscr{T}} g_i(\mathbf{x})^{n_i}$. We also write this expression as $\mathbf{g}^{\mathbf{n}}$.

DEFINITION 2.4. A colored set (E, f) is a set E with a function $f\colon E \to \mathscr{T}$. The color of an element $a \in E$ is the value $f(a)$. The colored set (E, f) is finite if E is a finite set. The cardinality of a finite colored set (E, f) is a multi-index $\operatorname{card}(E, f) = \mathbf{n}$ such that

$$n_i = |\{a \in E : f(a) = i\}|.$$

If $\operatorname{card}(E, f) = \mathbf{n}$ we say that (E, f) is an $\mathbf{n}$ set. When we need to speak about a generic colored set of cardinatlity $\mathbf{n}$, we write $\mathbf{n}$ for this generic set.

3. Colored Species

We introduce now the theory of colored species. This theory was developed in [M-N]. We only give a short sketch of definitions and main results. The reader interested in this subject is referred to [M-N].

Let $\mathbb{B}$ be the category of finite sets and bijections. Recall that a *species* is a functor from $\mathbb{B}$ to $\mathbb{B}$. Similarly we can define the category of colored sets.

DEFINITION 3.1. Let $\mathbb{B}_{\mathscr{T}}$ be the category of finite colored sets and bijections, which preserve color. A colored species M is functor from $\mathbb{B}_{\mathscr{T}}$ to $\mathbb{B}$.

Define the generating function of colored species M to be

$$\operatorname{card}(M; \mathbf{x}) = \sum_{\mathbf{n}} |M[\mathbf{n}]| \cdot \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!},$$

where $M[\mathbf{n}]$ is the species applied to a generic $\mathbf{n}$ set.

For $i \in \mathscr{T}$ define the colored species X_i by

$$X_i[(E, f)] = \begin{cases} \{E\} & \text{if} \quad \operatorname{card}(E, f) = \mathbf{e}_i \\ \varnothing & \text{otherwise} \end{cases}.$$

Observe that $\operatorname{card}(X_i; \mathbf{x}) = x_i$.

Define sum and product of two colored species by

$$(M + N)[(E, f)] = M[(E, f)] \cup N[(E, f)]$$
$$(M \cdot N)[(E, f)] = \dot{\bigcup_{E_1 + E_2 = E}} M[(E_1, f|_{E_1})] \times N[(E_2, f|_{E_2})].$$

For a colored species M and $i \in \mathscr{T}$ define the colored species $M'^{(i)}$ by

$$M'^{(i)}[(E, f)] = M[(E \cup \{*\}, f)],$$

where $*$ is a ghost element of color i. That is, we extend f such that $f(*) = i$. Moreover let

$$M^{\bullet(i)} = X_i \cdot M'^{(i)}.$$

That means that we mark an element of color i in the underlying set.

A *colored partition* of a colored set (E, f) is a partition π of the set E, with function $g: \pi \to \mathscr{T}$. Let $\Pi[(E, f)]$ be the set of all colored partitions of (E, f).

DEFINITION 3.2. A collection of colored species $\vec{\mathbf{M}}$ is a set of colored species indexed by the set $\mathscr{T}$. That is,

$$\vec{\mathbf{M}} = (M_i)_{i \in \mathscr{T}}.$$

A *summable* collection $\vec{\mathbf{M}}$ is a collection such that for every colored set (E, f) the set $M_i[(E, f)]$ is nonempty for only a finite number of $i \in \mathscr{T}$.

Observe that if $\vec{\mathbf{M}}$ is a summable collection of colored species; then $(\operatorname{card}(M_i; \mathbf{x}))_{i \in \mathscr{T}}$ is a summable collection of power series.

Let M be a colored species and let $\vec{\mathbf{N}}$ be a summable collection of colored species, such that $N_i[\varnothing]=\varnothing$ for all $i \in \mathscr{T}$. Define the divided power $\Gamma_{\mathbf{k}}(\vec{\mathbf{N}})$ as

$$(\Gamma_{\mathbf{k}}(\vec{\mathbf{N}}))[(E,f)] = \dot{\bigcup_{(\pi,g)\in \Pi[(E,f)],\ \operatorname{card}(\pi,g)=\mathbf{k}}} \prod_{B\in\pi} N_{g(B)}[(B, f|_B)].$$

Define the composition $M \circ \vec{\mathbf{N}}$ by

$$(M\circ\vec{\mathbf{N}})[(E,f)] = \dot{\bigcup_{(\pi,g)\in \Pi[(E,f)]}} M[(\pi,g)] \times \prod_{B\in\pi} N_{g(B)}[(B, f|_B)].$$

Observe that the summability condition of the collection $\vec{\mathbf{N}}$ implies that the two sets $(\Gamma_{\mathbf{k}}(\vec{\mathbf{N}}))[(E,f)]$ and $(M\circ\vec{\mathbf{N}})[(E,f)]$ are finite.

Note the following identity:

$$\prod_{i\in\mathscr{T}} X_i^{n_i} \circ \vec{\mathbf{N}} = \prod_{i\in\mathscr{T}} N_i^{n_i}.$$

We write the above expression as $\vec{\mathbf{N}}^{\mathbf{n}}$.

Now we can show the relationship between operations on colored species and operations on the respective generating functions.

PROPOSITION 3.1.

$$\operatorname{card}(M+N;\mathbf{x}) = \operatorname{card}(M;\mathbf{x}) + \operatorname{card}(N;\mathbf{x})$$

$$\operatorname{card}(M\cdot N;\mathbf{x}) = \operatorname{card}(M;\mathbf{x}) \cdot \operatorname{card}(N;\mathbf{x})$$

$$\operatorname{card}(M'^{(i)};\mathbf{x}) = \frac{\partial}{\partial x_i}\operatorname{card}(M;\mathbf{x})$$

$$\operatorname{card}(M^{\bullet(i)};\mathbf{x}) = x_i\frac{\partial}{\partial x_i}\operatorname{card}(M;\mathbf{x})$$

$$\operatorname{card}((M\circ\vec{\mathbf{N}};\mathbf{x}) = (\operatorname{card}(M;\mathbf{x})\circ((\operatorname{card}(N_i;\mathbf{x}))_{i\in\mathscr{T}}))(\mathbf{x})$$

$$\operatorname{card}(\Gamma_{\mathbf{k}}(\vec{\mathbf{N}});\mathbf{x}) = \left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}\circ((\operatorname{card}(N_i;\mathbf{x}))_{i\in\mathscr{T}})\right)(\mathbf{x})$$

$$= \frac{1}{\mathbf{k}!}\prod_{i\in\mathscr{T}} \operatorname{card}(N_i;\mathbf{x})^{k_i}.$$

4. Colored Functions and Colored Trees

Let $\vec{\mathbf{M}} = (M_i)_{i \in \mathcal{I}}$ be a collection species, not necessarily a summable collection.

Definition 4.1. An $\vec{\mathbf{M}}$-enriched function ϕ from (E, f) to (F, g) is a function ϕ from E to F such that for all $b \in F$ the colored set

$$(\{a \in E \mid \phi(a) = b\}, f) = (\phi^{-1}(b), f)$$

is enriched with an $M_{g(b)}$ structure.

Lemma 4.1. *Let (F, g) be a fixed colored set. The set of structures defined on the colored set* (E, f) *by*

$$\left(\prod_{b \in F} M_{g(b)}\right)[(E, f)]$$

is isomorphic to the set of $\vec{\mathbf{M}}$-enriched functions from (E, f) to (F, g).

Definition 4.2. A $\vec{\mathbf{M}}$-*enriched tree* (*forest*) on an $\mathbf{n}$ set (E, f) is a tree (forest) on E such that for every node $a \in E$ we put an $M_{f(a)}$ structure on its set of sons.

Let $A^{(i)}_{\vec{\mathbf{M}}}$ be the colored species of $\vec{\mathbf{M}}$-enriched colored trees with the root of color i. Let $\vec{\mathbf{A}}_{\vec{\mathbf{M}}}$ be the collection $(A^{(i)}_{\vec{\mathbf{M}}})_{i \in \mathcal{I}}$.

Proposition 4.2. *The collection $\vec{\mathbf{A}}_{\vec{\mathbf{M}}}$ is a summable collection of colored species and the colored species $A^{(i)}_{M}$ fulfills the functional equation*

$$A^{(i)}_{\vec{\mathbf{M}}} = X_i \cdot (M_i \circ \vec{\mathbf{A}}_{\vec{\mathbf{M}}}).$$

Proof. Look at the trees $A^{(i)}_{\vec{\mathbf{M}}}[(E, f)]$. The root of the trees has color i and the root lies in the colored set (E, f). But the colored set (E, f) has only a finite number of colors. Hence $A^{(i)}_{\vec{\mathbf{M}}}[(E, f)]$ will only be nonempty for a finite number of $i \in \mathcal{I}$. Thus the collection $\vec{\mathbf{A}}_{\vec{\mathbf{M}}}$ is summable.

The colored species $A^{(i)}_{\vec{\mathbf{M}}}$ puts an enriched root of color i on a colored set. Thus $(M_i \circ (\vec{\mathbf{A}}_{\vec{\mathbf{M}}})_{i \in \mathcal{I}})$ puts a colored partition on a set, where each block of color i receives the structure $A^{(i)}_{\vec{\mathbf{M}}}$, and the set of blocks receives an M_i structure. But note that $A^{(i)}_{\vec{\mathbf{M}}}$ is a colored tree. Moreover this colored tree has the root of color i. Since every block contains a unique root, we can view it as putting the M_i structure on the roots of all the trees. Thus $(M_i \circ \vec{\mathbf{A}}_{\vec{\mathbf{M}}})$ is a colored forest with an M_i structure on the roots.

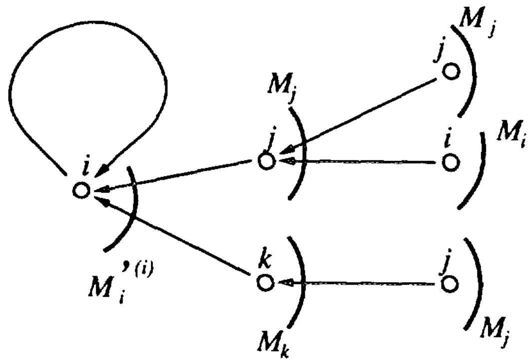

FIG. 1. An example of an $\vec{\mathbf{M}}$-enriched contraction.

Now by multiplying with X_i, we start by selecting an element of color i. Thus in order to get the set of structures the colored species $X_i \cdot (M \circ \vec{\mathbf{A}}_{\vec{\mathbf{M}}})$ describes, join this selected element to the roots of the forest. Thus we have a $\vec{\mathbf{M}}$-enriched colored tree, and the equation follows. ∎

The implicit species theorem [M-N] implies that the equation system $\vec{\mathbf{Y}} = \vec{\mathbf{X}} \cdot (\vec{\mathbf{M}} \circ \vec{\mathbf{Y}})$ has a unique solution $\vec{\mathbf{Y}}$, which is summable.

Define the colored species $\mathrm{End}_{\vec{\mathbf{M}}}$ by letting $\mathrm{End}_{\vec{\mathbf{M}}}[(E, f)]$ be all $\vec{\mathbf{M}}$-enriched colored functions from the colored set (E, f) to itself. Such a colored function is called an *$\vec{\mathbf{M}}$-enriched colored endofunction.*

An *$\vec{\mathbf{M}}$-enriched contraction* on a colored set (E, f) is and $\vec{\mathbf{M}}$-enriched function ϕ such that there exists a node $a \in E$, such that for all $b \in E$ there exists a positive integer k such that for all $n \geqslant k$ we have $\phi^n(b) = a$. Observe that $\phi(a) = a$. We call the vertex a the attracting point. The contraction has depth m if $\phi^m(b) = a$ for all $b \in E$.

LEMMA 4.3. *The colored species of $\vec{\mathbf{M}}$-enriched contractions with the attracting point of color i is described by*

$$X_i \cdot (M_i^{(i)} \circ \vec{\mathbf{A}}_{\vec{\mathbf{M}}}).$$

Proof. The colored species X_i chooses the attracting point a of color i. Then the set $(E - \{a\}, f|_{E - \{a\}})$ has the structure of an $\vec{\mathbf{M}}$-enriched colored forest. But the roots and the attracting vertex a has an M_i structure on them. This is equivalent to putting an $M_i^{\prime(i)}$ structure on the roots. ∎

LEMMA 4.4. *The colored species $M_i^{\bullet(i)}$ is naturally isomorphic to $\vec{\mathbf{M}}$-enriched contractions ϕ, which has depth* 1, *attracting vertex of color i and no structure put on $\phi^{-1}(b)$ if b is not the attracting vertex.*

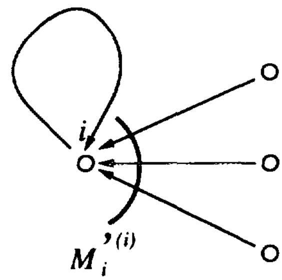

FIG. 2. An example of an $\vec{\mathbf{M}}$-enriched contraction of depth 1 without enriched leaves.

Let ϕ be a colored function from the colored set $(E, f|_E)$ to the colored set $(R \cup E, f)$. An element a in the set E is called periodic if there exists m, a positive integer such that $\phi^m(a) = a$.

DEFINITION 4.3. Define the species $\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}}$ of nonperiodic $\vec{\mathbf{M}}$-enriched colored functions as follows: let $\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}}[(E, f)]$ be the set of all $\vec{\mathbf{M}}$-enriched colored functions ϕ from the colored set (E, f) to the colored set $(E, f) \cup \mathbf{k}$, such that there is no periodic element in the colored set (E, f).

LEMMA 4.5. *Let $(R \cup E, f)$ be an $\mathbf{n}$ set and assume that $(R, f|_R)$ is a $\mathbf{k}$ set. The set of all $\vec{\mathbf{M}}$-enriched colored functions from $(E, f|_E)$ to $(R \cup E, f)$ is described by*

$$(\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}} \cdot \mathrm{End}_{\vec{\mathbf{M}}})[(E, f|_E)].$$

LEMMA 4.6. *For a collection of colored species $\vec{\mathbf{M}}$ we have that*

$$|(\Gamma_{\mathbf{k}}(\vec{\mathbf{A}}_{\vec{\mathbf{M}}}))[\mathbf{n}]| = \binom{\mathbf{n}}{\mathbf{k}} \cdot |\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}}[\mathbf{n} - \mathbf{k}]|.$$

Proof. $(\Gamma_{\mathbf{k}}(\vec{\mathbf{A}}_{\vec{\mathbf{M}}}))[\mathbf{n}]$ describes the set of forest on a $\mathbf{n}$ set such that there are k_i roots of color i. The set of roots R can be chosen in $\binom{\mathbf{n}}{\mathbf{k}}$ possible ways. Let E be the complemented set. Now the forest can be considered a nonperiodic $\vec{\mathbf{M}}$-enriched function from $(E, f|_E)$ to $(E, f|_E) \cup [\mathbf{k}]$. But the number of such functions are $|\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}}[\mathbf{n} - \mathbf{k}]|$, and the lemma follows. ∎

5. C-MONOIDS

In the following section we study how Good's inversion formula specializes to plethystic Lagrange's inversion formula. To do so we need to define the plethystic composition of two colored species and of two formal power series. We begin defining c-monoids, and from this concept we can define the general plethysm.

DEFINITION 5.1. A *c-monoid* $(\mathscr{T}, \cdot, 1)$ is a set $\mathscr{T}$ with an associative binary operation $\cdot$ with identity element $1 \in \mathscr{T}$ (that is, a monoid) which satisfies the additional properties:

(1) For all $i, j \in \mathscr{T}$, if $i \cdot j = 1$ then $i = j = 1$ (indivisibility of the identity).

(2) For all $i, j, j' \in \mathscr{T}$, if $i \cdot j = i \cdot j'$ then $j = j'$ (left cancellation law).

EXAMPLE 5.1. The c-monoid of natural integers under addition. Let $(\mathscr{T}, \cdot, 1) = (\mathbb{N}, +, 0)$. Clearly this is a c-monoid.

EXAMPLE 5.2. The c-monoid of positive integers under multiplication. Let $(\mathscr{T}, \cdot, 1) = (\mathbb{P}, \cdot, 1)$. Observe that this c-monoid is isomorphic to an infinite countable direct product of the c-monoid in the previous example.

EXAMPLE 5.3. The c-monoid of words over an alphabet. Let A be an alphabet. Denote A^* to be the set of all words with letters in A. Then $(A^*, \cdot, \varepsilon)$ is a c-monoid, where $\cdot$ is a concatenation and ε is the empty word.

DEFINITION 5.2. If $(\mathscr{T}, \cdot, 1)$ is a c-monoid we define the *divisibility* relation on $\mathscr{T}$ as the following: for $i, j \in \mathscr{T}$ we have that $i \leqslant j$ if and only if there is $k \in \mathscr{T}$ such that $j = i \cdot k$.

Note that $i \leqslant j$ implies that $k \cdot i \leqslant k \cdot j$.

EXAMPLE 5.4. In the first example above, the divisibility relation is the ordinary linear order on nonnegative integers. In the second example, the positive integer i is less than or equal to the positive integer j, if i divides j. In the last example the word i is less than or equal to the word j, if i is a prefix of j.

LEMMA 5.1. *Let $(\mathscr{T}, \cdot, 1)$ be a c-monoid, and let $\leqslant$ be the induced divisibility relation $\mathscr{T}$. Then:*

(1) $(\mathscr{T}, \leqslant)$ is a partial order on $\mathscr{T}$ with minimal element 1.

(2) For each $i \in \mathscr{T}$, the dual order ideal $\mathscr{T}_i = \{j \in \mathscr{T} : i \leqslant j\}$ is isomorphic to $\mathscr{T}$ via the function $\phi_i : \mathscr{T} \to \mathscr{T}_i$ given by $\phi_i(j) = i \cdot j$. In particular, if $\mathscr{T} \neq \{1\}$ then $\mathscr{T}$ is infinite.

DEFINITION 5.3. A finite factorization monoid (FF c-monoid)$(\mathscr{T}, \cdot, 1)$ is a monoid where every element has only a finite number of factorizations into different elements.

LEMMA 5.2. *Let* $(\mathcal{T}, \cdot, 1)$ *be a c-monoid, and let* $\leqslant$ *be the induced divisibility relation* $\mathcal{T}$. *Then the condition that* $(\mathcal{T}, \cdot, 1)$ *is a finite factorization monoid is equivalent to that the partial order* $(\mathcal{T}, \leqslant)$ *is locally finite.*

6. PLETHYSTIC COMPOSITION OF FORMAL POWER SERIES

Previously, we defined composition between a power series and a summable collection of power series. We also made a similar definition for colored species. If the index set $\mathcal{T}$ has the structure of a c-monoid, we are now able to define the plethystic composition between two power series. Likewise we define the plethystic composition between two colored species.

DEFINITION 6.1. Define the *Verschiebung operator* $\mathscr{V}_i$ on a multi-index by $\mathscr{V}_i(\mathbf{e}_j) = \mathbf{e}_{i \cdot j}$ and extend by linearity.

DEFINITION 6.2. Define the *Frobenius operator* $\mathscr{F}_i$ on formal power series by

$$\mathscr{F}_i(g((x_j)_{j \in \mathcal{T}})) = g((x_{i \cdot j})_{j \in \mathcal{T}}),$$

where $i \in \mathcal{T}$.

Note that $\mathscr{F}_i$ is an injective algebra homomorphism on formal power series. Moreover we have the fact

$$\mathscr{F}_i(\mathbf{x}^{\mathbf{n}}) = \mathbf{x}^{\mathscr{V}_i(\mathbf{n})}.$$

DEFINITION 6.3. Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be formal power series, such that $g(\mathbf{x})$ has no constant coeffcient. Define the plethystic composition $f * g$ as

$$(f * g)(\mathbf{x}) = f((\mathscr{F}_i(g(\mathbf{x})))_{i \in \mathcal{T}}).$$

Let $\mathbf{g}(\mathbf{x})$ be the collection of power series defined by $\mathbf{g}(\mathbf{x}) = (\mathscr{F}_i(g(\mathbf{x})))_{i \in \mathcal{T}}$. Since $g(\mathbf{x})$ do not have a constant term, the collection $\mathbf{g}(\mathbf{x})$ is summable. Observe now that

$$(f * g)(\mathbf{x}) = (f \circ \mathbf{g})(\mathbf{x}).$$

This identity connects the two different compositions.

EXAMPLE 6.1. For the c-monoid of positive integers and multiplication, $(\mathbb{P}, \cdot, 1)$, the plethystic composition is the classical plethysm defined by Polya

$$\begin{aligned}(f * g)&(x_1, x_2, x_3, \ldots)\\ &= f(g(x_1, x_2, x_3, \ldots), g(x_2, x_4, x_6, \ldots), g(x_3, x_6, x_9, \ldots), \ldots).\end{aligned}$$

EXAMPLE 6.2. For the c-monoid of natural integers and addition, $(\mathbb{N}, +, 0)$, the plethystic composition we obtain is the shift-plethysm

$$(f * g)(x_0, x_1, x_2, ...)$$
$$= f(g(x_0, x_1, x_2, ...), g(x_1, x_2, x_3, ...), g(x_2, x_3, x_4, ...), ...).$$

EXAMPLE 6.3. The c-monoid $(A^*, \cdot, \varepsilon)$ leads to an infinite family of plethysm, one for each cardinality of the alphabet A,

$$(f * g)((x_w)_{w \in A^*}) = f((g((x_{w'w})_{w \in A^*}))_{w' \in A}).$$

7. PLETHYSTIC COMPOSITION OF COLORED SPECIES

Define the Frobenius operator $\mathscr{F}_i$ on colored species by the identity

$$(\mathscr{F}_i(M))[(E, f)] = \begin{cases} M[(E, g)] & \text{if } \ i \cdot g(a) = f(a) \quad \text{for all} \quad a \in E \\ \varnothing & \text{otherwise.} \end{cases}$$

The Frobenius operator on colored species has the following combinatorial interpretation. Each structure in $\mathscr{F}_i(M)[(E, f)]$ is obtained from a unique structure in $M[(E, f)]$ by multiplying the colors of the underlying set (E, f) on the left by i.

We can rewrite this as

$$(\mathscr{F}_i(M))[(E, i \cdot f)] = M[(E, f)].$$

Directly we see that

$$\mathscr{F}_i(M + N) = \mathscr{F}_i(M) + \mathscr{F}_i(N),$$
$$\mathscr{F}_i(M \cdot N) = \mathscr{F}_i(M) \cdot \mathscr{F}_i(N).$$

Moreover, a *plethystic partition* of a colored set (E, f) is a partition π of the set E, with a function $g : \pi \to \mathscr{T}$ such that for all $B \in \pi$ and for all $e \in B$

$$g(B) \leqslant f(e).$$

Let $\Pi_p[(E, f)]$ be the set of all plethystic partitions of (E, f).

Let M and N be two colored species, such that $N[\varnothing] = \varnothing$. Define the divided power $\gamma_{\mathbf{k}}(N)$ as

$$(\gamma_{\mathbf{k}}(N))[(E, f)] = \dot{\bigcup_{(\pi, g) \in \Pi_p[(E, f)],\ \text{card}(\pi, g) = \mathbf{k}}} \prod_{B \in \pi} \mathscr{F}_{g(B)}(N)[(B, f|_B)].$$

Define the plethystic composition $M * N$ by

$$(M * N)[(E, f)] = \bigcup_{(\pi, g) \in \Pi_p[(E, f)]} M[(\pi, g)] \times \prod_{B \in \pi} \mathscr{F}_{g(B)}(N)[(B, f|_B)].$$

Let $\vec{\mathbf{N}}$ be the collection of colored species defined by $\vec{\mathbf{N}} = (\mathscr{F}_i(N))_{i \in \mathscr{T}}$. Since $N[\varnothing] = \varnothing$, the collection $\vec{\mathbf{N}}$ will be summable. It is now true that

$$M * N = M \circ \vec{\mathbf{N}}.$$

Moreover, it is also true that

$$\gamma_{\mathbf{k}}(N) = \Gamma_{\mathbf{k}}(\vec{\mathbf{N}}).$$

PROPOSITION 7.1.

$$\operatorname{card}(\mathscr{F}_i(M); \mathbf{x}) = \mathscr{F}_i(\operatorname{card}(M; \mathbf{x}))$$

$$\operatorname{card}((M * N); \mathbf{x}) = (\operatorname{card}(M; \mathbf{x}) * \operatorname{card}(N; \mathbf{x}))(\mathbf{x})$$

$$\operatorname{card}(\gamma_{\mathbf{k}}(N); \mathbf{x}) = \left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} * \operatorname{card}(N; \mathbf{x})\right)(\mathbf{x})$$

$$= \frac{1}{\mathbf{k}!} \prod_{i \in \mathscr{T}} \mathscr{F}_i(\operatorname{card}(N; \mathbf{x}))^{k_i}.$$

8. COLORED SPECIES OF PERMUTATIONS

Let S be the species of permutations on elements of color 1. Let S_0 be the species of nonempty permutations on elements of color 1. Similarly define L, L_0 as linear orders on elements of color 1. Their generating functions are

$$\operatorname{card}(S; \mathbf{x}) = \operatorname{card}(L; \mathbf{x}) = \frac{1}{1 - x_1}$$

$$\operatorname{card}(S_0; \mathbf{x}) = \operatorname{card}(L_0; \mathbf{x}) = \frac{1}{1 - x_1} - 1 = \frac{x_1}{1 - x_1}.$$

From [J] comes the following equipotent identity

LEMMA 8.1.

$$S_0 \equiv X_1 \cdot S.$$

Proof. We have that

$$S_0 \equiv L_0 = X_1 \cdot L \equiv X_1 \cdot S. \quad \blacksquare$$

Proof. It is easy to construct a natural bijection to see that the following is true:

$$S_0^\bullet = X_1 \cdot S + X_1 \cdot S^\bullet.$$

This can be written as

$$\begin{aligned} S_0^\bullet &= X_1 \cdot S + X_1 \cdot S^\bullet \\ &= X_1^\bullet \cdot S + X_1 \cdot S^\bullet \\ &= (X_1 \cdot S)^\bullet \end{aligned}$$

from which the result follows. ∎

DEFINITION 8.1. Let I be a subset of $\mathscr{T}$. Define the colored species S^I by

$$S^I = \prod_{i \in I} \mathscr{F}_i(S).$$

Similarly, for a finite subset I of $\mathscr{T}$, define the colored species S_0^I and X_I by

$$S_0^I = \prod_{i \in I} \mathscr{F}_i(S_0),$$

$$X_I = \prod_{i \in I} X_i.$$

The set of structures $S^I[(E, f)]$ consists of permutations on each fiber $f^{-1}(i)$, where $i \in I$. Recall that a permutation might be empty. Similarly the set of structures $S_0^I[(E, f)]$ consists of *nonempty* permutations on each fiber $f^{-1}(i)$, where $i \in I$. Observe then that the colored species $S_0^I \cdot S^{\mathscr{T} - I}$ puts a permutation on each fiber $f^{-1}(i)$ for $i \in \mathscr{T}$ and demands that there are elements with the colors of the set I.

LEMMA 8.2. *The following equipotent identity is true*:

$$S_0^I \cdot S^{\mathscr{T} - I} \equiv X_I \cdot S^{\mathscr{T}}.$$

Proof. Directly we have that

$$\begin{aligned} S_0^I \cdot S^{\mathscr{T} - I} &= \prod_{i \in I} \mathscr{F}_i(S_0) \cdot \prod_{i \in \mathscr{T} - I} \mathscr{F}_i(S) \\ &\equiv \prod_{i \in I} \mathscr{F}_i(X_1 \cdot S) \cdot \prod_{i \in \mathscr{T} - I} \mathscr{F}_i(S) \\ &= \prod_{i \in I} X_i \cdot \prod_{i \in \mathscr{T}} \mathscr{F}_i(S) \\ &= X_I \cdot S^{\mathscr{T}}. \end{aligned}$$ ∎

9. Plethystic Functions and Plethystic Trees

Let M be a colored species.

Definition 9.1. An *M-enriched plethystic function/contraction/tree/forest* is an $\vec{\mathbf{M}}$-enriched function/contraction/tree/forest, where the collection $\vec{\mathbf{M}}$ is defined by

$$\vec{\mathbf{M}} = (\mathscr{F}_i(M))_{i \in \mathscr{T}}.$$

Observe that an M-enriched plethystic function ϕ from the colored set (E, f) to the colored set (F, g) fulfills $f(a) \geqslant g(\phi(a))$ for all $a \in E$.

Note that if b is a node in a plethystic tree on a colored set (E, f), and if a is the father of the node b, then $f(a) \leqslant f(b)$. This is the same definition of plethystic trees as in [M-N].

We have three important colored species to define.

1. Let A_M be the colored species of M-enriched plethystic trees with the root of color 1. That is, $A_M = A^{(1)}_{\vec{\mathbf{M}}}$. Observe that $(\mathscr{F}_i(A_M))_{i \in \mathscr{T}} = \vec{\mathbf{A}}_{\vec{\mathbf{M}}}$.
2. Let End_M be the colored species of M-enriched plethystic endofunctions. That is, $\mathrm{End}_M = \mathrm{End}_{\vec{\mathbf{M}}}$.
3. Let $\mathcal{N}^{\mathbf{k}}_M$ be the colored species of nonperiodic M-enriched plethystic functions. That is, $\mathcal{N}^{\mathbf{k}}_M = \mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}}$.

Thus we can rewrite Lemma 4.1 to the following.

Lemma 9.1. *Let (E, f) and (F, g) be two colored sets. The set of all M-enriched plethystic functions from (E, f) to (F, g) is described by*

$$\left(\prod_{b \in F} \mathscr{F}_{g(b)}(M)\right)[(E, f)].$$

Example 9.1. Let the underlying c-monoid be positive integers and multiplication. An M-enriched plethystic tree, where M is a colored species, is an enriched plethystic tree as defined in [C].

Directly from Proposition 4.2 we obtain the following lemma.

Lemma 9.2. *The colored species A_M fulfills the functional equation*

$$A_M = X_1 \cdot (M * A_M).$$

Lemmas 4.3 and 4.4 translate into

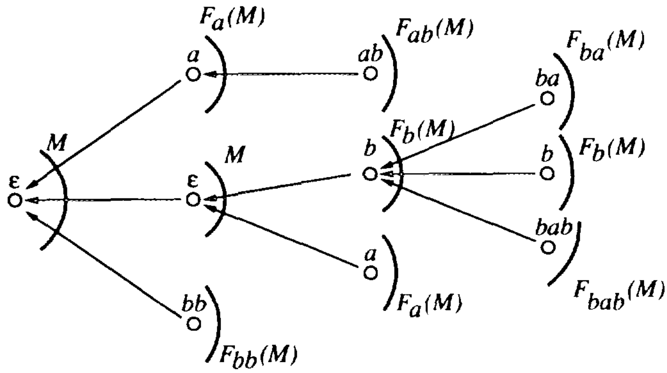

FIG. 3. An example of an M-enriched plethystic tree over the c-monoid $(\{a, b\}^*, \cdot, \varepsilon)$.

LEMMA 9.3. *The colored species of M-enriched contractions with the attracting point of color* 1 *is described by*

$$X_1 \cdot (M'^{(1)} * A_M).$$

LEMMA 9.4. *The colored species* $M^{\bullet(1)} = X_1 \cdot M'^{(1)}$ *is naturally isomorphic to M-enriched contractions* ϕ, *which has depth* 1, *attracting vertex of color* 1 *and no structure put on* $\phi^{-1}(b)$ *if b is not the attracting vertex.*

Let ϕ be a plethystic function from the colored set $(E, f|_E)$ to to the colored set $(R \cup E, f)$. Let a be a periodic element of color i. Since $f(a) \geqslant f(\phi(a)) \geqslant \cdots \geqslant f(\phi^2(a)) \geqslant \cdots \geqslant f(\phi^m(a))$ we know that all nodes in the same cycle as a have the color i.

LEMMA 9.5. *The colored species* End_M *is naturally isomorphic to*

$$S^{\mathscr{T}} * (X_1 \cdot (M'^{(1)} * A_M)).$$

LEMMA 9.6. *Let* $(R \cup E, f)$ *be an* $\mathbf{n}$ *set and assume that* $(R, f|_R)$ *is a* $\mathbf{k}$ *set. The set of all M-enriched plethystic functions from* $(E, f|_E)$ *to* $(R \cup E, f)$ *is described by*

$$(\mathscr{N}_M^{\mathbf{k}} \cdot \mathrm{End}_M)[(E, f|_E)].$$

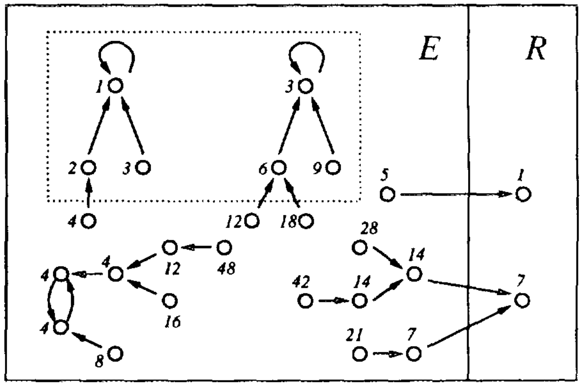

FIG. 4. An example of a plethystic function with marked contractions of depth 1.

10. LAGRANGE INVERSION FORMULA

Let $R \cup E$ be a colored $\mathbf{n}$ set and R a colored $\mathbf{k}$ set. Let $J = \{i : n_i \neq 0\}$, which is a finite set. Moreover, in this section, let $M' = M'^{(1)}$ and $M^{\bullet} = M^{\bullet(1)}$.

PROPOSITION 10.1. *Let I be a subset of J. Then there is a natural bijection between the set of M-enriched plethystic functions from E to $R \cup E$ such that there exists a cyclic point of color i for all $i \in I$, and the set*

$$\left(\prod_{i \in I} \mathcal{F}_i(M^{\bullet} M^{n_i - 1}) \cdot \prod_{i \in \mathcal{J} - I} \mathcal{F}_i(M^{n_i})\right)[(E, f)].$$

Proof. Observe that

$$\begin{aligned}\prod_{i \in I} \mathcal{F}_i(M^{\bullet} M^{n_i - 1}) \cdot \prod_{i \in \mathcal{J} - I} \mathcal{F}_i(M^{n_i}) \\ = \prod_{i \in I} \mathcal{F}_i(M^{\bullet}) \cdot \prod_{i \in I} \mathcal{F}_i(M^{n_i - 1}) \cdot \prod_{i \in \mathcal{J} - I} \mathcal{F}_i(M^{n_i}).\end{aligned}$$

By Lemma 9.4, $\mathcal{F}_i(M^{\bullet})$ chooses an M-enriched contraction of depth 1 with the attracting vertex of color i. Thus

$$\prod_{i \in I} \mathcal{F}_i(M^{\bullet})$$

chooses for each $i \in I$ an attracting vertex of color i, and to each of them a contraction of depth 1. Let C be the set of attracting vertices, and let E_1 be the underlying set on which these contractions are built. Hence we have chosen an M-enriched plethystic function from the colored set $(E_1, f|_{E_1})$ to the colored set $(C, f|_C)$.

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The cardinality of $(R \cup E) - C$ is $\mathbf{n} - \mathbf{e}_I$. Hence the colored species

$$\prod_{i \in I} \mathscr{F}_i(M^{n_i - 1}) \cdot \prod_{i \in \mathscr{T} - I} \mathscr{F}_i(M^{n_i})$$

chooses an M-enriched plethystic function from the colored set $(E_2, f|_{E_2})$ to the colored set $((R \cup E) - C, f|_{(R \cup E) - C})$, where $E_2 = E - E_1$.

Recall that $E = E_1 + E_2$; thus by joining these two plethystic functions we get an M-enriched plethystic function from the colored set (E, f) to the colored set $(R \cup E, f)$. Moreover we know that this function has an attracting vertex of color i for each $i \in I$.

The above set of structures can be written as

$$\left(\mathscr{N}_M^{\mathbf{k}} \cdot \prod_{i \in I} \mathscr{F}_i(X_1 \cdot (M' * A_M)) \cdot \mathrm{End}_M\right) [(E, f|_E)].$$

The first factor describes the structure of elements that image of repeated applications of the functions will be in R. The second part is all the contractions which have the attracting vertices of the given colors. The third part is the structure on those elements that will be in a cycle after repeated applications of the function.

The above colored species can be written as

$$\begin{aligned} \mathscr{N}_M^{\mathbf{k}} \cdot \prod_{i \in I} \mathscr{F}_i(X_1 \cdot (M' * A_M)) \cdot \mathrm{End}_M &= \mathscr{N}_M^{\mathbf{k}} \cdot (X_I * (X_1 \cdot (M' * A_M))) \cdot (S^{\mathscr{T}} * (X_1 \cdot (M' * A_M))) \\ &= \mathscr{N}_M^{\mathbf{k}} \cdot ((X_1 \cdot S^{\mathscr{T}}) * (X_1 \cdot (M' * A_M))). \end{aligned}$$

Since

$$X_I \cdot S^{\mathscr{T}} \equiv S_0^I \cdot S^{\mathscr{T} - I},$$

we conclude that the set of structures above is equipotent to

$$(\mathscr{N}_M^{\mathbf{k}} \cdot ((S_0^I \cdot S^{\mathscr{T} - I}) * (X_1 \cdot (M' * A_M))))[(E, f|_E)].$$

But this is the structure of enriched plethystic functions such that there will be at least one periodic element of color i for each $i \in I$. This concludes the proof of the proposition. ▌

THEOREM 1 (Lagrange Inversion Formula, Species Version). *Let M be a colored species and let A_M be the colored species of M-enriched species of M-enriched plethystic trees. Assume that $\mathbf{n} \geq \mathbf{k}$, and let $J = \{i \in \mathscr{T} : n_i \neq 0\}$. Then*

$$|(\gamma_{\mathbf{k}}(A_M))[\mathbf{n}]| \equiv \binom{\mathbf{n}}{\mathbf{k}} \cdot \left| \left(\prod_{i \in J} \mathscr{F}_i(M^{n_i} - M^{\bullet} \cdot M^{n_i - 1}) \right) [\mathbf{n} - \mathbf{k}] \right|,$$

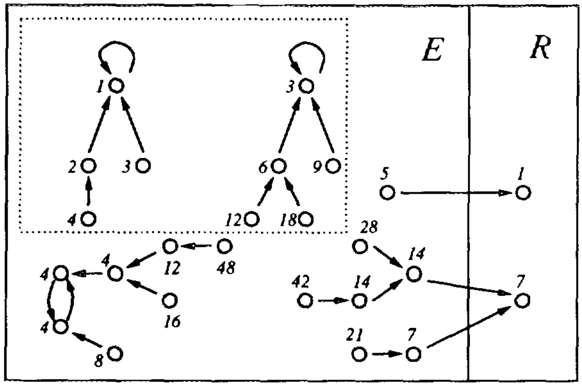

FIG. 5. An example of a plethystic function with marked contractions.

written by help of abuse of notation. (The notation could be made strict by using Möbius species [M-Y].)

Proof. By Proposition 10.1 we know that

$$\left|\left(\prod_{i\in I}\mathscr{F}_i(M^{\bullet}\cdot M^{n_i-1})\cdot\prod_{i\in J-I}\mathscr{F}_i(M^{n_i})\right)[(E,f|_E)]\right|$$

counts the plethystic functions which has periodic elements of color i for each $i\in I$. But we would like to count plethystic functions that do not have any periodic elements at all. By inclusion and exclusion the number of such plethystic functions is

$$\sum_{I\subseteq J}(-1)^{|I|}\left|\left(\prod_{i\in I}\mathscr{F}_i(M^{\bullet}\cdot M^{n_i-1})\cdot\prod_{i\in J-I}\mathscr{F}_i(M^{n_i})\right)[(E,f|_E)]\right|.$$

By abuse of notation we can write the above

$$\sum_{I\subseteq J}(-1)^{|I|}\left|\left(\prod_{i\in I}\mathscr{F}_i(M^{\bullet}M^{n_i-1})\prod_{i\in J-I}\mathscr{F}_i(M^{n_i})\right)[(E,f|_E)]\right|$$

$$=\left|\sum_{I\subseteq J}(-1)^{|I|}\left(\prod_{i\in I}\mathscr{F}_i(M^{\bullet}M^{n_i-1})\prod_{i\in J-I}\mathscr{F}_i(M^{n_i})\right)[(E,f|_E)]\right|$$

$$=\left|\left(\prod_{i\in J}\mathscr{F}_i(M^{n_i}-M^{\bullet}M^{n_i-1})\right)[(E,f|_E)]\right|.$$

This is the number of nonperiodic M-enriched functions from the set $(E, f|_E)$ to the set $(E, f|_E) \cup \mathbf{k}$. Thus we have the identity

$$|\mathcal{N}^{\mathbf{k}}_M[(E, f|_E)]| = \left| \left(\prod_{i \in J} \mathcal{F}_i(M^{n_i} - M \cdot M^{n_i - 1}) \right) [(E, f|_E)] \right|.$$

Now by Lemma 4.6 and the above identity, Lagrange inversion formula follows. ∎

By equating the coefficients of the equation $f(\mathbf{x}) = x_1 \cdot (G * f)(\mathbf{x})$ a system of recurrences occurs for the coefficients of $f(\mathbf{x})$, and this system is easily seen to have a unique solution. Hence the equation $f(\mathbf{x}) = x_1 \cdot (G * f)(\mathbf{x})$ uniquely determines the power series $f(\mathbf{x})$.

THEOREM 2 (Lagrange Inversion Formula). *Let $f(\mathbf{x})$ and $G(\mathbf{x})$ be power series in the variables $(x_i)_{i \in \mathcal{T}}$ such that*

$$f(\mathbf{x}) = x_1 \cdot (G * f)(\mathbf{x}).$$

Assume that $\mathbf{n} \geq \mathbf{k}$, and let $J = \{i \in \mathcal{T} : n_i \neq 0\}$. Then

$$[\mathbf{x}^{\mathbf{n}}](\mathbf{x}^{\mathbf{k}} * f)(\mathbf{x}) = [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \prod_{i \in J} H_i(\mathbf{x}),$$

where

$$H_i(\mathbf{x}) = \mathcal{F}_i\left(G(\mathbf{x})^{n_i} - x_1 \cdot \frac{\partial G(\mathbf{x})}{\partial x_1} \cdot G(\mathbf{x})^{n_i - 1} \right).$$

Proof. Let M be a colored species and $G(\mathbf{x})$ its generating function. That is, $G(\mathbf{x}) = \operatorname{card}(M; \mathbf{x})$. Let $\bar{f}(\mathbf{x}) = \operatorname{card}(A_M; \mathbf{x})$, where A_M is the colored species of M-enriched plethystic trees. Since $A_M = X_1 \cdot (M * A_M)$, we get

$$\begin{aligned} \bar{f}(\mathbf{x}) &= \operatorname{card}(A_M; \mathbf{x}) \\ &= \operatorname{card}(X_1 \cdot (M * A_M); \mathbf{x}) \\ &= x_1 \cdot (\operatorname{card}(M; \mathbf{x}) * \operatorname{card}(A_M; \mathbf{x})) \\ &= x_1 \cdot (G(\mathbf{x}) * \bar{f}(\mathbf{x})). \end{aligned}$$

But $f(\mathbf{x})$ is uniquely determined by the above equation. Hence $f(\mathbf{x}) = \bar{f}(\mathbf{x}) = \operatorname{card}(A_M; \mathbf{x})$.

Now the left-hand side of the species version of Lagrange inversion formula is equal to

$$\begin{aligned}|(\gamma_{\mathbf{k}}(A_M))[\mathbf{n}]| &= \left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right] \operatorname{card}(\gamma_{\mathbf{k}}(A_M); \mathbf{x}) \\ &= \left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right]\left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} * f\right)(\mathbf{x}) \\ &= \frac{\mathbf{n}!}{\mathbf{k}!}[\mathbf{x}^{\mathbf{n}}](\mathbf{x}^{\mathbf{k}} * f)(\mathbf{x}).\end{aligned}$$

And the right-hand side

$$\begin{aligned}&\binom{\mathbf{n}}{\mathbf{k}} \cdot \left|\left(\prod_{i \in J} \mathscr{F}_i(M^{n_i} - M^{\bullet} \cdot M^{n_i - 1})\right)[\mathbf{n}-\mathbf{k}]\right| \\ &= \binom{\mathbf{n}}{\mathbf{k}} \cdot \left[\frac{\mathbf{x}^{\mathbf{n}-\mathbf{k}}}{(\mathbf{n}-\mathbf{k})!}\right] \operatorname{card}\left(\prod_{i \in J} \mathscr{F}_i(M^{n_i} - M^{\bullet} \cdot M^{n_i - 1}); \mathbf{x}\right) \\ &= \frac{\mathbf{n}!}{\mathbf{k}!} \cdot [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \prod_{i \in J} \mathscr{F}_i(G(\mathbf{x})^{n_i} - x_1 G'(\mathbf{x}) \cdot G(\mathbf{x})^{n_i - 1}) \\ &= \frac{\mathbf{n}!}{\mathbf{k}!} \cdot [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \prod_{i \in J} H_i(\mathbf{x}).\end{aligned}$$

Thus we have proven the theorem for formal power series $G(\mathbf{x})$ such that $[\mathbf{x}^{\mathbf{n}}/\mathbf{n}!]\, G(\mathbf{x})$ is a nonnegative integer for all multi-indices $\mathbf{n}$. By the principle of extension of algebraic identities the theorem follows for all $G(\mathbf{x})$. ∎

We point out that one can also prove the above theorem by directly enumerating plethystic trees. This method is based on counting the number of plethystic trees, given a degree sequence in each color class. Observe that the degree of a node is a multi-index; hence a degree sequence is a function from multi-indices to nonnegative integers. This can be done in two possible ways. The first method is to count unlabeled plane plethystic trees. This corresponds to ordinary generating functions. This method is a generalization of Raney's proof of Lagrange inversion formula [R]. The second method is to count labeled plethystic trees, which correspond to exponential generating functions. This proof generalizes the third proof of the Lagrange inversion formula that appears in [S].

11. Good's Inversion Formula

Let $R \cup E$ be a colored $\mathbf{n}$ set and R a colored $\mathbf{k}$ set. Let $J = \{i : n_i \neq 0\}$.

Definition 11.1. Let I be a finite subset of $\mathscr{T}$, and let π be a permutation on I. Define the colored species $P^{\pi}_{\vec{\mathbf{M}}}$ by $P^{\pi}_{\vec{\mathbf{M}}}[(E, f)]$ is all $\vec{\mathbf{M}}$-enriched colored functions ϕ, such that there is a subset $\{e_i\}_{i \in I}$ of E such that $f(e_i) = i$, $\phi(e_i) = e_{\pi(i)}$ and for all $b \in E$ there exists a positive integer k such that $\phi^k(b) \in \{e_i\}_{i \in I}$. The elements e_i are called the attracting vertices.

If I only consists of one element, that is $I = \{i\}$, then $P^{\pi}_{\vec{\mathbf{M}}}$ is just a colored contraction, with the attracting vertex of color i. If I is the empty set, then $P^{\varnothing}_{\vec{\mathbf{M}}} = 1$.

Lemma 11.1. *The colored species*

$$\prod_{i \in I} M_i^{\bullet(\pi^{-1}(i))}$$

is naturally isomorphic to $\vec{\mathbf{M}}$-enriched endofunction ψ, such that there are elements e_i such that $f(e_i) = i$ and $\psi(e_i) = e_{\pi(i)}$ for all elements b we have that $\psi(b) \in \{e_i\}_{i \in I}$. The last condition makes the function ψ to be of depth 1. *Moreover, we do not put any M structure on the fiber $\psi^{-1}(b)$, if $b \notin \{e_i\}_{i \in I}$.*

Proof. By Lemma 4.4, $M_i^{\bullet(\pi^{-1}(i))}$ chooses a M_i-enriched contraction of depth 1 and the attracting vertex of color $\pi^{-1}(i)$. Thus

$$\prod_{i \in I} M_i^{\bullet(\pi^{-1}(i))}$$

chooses a function ϕ such for each $i \in I$ a fixpoint of color $\pi^{-1}(i)$, and to each of them a contradiction of depth 1. Now define $\omega(e_{\pi^{-1}(i)}) = e_i$. Define

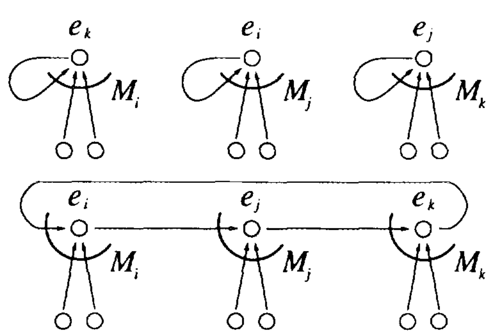

Fig. 6. The construction for the permutation $\pi(i) = j$, $\pi(j) = k$, and $\pi(k) = i$.

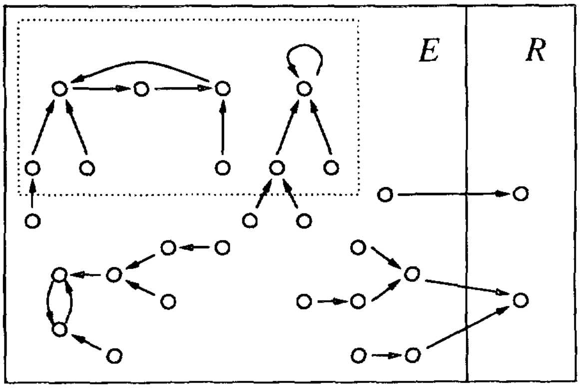

FIG. 7. An example with a colored function with the product $\prod_{i\in I} M_i^{\bullet(\pi^{-1}(i))}$ marked.

a new function ψ by $\psi(b)=\omega(\phi(b))$. Note that ψ is a colored function. Enrich the colored set $\psi^{-1}(b)$ with the structure $\phi^{-1}(\omega^{-1}(b))$. This completes the bijection. ▮

PROPOSITION 11.2. *Let I be a subset of J. Let π be a permutation on the set I. Then there is a natural bijection between the set*

$$\left(\prod_{i\in I} M_i^{\bullet(\pi^{-1}(i))} M_i^{n_i-1} \cdot \prod_{i\in \mathcal{T}-I} M_i^{n_i}\right)[(E, f|_E)].$$

and the set

$$(\mathcal{N}_{\mathbf{M}}^{\mathbf{k}} \cdot P_{\mathbf{M}}^{\pi} \cdot \mathrm{End}_{\vec{\mathbf{M}}})[(E, f|_E)].$$

Proof. We can write

$$\prod_{i\in I} M_i^{\bullet(\pi^{-1}(i))} M_i^{n_i-1} \cdot \prod_{i\in \mathcal{T}-I} M_i^{n_i} = \prod_{i\in I} M_i^{\bullet(\pi^{-1}(i))} \cdot \vec{\mathbf{M}}^{\mathbf{n}-\mathbf{e}_I}.$$

By Lemma 11.1 the first term in the above product chooses a $\vec{\mathbf{M}}$-enriched endofunction such that there are elements e_i so that $f(e_i)=i$ and $\psi(e_i)=e_{\pi(i)}$ for all $i\in I$, and for all elements b we have that $\phi(b)\in\{e_i\}_{i\in I}$. Let $C=\{e_i : i\in I\}$, and we call these elements the attracting vertices. Let E_1 be the underlying set of elements that this structure is built on.

The set $(R\cup E)-C$ has the cardinality $\mathbf{n}-\mathbf{e}_I$. Hence the colored species $\vec{\mathbf{M}}^{\mathbf{n}-\mathbf{e}_I}$ chooses an $\vec{\mathbf{M}}$-enriched function from the colored set $(E_2, f|_{E_2})$ to the colored set $((R\cup E)-C, f|_{(R\cup E)-C})$.

Thus by taking the union between these two functions we get an $\vec{\mathbf{M}}$-enriched colored function ψ from the colored set $(E, f|_E)$ to the colored

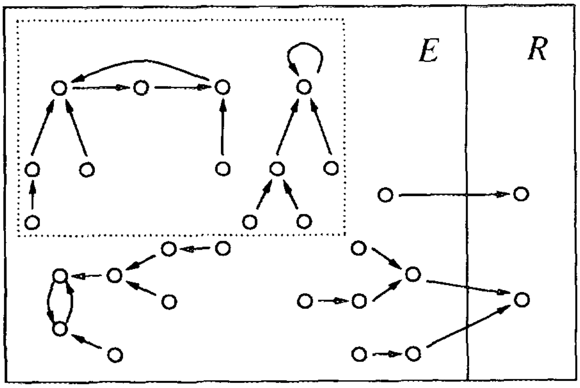

FIG. 8. An example with a colored function with the colored species $P^{\pi}_{\vec{\mathbf{M}}}$ marked.

set $(E \cup R, f)$ with attracting vertex e_i of color i for each $i \in I$, such that $\psi(e_i) = e_{\pi(i)}$ for all $i \in I$.

Now consider the right-hand side of the proposition.

$$\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}} \cdot P^{\pi}_{\vec{\mathbf{M}}} \cdot \mathrm{End}_{\vec{\mathbf{M}}}.$$

The first term $\mathcal{N}^{\mathbf{k}}_{\vec{\mathbf{M}}}$ describes the part of the structure of the function whose underlying elements will reach the colored set R after repeatedly application of the function. The second term $P^{\pi}_{\vec{\mathbf{M}}}$ describes the part of the structure whose underlying elements will reach the attracting vertices. Finally, the third term $\mathrm{End}_{\vec{\mathbf{M}}}$ describes the part of the function whose elements will reach cycles. Hence this is the same set of structures as above and thus the result follows. ∎

Define $c(\pi)$, where π is a permutation, as the number of cycles in π.

PROPOSITION 11.3. *Let $j \in \mathcal{T}$. Then we have that*

$$\sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} (P^{\pi}_{\vec{\mathbf{M}}} \cdot \mathrm{End}_{\vec{\mathbf{M}}})^{\bullet(j)} = 0.$$

Proof. Let T be the set

$$T = \dot{\bigcup_{I \subseteq J}} \dot{\bigcup_{\pi \in S[I]}} (P^{\pi}_{\vec{\mathbf{M}}} \cdot \mathrm{End}_{\vec{\mathbf{M}}})^{\bullet(j)} [(E, f)].$$

An element of T is written as (ϕ, π, K, c), where ϕ is the $\vec{\mathbf{M}}$-enriched function, π is the marked permutation of colors, K is elements which have the colors the permutation π acts upon, and c is the marked element of color I. Define the sign of (ϕ, π, K, c) by $\mathrm{sign}((\phi, \pi, K, c)) = (-1)^{c(\pi)}$.

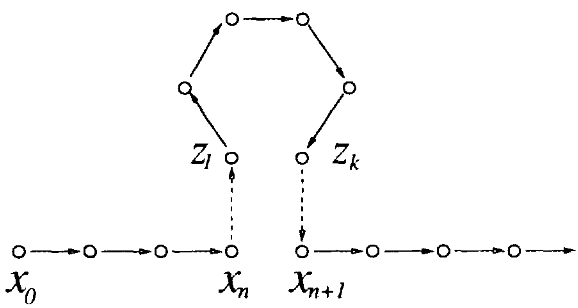

FIG. 9. The construction of ψ in case (iii).

We construct an involution χ on T that is sign reversing. That is, $\chi^2((\phi, \pi, K, c)) = (\phi, \pi, K, c)$, and sign $(\chi((\phi, \pi, K, c))) = -\text{sign}((\phi, \pi, K, c))$. From this the proposition follows since χ defines a bijection between the two sets

$$\{t \in T : \text{sign}(t) = 1\} \qquad \text{and} \qquad \{t \in T : \text{sign}(t) = -1\}.$$

Given $(\phi, \pi, K, c) \in T$, we now start constructing the involution $\chi((\phi, \pi, K, c)) = (\psi, \sigma, L, c)$.

Define the sequence $x_0, x_1, x_2, \ldots$ by the following rules. Let m be the smallest nonnegative integer such that $\phi^m(c)$ is a periodic element. Let $x_0 = \phi^m(c)$. Define $x_k = \phi^k(x_0)$ for $k \geqslant 1$.

Let n be the smallest nonnegative integer such that $f(x_n) \in \{f(x_0), \ldots, f(x_{n-1})\} \cup f(K)$. Observe that such an integer exists since x_0 is periodic element. Four cases can occur:

(i) $n = 0$ and $x_0 \in K$. Then remove the cycle $\{x_0, x_1, \ldots\}$ from the marked permutation. That is, $L = K - \{x_0, x_1, \ldots\}$. Restrict also π to the set $f(L) = f(K) - \{f(x_0), f(x_1), \ldots\}$, to obtain σ. But let $\psi = \phi$, and let them have the same marked element c.

(ii) $x_n = x_0$. Then add the cycle $\{x_0, x_1, \ldots, x_{n-1}\}$ to the marked permutation. That is, $L = K \cup \{x_0, x_1, \ldots, x_{n-1}\}$. Extend also π to the set $f(L) = f(K) \cup \{f(x_0), f(x_1), \ldots, f(x_{n-1})\}$, to obtain σ. But let $\psi = \phi$, and let them have the same marked element c.

(iii) $f(x_n) \in f(K)$. Assume that $f(x_n) = f(z_0)$ for some element z_0 such that z_0 is in the permutation $\tilde{\pi}$. Assume that the length of the cycle the element z_0 is in is k. Let $z_i = \tilde{\pi}^i(z_0)$. Thus $z_k = z_0$ and $\{z_1, \ldots, z_k\}$ are the elements of the cycle. Define ψ by

$$\psi(b) = \begin{cases} x_{n+1} & \text{if} \quad b = z_k \\ z_1 & \text{if} \quad b = x_n \\ \phi(b) & \text{if} \quad b \neq z_k, b \neq x_n. \end{cases}$$

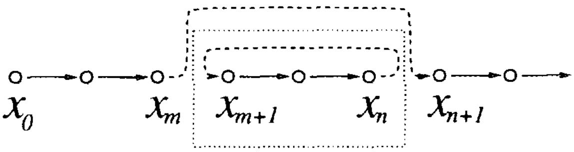

FIG. 10. The construction of ψ in case (iv).

As in the previous case, the colored sets $\phi^{-1}(b)$ and $\psi^{-1}(b)$ have the same cardinality for all $b \in E$ and thus the functions ϕ and ψ have the same enrichment.

But remove from the marked permutation the cycle $\{z_1, ..., z_k\}$. That is, $L = K - \{z_1, ..., z_k\}$. Restrict π to the set $f(L) = f(K) - \{f(z_1), ..., f(z_k)\}$, to obtain the permutation σ. Let c still be the marked periodic element of ϕ.

(iv) $f(x_n) \in \{f(x_0), ..., f(x_{n-1})\}$. Assume that $f(x_n) = f(x_m)$ for some integer m such that $0 \leqslant m \leqslant n-1$. Define ψ by

$$\psi(b) = \begin{cases} x_{m+1} & \text{if } \; b = x_n \\ x_{n+1} & \text{if } \; b = x_m \\ \phi(b) & \text{if } \; b \neq x_m, b \neq x_n. \end{cases}$$

Observe that the colored sets $\phi^{-1}(b)$ and $\psi^{-1}(b)$ have the same cardinality for all $b \in E$. Thus the functions ϕ and ψ have the same enrichment of the colored species $\vec{\mathbf{M}}$.

But extend the marked permutation with the cycle $\{x_{m+1}, ..., x_n\}$. Hence we can write $L = K \cup \{x_{m+1}, ..., x_n\}$. Extend also π to the set $f(L) = f(K) \cup \{f(x_{m+1}), ..., f(x_n)\}$, to obtain σ. Let c still be the marked periodic element of ϕ.

Clearly $\operatorname{sign}((\phi, \pi, K, c)) = -\operatorname{sign}((\psi, \sigma, L, c))$, since the difference in the number of cycles of π and σ is one.

It remains to show that χ is an involution. Now if we apply χ twice observe that we are going to add and remove the same cycle from the partial permutation π. This fact checks in all four cases above. Observe that cases (i) and (ii) are dual and that the cases (iii) and (iv) are dual. Thus χ is an involution, and the proposition follows. ∎

LEMMA 11.4.

$$\sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} P^{\pi}_{\vec{\mathbf{M}}} \cdot \operatorname{End}_{\vec{\mathbf{M}}} = 1.$$

Proof. Let

$$A(\mathbf{x}) = \operatorname{card}\left(\sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} P_{\vec{\mathbf{M}}}^{\pi} \cdot \operatorname{End}_{\vec{\mathbf{M}}} ; \mathbf{x}\right).$$

Observe for all $j \in \mathscr{T}$ that

$$x_j \cdot \frac{\partial}{\partial x_j} A(\mathbf{x}) = \operatorname{card}\left(\sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} (P_{\vec{\mathbf{M}}}^{\pi} \cdot \operatorname{End}_{\vec{\mathbf{M}}})^{\bullet(j)} ; \mathbf{x}\right) = 0.$$

Hence for all $j \in \mathscr{T}$ we have that $(\partial/\partial x_j)\, A(\mathbf{x}) = 0$. Thus solving for $A(\mathbf{x})$ by integration, we observe that $A(\mathbf{x})$ will be a constant. To find this constant observe that the colored species $P_{\varnothing}$ will contain the empty function. Thus the constant equals 1 and the result follows. ∎

LEMMA 11.5. *Let $(a_{i,j})_{i,j \in J}$ be a matrix. Then*

$$\det(\delta_{i,j} \cdot b_i - a_{i,j})_{i,j \in J} = \sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} \prod_{i \notin I} b_i \prod_{i \in I} a_{i,\pi^{-1}(i)}.$$

Proof.

$$\begin{aligned} \det(\delta_{i,j} \cdot b_i - a_{i,j})_{i,j \in J} &= \sum_{I \subseteq J} (-1)^{|I|} \prod_{i \notin I} b_i \det(a_{i,j})_{i,j \in I} \\ &= \sum_{I \subseteq J} (-1)^{|I|} \sum_{\pi \in S[I]} \operatorname{sign}(\pi) \prod_{i \notin I} b_i \prod_{i \in I} a_{i,\pi^{-1}(i)} \\ &= \sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} \prod_{i \notin I} b_i \prod_{i \in I} a_{i,\pi^{-1}(i)}. \end{aligned}$$ ∎

THEOREM 3 (Good's Inversion Formula, the Species Version). *Let $\vec{\mathbf{M}}$ be a collection of colored species. Let $A_{\vec{\mathbf{M}}}^{(i)}$ be the colored species of $\vec{\mathbf{M}}$-enriched trees, with the root of color i. Let $J = \{i \in \mathscr{T} : n_i \neq 0\}$ and assume that $\mathbf{n} \geqslant \mathbf{k}$. Then we have that*

$$|(\Gamma_{\mathbf{k}}(\vec{\mathbf{A}}_{\vec{\mathbf{M}}}))[\mathbf{n}]| \equiv \binom{\mathbf{n}}{\mathbf{k}} \cdot |(\det(\delta_{i,j} M_i^{n_i} - M_i^{\bullet(j)} \cdot M_i^{n_i - 1})_{i,j \in J})[\mathbf{n} - \mathbf{k}]|,$$

written by help of abuse of notation. (The notation could be made strict by using Möbius species [M-Y].)

Proof. By Lemma 11.5, Proposition 11.2, and Lemma 11.4 it follows that

$$\begin{aligned}
&|(\det(\delta_{i,j} M_i^{n_i} - M_i^{\bullet(j)} \cdot M_i^{n_i - 1})_{i,j \in J})[\mathbf{n}-\mathbf{k}]| \\
&\quad = \left| \left(\sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} \prod_{i \notin I} M_i^{n_i} \prod_{i \in I} M_i^{\bullet(\pi^{-s}(i))} M_i^{n_i - 1} \right) [\mathbf{n}-\mathbf{k}] \right| \\
&\quad = \left| \left(\sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} \mathcal{N}_{\vec{\mathbf{M}}}^{\mathbf{k}} \cdot P_\pi \cdot \mathrm{End}_{\vec{\mathbf{M}}} \right) [\mathbf{n}-\mathbf{k}] \right| \\
&\quad = \left| \left(\mathcal{N}_{\vec{\mathbf{M}}}^{\mathbf{k}} \cdot \sum_{I \subseteq J} \sum_{\pi \in S[I]} (-1)^{c(\pi)} P_\pi \cdot \mathrm{End}_{\vec{\mathbf{M}}} \right) [\mathbf{n}-\mathbf{k}] \right| \\
&\quad = |\mathcal{N}_{\vec{\mathbf{M}}}^{\mathbf{k}} [\mathbf{n}-\mathbf{k}]|.
\end{aligned}$$

Now by Lemma 4.6 the result follows. ∎

The Lagrange inversion formula follows easily from Good's inversion formula. To see this implication, use the collection $\vec{\mathbf{M}} = (\mathscr{F}_i(M))_{i \in \mathscr{T}}$. Observe that $i \not\leq j$ implies that $(\mathscr{F}_i(M))^{\bullet(j)} = 0$. Thus the determinant in Good's inversion formula is upper triangular, and it follows that its value is the product of the elements on the main diagonal. Thus the Lagrange inversion formula is proved.

By equating the coefficients of the equation system $\mathbf{f}(\mathbf{x}) = \mathbf{x} \cdot (\mathbf{G} \circ \mathbf{f})(\mathbf{x})$ a set of recurrences occur for the coeffients of $\mathbf{f}(\mathbf{x})$, and this set of recurrences is easily seen to have a unique solution. Hence the equations $f_i(\mathbf{x}) = x_i \cdot (G_i \circ \mathbf{f})(\mathbf{x})$ for $i \in \mathscr{T}$ uniquely determines the collection $\mathbf{f}(\mathbf{x})$. Moreover, it is easy to see that $\mathbf{f}(\mathbf{x})$ is a summable collection.

THEOREM 4 (Good's Inversion Formula). *Let* $\mathbf{f}(\mathbf{x})$ *be a summable collection of formal power series and let* $\mathbf{G}(\mathbf{x})$ *be a collection of formal power series, such that for* $i \in \mathscr{T}$

$$f_i(\mathbf{x}) = x_i \cdot (G_i \circ \mathbf{f}(\mathbf{x})).$$

Let $J = \{i \in \mathscr{T} : n_i \neq 0\}$ *and assume that* $\mathbf{n} \geq \mathbf{k}$. *Then we have that*

$$[\mathbf{x}^{\mathbf{n}}] \prod_{i \in \mathscr{T}} f_i(\mathbf{x})^{k_i} = [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \det\left(\delta_{i,j} \cdot G_i(\mathbf{x})^{n_i} - x_j \frac{\partial G_i(\mathbf{x})}{\partial x_j} G_i(\mathbf{x})^{n_i - 1} \right)_{i,j \in J}.$$

Proof. Let M_i be a colored species and $G_i(\mathbf{x})$ its generating function. That is, $G_i(\mathbf{x}) = \mathrm{card}(M_i; \mathbf{x})$. Let $\bar{f}_i(\mathbf{x}) = \mathrm{card}(A_{\vec{\mathbf{M}}}^{(i)}; \mathbf{x})$ where $A_{\vec{\mathbf{M}}}^{(i)}$ is the colored species of $\vec{\mathbf{M}}$-enriched colored trees with root of color i. Since $A_{\vec{\mathbf{M}}}^{(i)} = X_i \cdot (M_i \circ \vec{\mathbf{A}}_{\vec{\mathbf{M}}})$, we get

$$\begin{aligned}\bar{f}_i(\mathbf{x}) &= \operatorname{card}(A_{\vec{\mathbf{M}}}^{(i)}; \mathbf{x}) \\ &= \operatorname{card}(X_i \cdot (M_i \circ \vec{\mathbf{A}}_{\vec{\mathbf{M}}}); \mathbf{x}) \\ &= x_i \cdot (\operatorname{card}(M_i; \mathbf{x}) \circ \operatorname{card}(\vec{\mathbf{A}}_{\vec{\mathbf{M}}}; \mathbf{x})) \\ &= x_i \cdot (G_i \circ \bar{\mathbf{f}})(\mathbf{x}).\end{aligned}$$

But $f_i(\mathbf{x})$ is uniquely determined by the above equation. Hence $f_i(\mathbf{x}) = \bar{f}_i(\mathbf{x}) = \operatorname{card}(A_{\vec{\mathbf{M}}}^{(i)}; \mathbf{x})$.

Now the left-hand side of the species version of Good's inversion formula is equal to

$$\begin{aligned}|(\Gamma_{\mathbf{k}}(\vec{\mathbf{A}}_{\vec{\mathbf{M}}}))[\mathbf{n}]| &= \left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right] \operatorname{card}(\Gamma_{\mathbf{k}}(\vec{\mathbf{A}}_{\vec{\mathbf{M}}}); \mathbf{x}) \\ &= \left[\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}\right]\left(\frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} \circ \mathbf{f}\right)(\mathbf{x}) \\ &= \frac{\mathbf{n}!}{\mathbf{k}!}[\mathbf{x}^{\mathbf{n}}](\mathbf{x}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{x}).\end{aligned}$$

And the right-hand side

$$\begin{aligned}&\binom{\mathbf{n}}{\mathbf{k}} \cdot |(\det(\delta_{i,j} M_i^{n_i} - M_i^{\bullet(j)} \cdot M_i^{n_i - 1})_{i,j \in J})[\mathbf{n} - \mathbf{k}]| \\ &\quad = \binom{\mathbf{n}}{\mathbf{k}} \cdot \left[\frac{\mathbf{x}^{\mathbf{n}-\mathbf{k}}}{(\mathbf{n}-\mathbf{k})!}\right] \operatorname{card}(\det(\delta_{i,j} M_i^{n_i} - M_i^{\bullet(j)} \cdot M_i^{n_i - 1})_{i,j \in J}; \mathbf{x}) \\ &\quad = \frac{\mathbf{n}!}{\mathbf{k}!} \cdot [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \det\left(\delta_{i,j} G_i(\mathbf{x})^{n_i} - x_j \cdot \frac{\partial G_i(\mathbf{x})}{\partial x_j} \cdot G_i(\mathbf{x})^{n_i - 1}\right)_{i,j \in J}.\end{aligned}$$

Thus we have proven the theorem for formal power series $G_i(\mathbf{x})$ such that $[\mathbf{x}^{\mathbf{n}}/\mathbf{n}!]\, G_i(\mathbf{x})$ is a nonnegative integer for all multi-indices $\mathbf{n}$. By the principle of extension of algebraic identities the theorem follows for all $\mathbf{G}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$. ∎

12. Umbral Calculus

We give here a short review of some result in the infinite variated multivariable umbral calculus. The theory was developed in [C] by W. Chen. We only present those results that we need for this presentation.

Let $\mathscr{K}[\mathbf{x}]$ be the set of all polynomials in the variables $(x_i)_{i \in \mathscr{T}}$. An operator M on $\mathscr{K}[\mathbf{x}]$ is a linear map from $\mathscr{K}[\mathbf{x}]$ to itself. Three classical operators are

(i) The partial differentiation with respect to x_i. That is, the map

$$D_i p(\mathbf{x}) = \frac{\partial p(\mathbf{x})}{\partial x_i}.$$

(ii) The multiplication with respect to x_i,

$$\underline{\mathbf{x}}_i p(\mathbf{x}) = x_i p(\mathbf{x}).$$

(iii) The shift operator. Let $\mathbf{a}$ be a vector, then the shift is defined to be

$$E^{\mathbf{a}} p(\mathbf{x}) = p(\mathbf{x} + \mathbf{a}).$$

We say that an operator T is invertible if there is another operator S such that $TS = 1$, where 1 is the identity operator.

DEFINITION 12.1. An operator T is called *shift invariant* if it commutes with all shift operators; that is, for every vector $\mathbf{a}$,

$$TE^{\mathbf{a}} = E^{\mathbf{a}} T.$$

From [C] we have the following classification of shift invariant operators.

PROPOSITION 12.1. *An operator T on $\mathscr{K}[\mathbf{x}]$ is shift invariant if and only if it is a formal power series in the differential operators $(D_i)_{i \in \mathscr{T}}$. Thus we can write*

$$T = \sum_{\mathbf{n}} a_{\mathbf{n}} \frac{\mathbf{D}^{\mathbf{n}}}{\mathbf{n}!}.$$

The formal power series $\sum_{\mathbf{n}} a_{\mathbf{n}} (\mathbf{t}^{\mathbf{n}}/\mathbf{n}!)$ is called the indicator series of T.

DEFINITION 12.2. A *delta operator* is a shift invariant operator Q such that $Q1 = 0$.

DEFINITION 12.3. A *summable collection* of delta operators $\mathbf{Q} = (Q_i)_{i \in \mathscr{T}}$ is a set of delta operators Q_i indexed by the set $\mathscr{T}$, such that their indicator sequences are summable.

We say that a summable collection of delta opererators $\mathbf{Q}$ is *admissible* if there exists a summable collection of formal power series $\mathbf{g}(\mathbf{t}) = (g_i(\mathbf{t}))_{i \in \mathcal{I}}$ such that

$$(\mathbf{q} \circ \mathbf{g})(\mathbf{t}) = \mathbf{t},$$

or

$$(q_i \circ \mathbf{g})(\mathbf{t}) = t_i,$$

where $q_i(\mathbf{t})$ is the indicator sequence of Q_i. If $\mathbf{Q}$ is admissible, we denote $g_i(\mathbf{t})$ by $q_i^{\langle -1 \rangle}(\mathbf{t})$ and $\mathbf{g}(\mathbf{t})$ by $\mathbf{q}^{\langle -1 \rangle}(\mathbf{t})$.

PROPOSITION 12.2. *Let $\mathbf{Q} = (Q_i)_{i \in \mathcal{I}}$ be a summable collection of delta operators. Then there exists a unique polynomial sequence $p_{\mathbf{n}}(\mathbf{x})$, indexed by multi-indices $\mathbf{n}$, such that*

$$Q_i p_{\mathbf{n}}(\mathbf{x}) = n_i p_{\mathbf{n} - \mathbf{e}_i}(\mathbf{x}),$$

and

$$p_{\mathbf{n}}(0) = \delta_{\mathbf{n},0}.$$

Such a sequence is called a basic sequence of the collection of delta operators $\mathbf{Q}$.

PROPOSITION 12.3. *Let $\mathbf{Q}$ be a summable collection of delta operators, which is admissible. Assume that Q_i has indicator series $q_i(\mathbf{t})$. Let $(p_{\mathbf{n}}(\mathbf{x}))$ be the basic sequence of $\mathbf{Q}$. Then we have*

$$\sum_{\mathbf{n}} p_{\mathbf{n}}(\mathbf{x}) \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \exp\left(\sum_{i \in \mathcal{I}} x_i q_i^{\langle -1 \rangle}(\mathbf{t}) \right),$$

where $q_i^{\langle -1 \rangle}(\mathbf{t})$ is the inverse defined above.

13. THE GENERAL TRANSFER FORMULA AND GOOD'S INVERSION FORMULA

DEFINITION 13.1. The *Pincherle derivate* of an operator T is defined by

$$\partial_i(T) = T\underline{\mathbf{x}}_i - \underline{\mathbf{x}}_i T.$$

Observe that the indicator sequence of $\partial_i(T)$ is the derivative of the indicator sequence of T with respect to t_i.

LEMMA 13.1. *For a formal power series $M(\mathbf{x})$ we have that*

$$[\mathbf{x}^{\mathbf{k}}]\, M(\mathbf{D})\, \mathbf{x}^{\mathbf{n}} = [\mathbf{x}^{\mathbf{n}-\mathbf{k}}]\, \frac{\mathbf{n}!}{\mathbf{k}!} M(\mathbf{x}).$$

Proof. Since the identity is linear in $M(\mathbf{x})$, it is enough to consider the case $M(\mathbf{x}) = \mathbf{x}^{\mathbf{h}}$

$$[\mathbf{x}^{\mathbf{k}}]\, \mathbf{D}^{\mathbf{h}}\mathbf{x}^{\mathbf{n}} = [\mathbf{x}^{\mathbf{k}}](\mathbf{n})_{\mathbf{h}}\, \mathbf{x}^{\mathbf{n}-\mathbf{h}}$$
$$= \delta_{\mathbf{k},\mathbf{n}-\mathbf{h}} \cdot (\mathbf{n})_{\mathbf{h}}$$
$$= \delta_{\mathbf{n}-\mathbf{k},\mathbf{h}} \cdot (\mathbf{n})_{\mathbf{n}-\mathbf{k}}$$
$$= [\mathbf{x}^{\mathbf{n}-\mathbf{k}}]\, \frac{\mathbf{n}!}{\mathbf{k}!}\, \mathbf{x}^{\mathbf{h}}. \quad ∎$$

LEMMA 13.2. *Let* $\mathbf{Q}$ *be a summable collection of delta operators, where* Q_i *has indicator series* $f_i^{\langle -1 \rangle}(\mathbf{t})$. *Assume that* $\mathbf{Q}$ *is admissible. Then we have*

$$p_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}\, \frac{\mathbf{n}!}{\mathbf{k}!}\, [\mathbf{t}^{\mathbf{n}}](\mathbf{t}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{t}),$$

where $(p_{\mathbf{n}}(\mathbf{x}))$ *is the basic sequence associated with* $\mathbf{Q}$.

Proof. We have by Proposition 12.3 that

$$\sum_{\mathbf{n}} p_{\mathbf{n}}(\mathbf{x}) \cdot \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} = \exp\left(\sum_{i \in \mathscr{T}} x_i \cdot f_i(\mathbf{t}) \right).$$

Thus by looking at the coefficient of $\mathbf{t}^{\mathbf{n}}/\mathbf{n}!$ we get

$$p_{\mathbf{n}}(\mathbf{x}) = \left[\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \right] \exp\left(\sum_{i \in \mathscr{T}} x_i \cdot f_i(\mathbf{t}) \right)$$
$$= \mathbf{n}!\, [\mathbf{t}^{\mathbf{n}}] \sum_{\mathbf{k}} \prod_{i \in \mathscr{T}} \frac{(x_i \cdot f_i(\mathbf{t}))^{k_i}}{k_i!}$$
$$= \mathbf{n}!\, [\mathbf{t}^{\mathbf{n}}] \sum_{\mathbf{k}} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} \prod_{i \in \mathscr{T}} f_i(\mathbf{t})^{k_i}$$
$$= \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} [\mathbf{t}^{\mathbf{n}}]\, \frac{\mathbf{n}!}{\mathbf{k}!}\, (\mathbf{t}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{t}). \quad ∎$$

THEOREM 5 (The General Transfer Formula). *Let* $\mathbf{Q} = (Q_i)_{i \in \mathscr{T}}$ *be a summable collection of delta operators, such that* $\mathbf{Q}$ *is admissible. Assume that we can write* $Q_i = D_i P_i$, *where* P_i *is an invertible shift invariant operator. Let* $(p_{\mathbf{n}}(\mathbf{x}))$ *be the basic sequence of the summable collection* $\mathbf{Q}$. *Let* $J = \{ i \in \mathscr{T} : n_i \neq 0 \}$. *Then we have*

$$p_{\mathbf{n}}(\mathbf{x}) = \det(\delta_{i,j} \cdot P_i^{-n_i} - D_j\, \partial_j(P_i^{-1})\, P_i^{-n_i+1})_{i,j \in J}\, \mathbf{x}^{\mathbf{n}}.$$

Moreover, this formula is equivalent to Good's inversion formula.

Proof. First we prove that Good's inversion formula implies the general transfer formula. Let $h_i(\mathbf{t})$ be the indicator series of Q_i. Since Q_i is a delta operator we know that $h_i^{\langle -1\rangle}(\mathbf{t}) = f_i(\mathbf{t})$ exists. Moreover since P_i is invertible let $G_i(\mathbf{t})$ be the indicator series of P_i^{-1}. Hence $G_i(\mathbf{D}) = P_i^{-1}$. Thus we have

$$f_i^{\langle -1\rangle}(\mathbf{t}) = t_i \cdot G_i^{-1}(\mathbf{t}).$$

This equation is equivalent to

$$\mathbf{f}_i^{\langle -1\rangle}(\mathbf{t}) = \mathbf{t} \cdot \mathbf{G}^{-1}(\mathbf{t}),$$

which can be written as

$$\mathbf{f}(\mathbf{t}) = \mathbf{t} \cdot (\mathbf{G} \circ \mathbf{f})(\mathbf{t}).$$

Thus $f_i(\mathbf{t}) = t_i \cdot (G_i \circ \mathbf{f})(\mathbf{t})$. Let $(p_{\mathbf{n}}(\mathbf{x}))$ be the basic sequence associated with $\mathbf{Q}$. Thus by Lemma 13.2, by Good's inversion formula, and by Lemma 13.1, we have that

$$\begin{aligned} p_{\mathbf{n}}(\mathbf{x}) &= \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \frac{\mathbf{n}!}{\mathbf{k}!} [\mathbf{t}^{\mathbf{n}}](\mathbf{t}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{t}) \\ &= \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \frac{\mathbf{n}!}{\mathbf{k}!} [\mathbf{x}^{\mathbf{n}}](\mathbf{x}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{x}) \\ &= \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \frac{\mathbf{n}!}{\mathbf{k}!} [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \det\left(\delta_{i,j} \cdot G_i(\mathbf{x})^{n_i} - x_j \frac{\partial G_i(\mathbf{x})}{\partial x_j} G_i(\mathbf{x})^{n_i - 1}\right)_{i,j \in J} \\ &= \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} [\mathbf{x}^{\mathbf{k}}] \det(\delta_{i,j} \cdot G_i(\mathbf{D})^{n_i} - D_j \,\partial_j(G_i(\mathbf{D}))\, G_i(\mathbf{D})^{n_i - 1})_{i,j \in J}\, \mathbf{x}^{\mathbf{n}} \\ &= \det(\delta_{i,j} \cdot G_i(\mathbf{D})^{n_i} - D_j \,\partial_j(G_i(\mathbf{D}))\, G_i(\mathbf{D})^{n_i - 1})_{i,j \in J}\, \mathbf{x}^{\mathbf{n}} \\ &= \det(\delta_{i,j} \cdot P_i^{-n_i} - D_j \,\partial_j(P_i^{-1})\, P_i^{-n_i + 1})_{i,j \in J}\, \mathbf{x}^{\mathbf{n}}, \end{aligned}$$

which proves the general formula.

It is easy to see that the general transfer formula implies Good's inversion formula. First assume that $G_i(\mathbf{x})^{-1}$ exists for all $i \in \mathscr{T}$. Since $f_i(\mathbf{x}) = x_i \cdot (G \circ \mathbf{f})(\mathbf{x})$, we know that $\mathbf{f}^{\langle -1\rangle}(\mathbf{x})$ exist. Let Q be the plethystic delta operator with indicator series $f^{\langle -1\rangle}(\mathbf{t})$. Then by the same list of equalities as above we conclude that

$$\begin{aligned} &\sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \frac{\mathbf{n}!}{\mathbf{k}!} [\mathbf{x}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{x}) \\ &\qquad = \sum_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \frac{\mathbf{n}!}{\mathbf{k}!} [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \det\left(\delta_{i,j} \cdot G_i(\mathbf{x})^{n_i} - x_j \frac{\partial G_i(\mathbf{x})}{\partial x_j} G_i(\mathbf{x})^{n_i - 1}\right)_{i,j \in J}. \end{aligned}$$

Take the coefficient of $\mathbf{x}^{\mathbf{k}}$ on both sides and we obtain the Good's inversion formula,

$$[\mathbf{x}^{\mathbf{n}}](\mathbf{x}^{\mathbf{k}} \circ \mathbf{f})(\mathbf{x}) = [\mathbf{x}^{\mathbf{n}-\mathbf{k}}] \det\left(\delta_{i,j} \cdot G_i(\mathbf{x})^{n_i} - x_j \frac{\partial G_i(\mathbf{x})}{\partial x_j} G_i(\mathbf{x})^{n_i - 1}\right)_{i,j \in J}.$$

Thus the implication is proved in the case when $G_i(\mathbf{x})^{-1}$ exists for all $i \in \mathscr{T}$.

To complete the proof of the implication, consider the coefficients $b_{\mathbf{n},m}$ of the collection $\mathbf{G}(\mathbf{x}) = (G_m(\mathbf{x}))_{m \in \mathscr{T}}$ as indeterminates. That is, $G_m(\mathbf{x}) = \sum_{\mathbf{n}} b_{\mathbf{n},m} \mathbf{x}^{\mathbf{n}}$. Let now

$$a_{\mathbf{n},m} = [\mathbf{x}^{\mathbf{n}-\mathbf{e}_m}] \det\left(\delta_{i,j} \cdot G_i(\mathbf{x})^{n_i} - x_j \frac{\partial G_i(\mathbf{x})}{\partial x_j} G_i(\mathbf{x})^{n_i - 1}\right)_{i,j \in J}.$$

Observe that $a_{\mathbf{n},m}$ is by the above equation expressed as a polynomial in the indeterminates $b_{\mathbf{n},m}$. Let $\bar{f}_m(\mathbf{x}) = \sum_{\mathbf{n}} a_{\mathbf{n},m} \mathbf{x}^{\mathbf{n}}$. We claim that $\bar{f}_m(\mathbf{x}) = x_m \cdot (G_m \circ \mathbf{f})(\mathbf{x})$. Compare coefficients of both sides. The equations that arise are polynomial identities in the indeterminates $b_{\mathbf{n},m}$. But we have shown above that these identities are true in the case when $b_{0,m}$ is nonzero for each $m \in \mathscr{T}$. By the principle of extension of algebraic identities, the polynomial identities follow and the claim is proved. By the uniqueness of the collection $\mathbf{f}$ we conclude that $f_m(\mathbf{x}) = \bar{f}_m(\mathbf{x})$. Now to prove Good's inversion formula, use that it is a polynomial identity in $b_{\mathbf{n},m}$ and apply again the principle of extension of algebraic identities. This argument finishes up the implication and thus the equivalence is proved. ∎

14. The Plethystic Umbral Calculus

Assume now that our index set $\mathscr{T}$ is given the structure of a c-monoid.

Definition 14.1. The *Frobenius operator* of a shift invariant operator $T = \sum_{\mathbf{n}} a_{\mathbf{n}}(\mathbf{D}^{\mathbf{n}}/\mathbf{n}!)$ is defined to be

$$\mathscr{F}_i(T) = \sum_{\mathbf{n}} a_{\mathbf{n}} \frac{\mathbf{D}^{v_i(\mathbf{n})}}{\mathbf{n}!}.$$

Thus we have that

$$\mathscr{F}_i(D_j) = D_{i \cdot j}.$$

Definition 14.2. A *plethystic delta operator* is a shift invariant operator Q such that $Q1 = 0$ and Qx_1 is a nonzero constant.

Observe that $\mathbf{Q} = (\mathscr{F}_i(Q))_{i \in \mathscr{T}}$ is a summable sequence of delta operators. Hence by Proposition 12.2 there exists a unique polynomial sequence $p_{\mathbf{n}}(\mathbf{x})$, index by multi-indices $\mathbf{n}$, such that

$$\mathscr{F}_i(Q)\, p_{\mathbf{n}}(\mathbf{x}) = n_i\, p_{\mathbf{n} - \mathbf{e}_i}(\mathbf{x}),$$

and

$$p_{\mathbf{n}}(0) = \delta_{\mathbf{n},0}.$$

Such a sequence is called a *plethystic basic sequence* of the plethystic delta operator Q.

15. The Plethystic Transfer Formula and Lagrange Inversion Formula

Choose a linear ordering $(\mathscr{T}, \preccurlyeq)$ of the c-monoid $\mathscr{T}$, which is compatible with the divisibility ordering $(\mathscr{T}, \preccurlyeq)$. That is, $(\mathscr{T}, \preccurlyeq)$ is a total order such that $i \leqslant j$ implies $i \preccurlyeq j$.

Example 15.1. In the c-monoid of positive integers under multiplication, $(\mathbb{P}, \cdot, 1)$ we can choose the linear ordering $(\mathbb{P}, \preccurlyeq)$ to be the natural linear ordering on positive integers. Note that this linear ordering is compatible with the divisibility ordering.

We employ the following rule when multiplying noncommutative products over the index set J, where J is a finite subset of $\mathscr{T}$. Multiply the factors in the order given by $(\mathscr{T}, \preccurlyeq)$. That is

$$\prod_{i \in J} A_i = A_{i_1} \cdot A_{i_2} \cdots A_{i_m},$$

where $J = \{i_1, i_2, ..., i_m\}$ and $i_2 \prec i_2 \prec \cdots \prec i_m$.

Note that the plethystic inverse of $f(\mathbf{x})$ exists, that is, $f^{\langle -1 \rangle}(\mathbf{x})$, is equivalent to that the inverse of $G(\mathbf{x})$ exists, which is $G^{-1}(\mathbf{x})$.

Proposition 15.1. *Let $G(\mathbf{x})$ be an invertible formal power series, and suppose $f(\mathbf{x}) = x_1 \cdot (G \circ f)(\mathbf{x})$. Let P be a shift invariant operator with indicator series $G^{-1}(\mathbf{x})$. Then*

$$\left(\prod_{i \in J} \underline{\mathbf{x}}_i P_i^{-n_i} \underline{\mathbf{x}}_i^{n_i - 1}\right) 1 = \left(\prod_{i \in J} H_i(\mathbf{D})\right) \mathbf{x}^{\mathbf{n}},$$

where

$$H_i(\mathbf{x}) = \mathscr{F}_i\left(G(\mathbf{x})^{n_i} - x_1 \cdot \frac{\partial G(\mathbf{x})}{\partial x_1} \cdot G(\mathbf{x})^{n_i - 1}\right).$$

Proof. By the definition of Pincherle derivate, we have that

$$\prod_{i \in J} \underline{\mathbf{x}}_i P_i^{-n_i} \underline{\mathbf{x}}_i^{x_i - 1} = \prod_{i \in J} (P_i^{-n_i} \underline{\mathbf{x}}_i^{n_i} - \partial_i(P_i^{-n_i})\, \underline{\mathbf{x}}_i^{n_i - 1}). \tag{1}$$

Let

$$K_{i,I} = \begin{cases} \partial_i(P_i^{-n_i}), & \text{if } i \in I \\ P_i^{-n_i} & \text{if } i \in J - I \end{cases},$$

and

$$L_{i,L} = \begin{cases} \underline{\mathbf{x}}_i^{n_i - 1} & \text{if } i \in I \\ \underline{\mathbf{x}}_i^{n_i} & \text{if } i \in J - I \end{cases}.$$

Observe that if $i \prec j$, then $i \not\succcurlyeq j$, so the operators $L_{i,I}$ and $K_{j,I}$ commutes. That is,

$$L_{i,I} K_{j,I} = K_{j,I} L_{i,I}.$$

Note also that the $K_{i,I}$ and $K_{j,I}$ commutes and that the $L_{i,I}$ and $L_{j,I}$ commutes. Thus we can expand the right-hand side of (1), and using these commuting relations

$$\begin{aligned}
\prod_{i \in J} \underline{\mathbf{x}}_i P_i^{-n_i} \underline{\mathbf{x}}_i^{n_i - 1} &= \sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in J} K_{i,I} L_{i,I} \\
&= \sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in J} K_{i,I} \cdot \prod_{i \in J} L_{i,I} \\
&= \sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in I} K_{i,I} \cdot \prod_{i \in J - I} K_{i,I} \cdot \prod_{i \in J} L_{i,I}.
\end{aligned}$$

Apply now the above operator identity to the polynomial 1, and we obtain

$$\left(\prod_{i \in J} \underline{\mathbf{x}}_i P_i^{-n_i} \underline{\mathbf{x}}_i^{n_i - 1}\right) 1 = \left(\sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in I} K_{i,I} \prod_{i \in J - I} K_{i,I}\right) \mathbf{x}^{\mathbf{n} - e_I}. \tag{2}$$

For $i \in I$ we then have that

$$\begin{aligned}
K_{i,I} &= \partial_i(P_i^{-n_i}) \\
&= \mathscr{F}_i(\partial_1(P^{-n_i})) \\
&= \mathscr{F}_i(\partial_1(G(\mathbf{D})^{n_i})) \\
&= \mathscr{F}_i(n_i G'(\mathbf{D})\, G(\mathbf{D})^{n_i - 1}).
\end{aligned}$$

Similarly for $i \in J - I$,

$$K_{i,I} = \mathscr{F}_i(G(\mathbf{D})^{n_i}).$$

Apply this now to Eq. (2):

$$\left(\prod_{i \in J} \underline{\mathbf{x}}_i P_i^{-n_i} \underline{\mathbf{x}}_i^{n_i - 1}\right) 1$$

$$= \left(\sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in I} \mathscr{F}_i(n_i G'(\mathbf{D})\, G(\mathbf{D})^{n_i - 1}) \cdot \prod_{i \in J} \mathscr{F}_i(G(\mathbf{D})^{n_i})\right) \mathbf{x}^{\mathbf{n} - e_I}$$

$$= \left(\sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in I} \mathscr{F}_i(n_i G'(\mathbf{D})\, G(\mathbf{D})^{n_i - 1})\right.$$

$$\left. \cdot \prod_{i \in J - I} \mathscr{F}_i(G(\mathbf{D})^{n_i}) \cdot \prod_{i \in I} \frac{1}{n_i} D_i\right) \mathbf{x}^{\mathbf{n}}$$

$$= \left(\sum_{I \subseteq J} (-1)^{|I|} \prod_{i \in I} \mathscr{F}_i(D_1 G'(\mathbf{D})\, G(\mathbf{D})^{n_i - 1}) \cdot \prod_{i \in J - I} \mathscr{F}_i(G(\mathbf{D})^{n_i})\right) \mathbf{x}^{\mathbf{n}}$$

$$= \left(\prod_{i \in J} \mathscr{F}_i(G(\mathbf{D})^{n_i} - D_1 G'(\mathbf{D})\, G(\mathbf{D})^{n_i - 1})\right) \mathbf{x}^{\mathbf{n}}$$

$$= \left(\prod_{i \in J} H_i(\mathbf{D})\right) \mathbf{x}^{\mathbf{n}}.$$

That completes the proof of the proposition. ∎

William Chen obtains in [C] the plethystic transfer formula.

THEOREM 6 (The Plethystic Transfer Formula). *Let P be an invertible shift invariant operator, and let $(p_{\mathbf{n}}(\mathbf{x}))$ be the plethystic basic sequence of the plethystic delta operator $Q = D_1 P$. Let P_i denote $\mathscr{F}_i(P)$. Then we have*

$$p_{\mathbf{n}}(\mathbf{x}) = \left(\prod_{i \in J} \underline{\mathbf{x}}_i P_i^{-n_i} \underline{\mathbf{x}}_i^{n_i - 1}\right) 1.$$

THEOREM 7. *The plethystic transfer formula and the plethystic Lagrange inversion formula are equivalent.*

Proof. Apply the plethystic case to Theorem 5. Observe that $i \not\leq j$ implies that $\partial_j(\mathscr{F}_i(P)^{-1}) = 0$. Hence the determinant will be upper triangular. Recall that the determinant of an upper triangular matrix is equal to the product of the elements on the main diagonal. Thus we know that the plethystic Lagrange inversion formula is equivalent to the formula

$$\begin{aligned} p_{\mathbf{n}}(\mathbf{x}) &= \prod_{i \in J} (P_i^{-n_i} - D_i \partial_i (P_i^{-n_i+1}) \mathbf{x}^{\mathbf{n}} \\ &= \prod_{i \in J} \mathscr{F}_i (P^{-n_i} - D_1 \partial_1 (P^{-1}) P^{-n_i+1}) \mathbf{x}^{\mathbf{n}} \\ &= \prod_{i \in J} \mathscr{F}_i (G(\mathbf{D})^{n_i} - D_1 G'(\mathbf{D}) G(\mathbf{D})^{n_i-1}) \mathbf{x}^{\mathbf{n}} \\ &= \prod_{i \in J} H_i(\mathbf{D}) \mathbf{x}^{\mathbf{n}}. \end{aligned}$$

Thus by the identity in Proposition 15.1 we conclude that the plethystic Lagrange inversion formula is equivalent to the plethystic transfer formula. ∎

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