# Lifting inequalities for polytopes*

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**Abstract**

We present a method of lifting linear inequalities for the flag $f$-vector of polytopes to higher dimensions. Known inequalities that can be lifted using this technique are the non-negativity of the toric $g$-vector and that the simplex minimizes the cd-index. We obtain new inequalities for 6-dimensional polytopes. In the last section we present the currently best known inequalities for dimensions 5 through 8.

## 1 Introduction

The flag $f$-vector of a convex polytope contains all the enumerative incidence information between the faces. Thus to classify the set of all possible flag $f$-vectors is one of the great open problems in discrete geometry. To date only partial results to this problem have been obtained. For the case when the polytopes are simplicial (and dually, simple), the problem reduces to classifying the $f$-vectors of simplicial polytopes. This major step was solved by the combined effort of Billera and Lee [7] and Stanley [19]. Returning to the general case, the classification of flag $f$-vectors of three-dimensional polytopes was done by Steinitz [24] almost one hundred years ago. By Euler's relation the number of edges $f_1$ is determined by the number vertices $f_0$ and the number of faces $f_2$. Steinitz proved that $f_0$ and $f_2$ satisfy the two inequalities

$$f_2 \leq 2 \cdot f_0 - 4 \quad \text{and} \quad f_0 \leq 2 \cdot f_2 - 4. \tag{1.1}$$

Interestingly, the reverse is also true. Given two integers $f_0$ and $f_2$ that satisfy the two inequalities in (1.1), there is a three-dimensional polytope with $f_0$ vertices and $f_2$ faces. For four-dimensional polytopes the problem remains open. The article by Bayer [1] contains the current state of knowledge for four-dimensional polytopes.

The first step toward classifying flag $f$-vectors was taken by Bayer and Billera [2]. They described all the linear redundancies occurring among the flag $f$-vector entries of a polytope. These relations are known as the generalized Dehn-Somerville relations. They imply that flag $f$-vectors of polytopes lie in a subspace of dimension $F_n$, where $F_n$ denotes the $n$th Fibonacci number.

The next natural step is to look for linear inequalities that the flag vectors of polytopes satisfy. One such example is the toric $g$-vector. It measures the intersection homology Betti numbers of the toric

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variety associated with a rational polytope. The entries of the toric $g$-vector are linear combinations of the entries of the flag $f$-vector. Stanley [21] proved that the toric $g$-vector of a rational polytope is non-negative using the hard Lefschetz theorem. Using rigidity theory Kalai [12] proved that the second entry of the toric $g$-vector of any polytope $P$ is non-negative. Recently Karu [14] proved the hard Lefschetz theorem for combinatorial intersection cohomology, and as consequence the toric $g$-vector is non-negative for all polytopes. More inequalities can be obtained by using a convolution due to Kalai [13]. However, this is far from being an exhaustive list. See the work of Stenson [25].

A different direction of research involves the **cd**-index, a non-commutative polynomial which encodes the flag $f$-vector of a polytope without linear redundancies [4]. Stanley [22] proved that the **cd**-index of a polytope has non-negative coefficients. This important result foreshadowed the central role the **cd**-index would later play in advancing the frontiers of polytopal inequalities. The next step was taken by Billera and Ehrenborg who proved that the **cd**-index is minimized coefficientwise on the $n$-dimensional simplex $\Sigma_n$ [5]. This gives a sharpening of Stanley's inequalities.

The purpose of this paper is to describe a new lifting technique for polytopal inequalities; see Theorem 3.1. Given a linear inequality on $k$-dimensional polytopes, we can produce inequalities in dimensions larger than $k$. For instance, when applying the lifting technique to the minimization inequalities of Billera-Ehrenborg, we obtain a large class of inequalities; see Theorem 3.7. One consequence is that the coefficients of the **cd**-index are increasing when replacing $\mathbf{c}^2$ with $\mathbf{d}$. Hence the **cd**-monomial with the largest coefficient in the **cd**-index of a polytope has no consecutive **c**'s; see Corollary 3.9. Another inequality that will generate more inequalities when lifted is the non-negativity of the toric $g$-vector; see Theorem 4.4.

Using our lifting technique we can now explicitly state the currently best known inequalities for polytopes of low dimensions. Dimension 4 has been described by Bayer [1]. We describe the inequalities for 5-dimensional polytopes in Section 5. Since one can deduce many inequalities by applying the Kalai convolution, we only present the irreducible inequalities for polytopes in dimensions 6 through 8. In the last section we discuss open problems and further research.

## 2 Preliminaries

Let $P$ be an $n$-dimensional polytope. For $S = \{s_1, \ldots, s_k\}$ a subset of $\{0, 1, \ldots, n-1\}$, define $f_S$ to be the number of flags (chains) of faces $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ such that $\dim(F_i) = s_i$. The $2^n$ values $f_S$ constitute the flag $f$-vector of the polytope $P$. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutative variables. For $S$ a subset of $\{0, \ldots, n-1\}$ define a polynomial $v_S$ of degree $n$ by letting $v_S = v_0 v_1 \cdots v_{n-1}$ where $v_i = \mathbf{a} - \mathbf{b}$ if $i \notin S$ and $v_i = \mathbf{b}$ otherwise. The **ab**-index $\Psi(P)$ of a polytope $P$ is defined by

$$\Psi(P) = \sum_S f_S \cdot v_S,$$

where $S$ ranges over all subsets of $\{0, \ldots, n-1\}$. The **ab**-index encodes the flag $f$-vector of a polytope $P$. Its use is demonstrated by the following theorem, due to Bayer and Klapper [4].

**Theorem 2.1** *Let $P$ be polytope. Then the* **ab***-index of $P$, $\Psi(P)$, can be written in terms of* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When $\Psi(P)$ is expressed in terms of $\mathbf{c}$ and $\mathbf{d}$, it is called the **cd**-index. Observe that $\mathbf{c}$ has degree 1 and $\mathbf{d}$ has degree 2. Hence there are $F_n$ **cd**-monomials of degree $n$, where $F_n$ is the $n$th Fibonacci number. The flag $f$-vector information is encoded as the coefficients of these monomials. Also knowing the **cd**-index of a polytope is the same as knowing the flag $f$-vector.

The existence of the **cd**-index is equivalent to the generalized Dehn-Somerville relations due to Bayer and Billera [2]. These relations are all the linear relations that hold among the entries of the flag $f$-vector. The **cd**-monomials offer an explicit linear basis for the subspace cut out by the generalized Dehn-Somerville relations.

In order to discuss inequalities for polytopes, define a bilinear form $\langle\cdot|\cdot\rangle : \mathbb{R}\langle\mathbf{c},\mathbf{d}\rangle \times \mathbb{R}\langle\mathbf{c},\mathbf{d}\rangle \longrightarrow \mathbb{R}$ by $\langle u|v\rangle = \delta_{u,v}$ for all **cd**-monomials $u$ and $v$. A linear functional $L$ on the flag $f$-vectors of $n$-dimensional polytopes can now be written in terms of the bilinear form as $L(P) = \langle z|\Psi(P)\rangle$, where $z$ is a **cd**-polynomial homogeneous of degree $n$.

Kalai's convolution is defined as follows; see [13]. Let $M$ and $L$ be two linear functionals on flag $f$-vectors of $m$- and $n$-dimensional polytopes, respectively. Define the linear functional $M * L$ on $(m+n+1)$-dimensional polytopes $P$ by

$$(M * L)(P) = \sum_{F} M(F) \cdot L(P/F),$$

where $F$ ranges over all $m$-dimensional faces of $P$ and $P/F$ denotes the face figure of $F$. It is straightforward to see that if $M$ and $L$ are non-negative on all polytopes then so is their convolution $M * L$.

Kalai's convolution defines a convolution on $\mathbb{R}\langle\mathbf{c},\mathbf{d}\rangle$ by

$$\langle z * w|\Psi(P)\rangle = \sum_{F} \langle z|\Psi(F)\rangle \cdot \langle w|\Psi(P/F)\rangle .$$

This convolution has an explicit expression in terms of **cd**-polynomials. The following result is independently due to Mahajan [15], Reading [18], and Stenson [25].

**Proposition 2.2** *For two* **cd**-*monomials $u$ and $v$ we have*

$$\begin{aligned} u\mathbf{c} * \mathbf{c}v &= 2 \cdot u\mathbf{c}^3 v + u\mathbf{dc}v + u\mathbf{cd}v, \\ u\mathbf{d} * \mathbf{c}v &= 2 \cdot u\mathbf{dc}^2 v + u\mathbf{d}^2 v, \\ u\mathbf{c} * \mathbf{d}v &= 2 \cdot u\mathbf{c}^2\mathbf{d}v + u\mathbf{d}^2 v, \\ u\mathbf{d} * \mathbf{d}v &= 2 \cdot u\mathbf{dcd}v. \end{aligned}$$

*Also we have* $1*1 = 2\cdot\mathbf{c}$, $1*\mathbf{c}v = 2\cdot\mathbf{c}^2 v+\mathbf{d}v$, $1*\mathbf{d}v = 2\cdot\mathbf{cd}v$, $u\mathbf{c}*1 = 2\cdot u\mathbf{c}^2+u\mathbf{d}$ *and* $u\mathbf{d}*1 = 2\cdot u\mathbf{dc}$.

**Sketch of proof:** Consider the coproduct $\Delta$ on $\mathbb{R}\langle\mathbf{c},\mathbf{d}\rangle$ that first appeared in [11]. It is defined by $\Delta(\mathbf{c}) = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{d}) = \mathbf{c} \otimes 1 + 1 \otimes \mathbf{c}$ and satisfies the Newtonian identity $\Delta(u \cdot v) =$

$\sum_u u_{(1)} \otimes u_{(2)} \cdot v + \sum_v u \cdot v_{(1)} \otimes v_{(2)}$. It is now enough to observe that the bilinear form $\langle \cdot | \cdot \rangle$ is a Laplace pairing, that is,

$$\langle u * v | w \rangle = \sum_w \langle u | w_{(1)} \rangle \cdot \langle v | w_{(2)} \rangle ;$$

see [10]. From these facts all the relations in the proposition follow. □

Proposition 2.2 can be rewritten into the following more compact form. Factor the monomial $u$ as $u = u_1 u_2$ where $u_2 = \mathbf{c}$ if $u$ ends with a $\mathbf{c}$ and $u_2 = 1$ otherwise. Similarly, factor $v = v_1 v_2$ where $v_1 = \mathbf{c}$ if $v$ begins with a $\mathbf{c}$ and $v_1 = 1$ otherwise. Then the Kalai convolution $u * v$ is equal to $u_1 p v_2$ where $p$ is determined by the table

| $u_2$ | $v_1$ | $p$ |
|---|---|---|
| 1 | 1 | $2\mathbf{c}$ |
| 1 | $\mathbf{c}$ | $2\mathbf{c}^2 + \mathbf{d}$ |
| $\mathbf{c}$ | 1 | $2\mathbf{c}^2 + \mathbf{d}$ |
| $\mathbf{c}$ | $\mathbf{c}$ | $2\mathbf{c}^3 + \mathbf{dc} + \mathbf{cd}$ |

As a corollary we obtain the following result:

**Corollary 2.3** *Let $u$, $q$, $r$ and $v$ be four* **cd***-monomials such that $u$ does not end in* **c** *and $v$ does not begin with* **c**. *Then the following associative law holds between the product and the Kalai convolution:*

$$u \cdot (q * r) \cdot v = (u \cdot q) * (r \cdot v).$$

As a remark, when $q$ differs from 1 we can omit the condition that $u$ does not end in **c**. Similarly, when $r$ differs from 1 we can omit the condition that $v$ does not begin with **c**. However, in what follows we will not be needing this slightly more general setting.

On the algebra $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ there is a natural antiautomorphism $w \longmapsto w^*$ defined by reversing each monomial; see [11]. This is also an antiautomorphism with respect to the Kalai convolution. On the geometric level it corresponds to the dual polytope $P^*$, that is, $\Psi(P^*) = \Psi(P)^*$. Hence for an inequality $\langle H | \Psi(P) \rangle \geq 0$ we also have the dual inequality $\langle H^* | \Psi(P) \rangle \geq 0$.

## 3 The lifting theorem

We now present our lifting theorem. It allows us to obtain more inequalities on the flag $f$-vectors of polytopes.

**Theorem 3.1** *Let $H$ be a* **cd***-polynomial such that the inequality $\langle H | \Psi(P) \rangle \geq 0$ holds for all (rational) polytopes $P$. Then for all (rational) polytopes $P$ we have the inequality*

$$\langle u \cdot H \cdot v | \Psi(P) \rangle \geq 0,$$

*where $u$ and $v$ are* **cd***-monomials such that $u$ does not end in* **c** *and $v$ does not begin with* **c**.

In order to prove this theorem, let us introduce two partial orders on **cd**-polynomials.

**Definition 3.2** *Let $H$, $z$ and $w$ be three* **cd**-*polynomials.*

1. *Define the relation $z \preceq_H w$ if we have $\langle u \cdot H \cdot v | w - z \rangle \geq 0$ for all* **cd**-*monomials $u$ and $v$ such that $u$ does not end with* **c** *and $v$ does not begin with* **c**.
2. *Define the relation $z \preceq'_H w$ if we have $\langle u \cdot H \cdot v | w - z \rangle \geq 0$ for all* **cd**-*monomials $u$ and $v$ such that $u$ does not end with* **c**, *$v$ does not begin with* **c** *and $v$ is different from $1$.*

Observe that in the definition of the relation $z \preceq'_H w$ the requirement that $v \neq 1$ implies that $v$ begins with a $\mathbf{d}$. Moreover, the conclusion of Theorem 3.1 can now be stated as $\Psi(P) \succeq_H 0$.

**Proposition 3.3** *The two relations $z \succeq'_H 0$ and $w \succeq_H 0$ together imply that $z \cdot \mathbf{c} + w \cdot \mathbf{d} \succeq'_H 0$.*

**Proof:** We would like to verify that $\langle u \cdot H \cdot v | z \cdot \mathbf{c} + w \cdot \mathbf{d} \rangle \geq 0$ for all **cd**-monomials $u$ and $v$ such that $u$ does not end with $\mathbf{c}$, $v$ does not begin with $\mathbf{c}$ and $v$ is different from 1. If $v = v' \cdot \mathbf{c}$ then $v' \neq 1$ and the left-hand side is given by $\langle u \cdot H \cdot v' | z \rangle$, which is non-negative by the assumption $z \succeq'_H 0$. If $v$ ends with a $\mathbf{d}$ then the left-hand side is non-negative by the relation $w \succeq_H 0$. □

**Proof of Theorem 3.1:** Assume without loss of generality that $H$ is homogeneous of degree $k$. Let $P$ be an $n$-dimensional polytope. Using the result of Bruggesser and Mani [9], there is a line shelling $F_1, \ldots, F_m$ of the polytope $P$, where $F_1, \ldots, F_m$ are the facets of $P$. Consider the following two statements:

$(a)$ The **cd**-index $\Psi(P)$ satisfies $\Psi(P) \succeq_H 0$.

$(b)$ The following string of inequalities holds, where $\Gamma'$ denotes the semisuspension of the cell complex $\Gamma$; see [5, 22]:

$$0 \preceq'_H \Psi(F'_1) \preceq'_H \Psi((F_1 \cup F_2)') \preceq'_H \cdots \preceq'_H \Psi((F_1 \cup \cdots \cup F_{m-1})') = \Psi(P).$$

We will prove these two statements by induction on the dimension $n$. The induction basis is $n \leq k+1$. In that case observe that there is nothing to prove in statement $(b)$. In statement $(a)$ there is nothing to prove, unless $n = k$, in which the statement is just the assumption of the theorem.

We next prove $(a)$ in dimension $n-2$ and $(b)$ in dimension $n-1$ imply $(b)$ in dimension $n$. By [5, Lemma 4.2] (also in the work of Stanley [22]) we have that

$$\Psi((F_1 \cup \cdots \cup F_r)') - \Psi((F_1 \cup \cdots \cup F_{r-1})') \quad = \quad (\Psi(F_r) - \Psi(\Lambda')) \cdot \mathbf{c} + \Psi(\partial\Lambda) \cdot \mathbf{d},$$

where $\Lambda = (F_1 \cup \cdots \cup F_{r-1}) \cap F_r$. By induction we know that $\Psi(F_r) - \Psi(\Lambda') \succeq'_H 0$. Now consider the set $\partial\Lambda$. We know that $\Lambda$ is the union of the facets of $F_r$ that form the beginning of a line shelling.

Thus $\partial\Lambda$ is combinatorially equivalent to an $(n-2)$-dimensional polytope and hence by induction $\Psi(\partial\Lambda) \succeq_H 0$. Now by Proposition 3.3 we obtain that

$$\Psi((F_1 \cup \cdots \cup F_r)') - \Psi((F_1 \cup \cdots \cup F_{r-1})') \succeq'_H 0,$$

completing the proof of $(b)$.

We prove $(b)$ in dimension $n$ implies $(a)$ in dimension $n$ by two cases. The first case when $v$ is different from 1 follows directly by transitivity of all the order relations in $(b)$, that is, we have $0 \preceq'_H \Psi(P)$. For the second case when $v$ is equal to 1 we have $u$ is different from 1 since $\deg(u) + \deg(v) = n-k \geq 2$. Now the result follows by applying the inequality $0 \preceq'_H \Psi(P)$ to the dual polytope $P^*$ using the dual order $\preceq'_{H^*}$.

Observe that when $\langle H|\Psi(P)\rangle \geq 0$ holds for rational polytopes $P$, the presented proof holds with a few remarks. In the first part observe that $\Lambda'$ is a shelling component of the rational polytope $F_r$, hence $\Psi(F_r) - \Psi(\Lambda') \succeq'_H 0$. Moreover, $\partial\Lambda$ can be obtained by a rational projection so that it is combinatorially equivalent to a rational polytope. Hence the first part of the proof holds in the rational case. Since the dual polytope of a rational polytope is also rational we have that second part of the proof also holds for rational polytopes. □

We present two examples of Theorem 3.1.

**Example 3.4** We have that $\langle \mathbf{c}^k|\Psi(P)\rangle = \delta_{k,\dim(P)} \geq 0$. Since every **cd**-monomial $w$ factors into the form $w = \mathbf{c}^k \cdot v$, where $v$ does not begin $\mathbf{c}$, we have that $\langle w|\Psi(P)\rangle \geq 0$. This is Stanley's result that the **cd**-index of a polytope has non-negative coefficients; see [22].

The next example shows that it is not necessary to lift inequalities obtained by the Kalai convolution. Instead, it is better to first lift each term and then convolve the lifted inequalities.

**Example 3.5** Assume that for $i = 1, 2$ we have the inequalities $\langle H_i|\Psi(P)\rangle \geq 0$. By Corollary 2.3 the lifting of the convolved inequality gives

$$\langle (u \cdot H_1) * (H_2 \cdot v)|\Psi(P)\rangle = \langle u \cdot (H_1 * H_2) \cdot v|\Psi(P)\rangle \geq 0. \tag{3.1}$$

Now instead lift each of the inequalities and then convolute. This gives

$$\langle (u_1 \cdot H_1 \cdot v_1) * (u_2 \cdot H_2 \cdot v_2)|\Psi(P)\rangle \geq 0. \tag{3.2}$$

Observe that the inequality in (3.1) is a special case of inequality in (3.2).

We end this section with a large class of inequalities. For $q$ a **cd**-monomial of degree $k$, let $\Delta_q$ denote the coefficient of $q$ in the **cd**-index of the $k$-dimensional simplex, $\Psi(\Sigma_k)$.

**Lemma 3.6** *For a* **cd***-monomial $q$ and non-negative integers $i$ and $j$, we have*

$$\Delta_{\mathbf{c}^i \cdot q \cdot \mathbf{c}^j} \geq \Delta_q.$$

**Proof:** By symmetry it is enough to prove that $\Delta_{q\cdot\mathbf{c}} \geq \Delta_q$. We have

$$\begin{aligned} \langle q\cdot\mathbf{c}|\Psi(\Sigma_{k+1})\rangle &= \langle q\cdot\mathbf{c}|\Psi(\Sigma_k)\cdot\mathbf{c} + G(\Psi(\Sigma_k))\rangle \\ &\geq \langle q\cdot\mathbf{c}|\Psi(\Sigma_k)\cdot\mathbf{c}\rangle \\ &= \langle q|\Psi(\Sigma_k)\rangle , \end{aligned}$$

where the first step is the pyramid operation developed in [11] and the second step uses that the derivation $G$ introduced in [11] preserves non-negativity. □

**Theorem 3.7** *Let $P$ be a polytope of dimension $n$ and let $u$, $q$ and $v$ be three* **cd***-monomials such the sum of the degrees of $u$, $q$ and $v$ is $n$ and the degree of $q$ is $k$. Then we have*

$$\langle u\cdot q\cdot v|\Psi(P)\rangle \geq \Delta_q \cdot \left\langle u\cdot\mathbf{c}^k\cdot v|\Psi(P)\right\rangle . \tag{3.3}$$

**Proof:** Factor $u$ and $v$ so that $u = u'\cdot\mathbf{c}^i$, $v = \mathbf{c}^j\cdot v'$, and $u'$ does not end in $\mathbf{c}$ and $v'$ does not begin with $\mathbf{c}$. Finally, let $q' = \mathbf{c}^i\cdot q\cdot\mathbf{c}^j$ and $k' = k+i+j$. Thus the monomial $q'$ has degree $k'$. Billera and Ehrenborg [5] proved that the **cd**-index over all $k'$-dimensional polytopes is coefficientwise minimized on the $k'$-dimensional simplex $\Sigma_{k'}$. Apply this to the **cd**-monomial $q'$, we have $\langle q'|\Psi(P)\rangle \geq \langle q'|\Psi(\Sigma_{k'})\rangle = \Delta_{q'} = \Delta_{q'}\cdot\left\langle\mathbf{c}^{k'}|\Psi(P)\right\rangle$. Thus we can write $\left\langle q' - \Delta_{q'}\cdot\mathbf{c}^{k'}|\Psi(P)\right\rangle \geq 0$. Lifting this inequality we have $\left\langle u'\cdot(q' - \Delta_{q'}\cdot\mathbf{c}^{k'})\cdot v'|\Psi(P)\right\rangle \geq 0$. Expanding this inequality in terms of $u$, $q$ and $v$ and appling Lemma 3.6, we obtain the desired result. □

The first dimension that Theorem 3.7 says something new about polytopes is dimension 6. This is the case when $u = 1$, $q = \mathbf{dc}^2$ and $v = \mathbf{d}$, and the dual case $u = \mathbf{d}$, $q = \mathbf{c}^2\mathbf{d}$ and $v = 1$. See inequalities (5.6.3) and (5.6.3*) in Theorem 5.6. Moreover, allowing the two monomials $u$ and $v$ in Theorem 3.7 to end, respectively begin, with a $\mathbf{c}$ does not give any sharper inequalities.

We have two direct corollaries of Theorem 3.7:

**Corollary 3.8** *The* **cd***-index of a polytope $P$ satisfies then following inequalities:*

$$\begin{aligned} \left\langle u\cdot\mathbf{c}^i\mathbf{d}\mathbf{c}^j\cdot v|\Psi(P)\right\rangle &\geq \left(\binom{i+j+2}{i+1} - 1\right)\cdot\left\langle u\cdot\mathbf{c}^{i+j+2}\cdot v|\Psi(P)\right\rangle , \\ \left\langle u\cdot\mathbf{d}^i\cdot v|\Psi(P)\right\rangle &\geq E_{2i+1}/2^i\cdot\left\langle u\cdot\mathbf{c}^{2i}\cdot v|\Psi(P)\right\rangle , \end{aligned}$$

*for any two* **cd***-monomials $u$ and $v$ and where $E_n$ denotes the $n$th Euler number.*

**Proof:** By Theorem 3.7 it is enough to observe that $\Delta_{\mathbf{c}^i\mathbf{d}\mathbf{c}^j} = \binom{i+j+2}{i+1} - 1$, and $\Delta_{\mathbf{d}^i} = E_{2i+1}/2^i$. The second statement follows from [6, Proposition 8.2]. □

**Corollary 3.9** *Let $P$ be a polytope. Then the largest* **cd**-*coefficient in $\Psi(P)$ corresponds to a* **cd**-*monomial having no consecutive* **c**'*s.*

**Proof:** Apply Theorem 3.7 with $q = \mathbf{d}$ recalling that $\Delta_{\mathbf{d}} = 1$. $\Box$

Observe that the maximum is not necessarily unique, as demonstrated by the **cd**-index of a triangle, $\Psi(\Sigma_2) = \mathbf{c}^2 + \mathbf{d}$.

# 4 Lifting the toric $g$-vector

We now turn our attention to the toric $g$-vector. It is defined by a recursion; see for instance Stanley [20, Chapter 3.14]. However we build on the work of Bayer-Ehrenborg who described the toric $g$-polynomial in terms of the **cd**-index. Recall that the toric $g$-vector is formed from the coefficients of the $g$-polynomial, that is,

$$g(P, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} g_i^n(P) \cdot x^i.$$

Before we begin, a few definitions are necessary. Define $p(k, j)$ to denote the difference $\binom{k}{j} - \binom{k}{j-1}$. Also we need two polynomial sequences. First define $Q_k(x)$ by $Q_k(x) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} (-1)^j \cdot p(k-1, j) \cdot x^j$. Now define $T_k(x)$ for $k$ odd as $T_k(x) = (-1)^{(k-1)/2} \cdot C_{(k-1)/2} \cdot x^{(k-1)/2}$, where $C_n = p(2n, n)$ denotes the $n$th Catalan number. For even $k$, let $T_k(x) = 0$. We are now able to state the result of Bayer and Ehrenborg [3, Theorem 4.2].

**Theorem 4.1** *Let $g$ be the linear map from $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ to $\mathbb{R}[x]$ such that*

$$g(\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k}) = x^r \cdot Q_{k+1}(x) \cdot \prod_{j=1}^{r} T_{k_j+1}(x). \tag{4.1}$$

*Then the toric $g$-polynomial of a polytope $P$ is described by $g(\Psi(P)) = g(P, x)$.*

Observe that the entry $g_i^n$ in the toric $g$-vector is a linear functional on **cd**-polynomials of degree $n$. Hence we view $g_i^n$ as a homogeneous **cd**-polynomial of degree $n$ such that

$$\langle g_i^n | \Psi(P) \rangle = g_i^n(P),$$

for all $n$-dimensional polytopes $P$.

For $v$ a **cd**-monomial of degree $2i$ we define a polynomial $b(v, n)$ in the variable $n$. If $v$ cannot be written in terms of $\mathbf{c}^2$ and $\mathbf{d}$ then $b(v, n) = 0$. Otherwise let

$$b(v, n) = (-1)^{i-r} \cdot \prod_{j=1}^{r} C_{\ell_j} \cdot p(n - 2i + 2\ell_{r+1}, \ell_{r+1})$$

where $v = \mathbf{c}^{2\ell_1}\mathbf{d}\mathbf{c}^{2\ell_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{2\ell_{r+1}}$.

**Theorem 4.2** *The toric* **cd***-polynomial* $g_i^n$ *is described by*

$$g_i^n = \left( \sum_v b(v, n) \cdot v \right) \cdot \mathbf{c}^{n-2i},$$

*where the sum ranges over all* **cd***-monomials* $v$ *of degree* $2i$*.*

**Proof:** Let $[x^i]p(x)$ denote the coefficient of $x^i$ in the polynomial $p(x)$. To expand the toric **cd**-polynomial $g_i^n$ in terms of the monomial basis, we need to calculate

$$\begin{aligned} \left\langle g_i^n | \mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k} \right\rangle &= [x^i] g(\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k}) \\ &= [x^i] x^r \cdot Q_{k+1}(x) \cdot \prod_{j=1}^{r} T_{k_j+1}(x). \end{aligned} \tag{4.2}$$

Observe first if any of the $k_1, \ldots, k_r$ are odd, the expression vanishes. Thus we may assume that $k_1, \ldots, k_r$ are all even. Observe that the smallest power of $x$ appearing in (4.2) is $r + \sum_{j=1}^{r} k_j/2 = (n-k)/2$. Hence for $i < (n-k)/2$ the coefficient of $x^i$ is equal to zero. Thus for $k < n - 2i$ we have that

$$\left\langle g_i^n | \mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k} \right\rangle = 0.$$

Thus the only **cd**-monomials that appear in the **cd**-polynomial $g_i^n$ must have $k \geq n - 2i$ and all the $k_1, \ldots, k_r$ even.

Let $k_{r+1}$ be $2i - 2r - \sum_{j=1}^{r} k_j$ such that $v = \mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k_{r+1}}$ has degree $2i$ and let $\ell_j = k_j/2$. Continuing to expand (4.2) we have

$$\begin{aligned} [x^i] x^r \cdot Q_{k+1}(x) \cdot \prod_{j=1}^{r} T_{k_j+1}(x) &= (-1)^{\sum_{j=1}^{r} k_j/2} \cdot \prod_{j=1}^{r} C_{k_j/2} \cdot [x^{k_{r+1}/2}] Q_{k+1}(x) \\ &= (-1)^{\sum_{j=1}^{r+1} \ell_j} \cdot \prod_{j=1}^{r} C_{\ell_j} \cdot p(k, \ell_{r+1}). \end{aligned}$$

This expression is $b(v, n)$ since $k = n - 2i + 2\ell_{r+1}$. $\square$

The three first examples of Theorem 4.2 are $g_0^n = \mathbf{c}^n$, $g_1^n = \mathbf{d}\mathbf{c}^{n-2} - (n-1) \cdot \mathbf{c}^n$ and

$$g_2^n = \mathbf{d}^2\mathbf{c}^{n-4} - \mathbf{c}^2\mathbf{d}\mathbf{c}^{n-4} - (n-3) \cdot \mathbf{d}\mathbf{c}^{n-2} + \left( \binom{n-1}{2} - 1 \right) \cdot \mathbf{c}^n.$$

Observe that $b(v, 2i) = b(v^*, 2i)$ for $v$ of degree $2i$. From this the classical duality $g_i^{2i} = g_i^{2i*}$ follows.

**Proposition 4.3** *The toric* **cd***-polynomial* $g_i^k$ *satisfies the following identity*

$$g_i^k \cdot \mathbf{c}^j = \sum_{m=0}^{i} \binom{j+i-m-1}{i-m} \cdot g_m^{k+j}.$$

**Proof:** Observe that there is nothing to prove when $j = 0$. Assuming that the statement is true when $j = 1$, by a straightforward induction the cases $j \geq 2$ follow. Thus it is enough to prove the case $j = 1$:

$$g_i^n \cdot \mathbf{c} = \sum_{m=0}^{i} g_m^{n+1}.$$

This is equivalent to proving

$$\langle g_i^n \cdot \mathbf{c} | w \rangle = \left\langle \sum_{m=0}^{i} g_m^{n+1} | w \right\rangle,$$

where $w$ is a **cd**-monomial of degree $n+1$. Clearly this is true when $w$ ends with a $\mathbf{d}$. Thus consider the case when $w = v \cdot \mathbf{c}$, where $v$ is a **cd**-monomial of degree $n$. For a polynomial $p(x) = \sum_{i=0}^{\deg(p)} a_i \cdot x^i$ let $U_{\leq m}[p(x)]$ be the polynomial $\sum_{i=0}^{m} a_i \cdot x^i$. Now we have

$$\begin{aligned}
\left\langle \sum_{m=0}^{i} g_m^{n+1} | v \cdot \mathbf{c} \right\rangle &= \sum_{m=0}^{i} [x^m] g(v \cdot \mathbf{c}) \\
&= \sum_{m=0}^{i} [x^m] U_{\leq \lfloor (n+1)/2 \rfloor} \left[ (1-x) \cdot g(v) \right] \\
&= \sum_{m=0}^{i} [x^m](1-x) \cdot g(v) \\
&= \sum_{m=0}^{i} \left( [x^m] g(v) - [x^{m-1}] g(v) \right) \\
&= [x^i] g(v) = \langle g_i^n | v \rangle,
\end{aligned}$$

where the second step is by [3, Proposition 7.10] and the third step by the inequality $m \leq i \leq \lfloor n/2 \rfloor \leq \lfloor (n+1)/2 \rfloor$. $\square$

Applying our main result Theorem 3.1 to $H = g_i^k \cdot \mathbf{c}^j$ we have the following result.

**Theorem 4.4** *Let $P$ be a polytope of dimension $n$, let $u$ and $v$ be any two* **cd***-monomials such that $u$ does not end in* $\mathbf{c}$*, the sum of the degrees of $u$ and $v$ is $n-k$ and $2 \leq i \leq n/2$. Then*

$$\left\langle u \cdot g_i^k \cdot v | \Psi(P) \right\rangle \geq 0.$$

Theorem 4.4 gives a new inequality in dimension 8; see Theorem 5.8 inequality (5.8.10). Similar to Theorem 3.7, we do not get any sharper inequalities in Theorem 4.4 by allowing the monomial $v$ to begin with a $\mathbf{c}$.

## 5 Inequalities for 5 through 8-dimensional polytopes

The purpose of this section is to present the currently best-known linear inequalities for polytopes of dimensions 5 through 8. We introduce two notations to simplify the presentation. First we will write $w \geq 0$ instead of the longer $\langle w|\Psi(P)\rangle \geq 0$. Second for a **cd**-monomial $q$ of degree $k$ let $\sigma^k(q)$ denote the polynomial $q - \Delta_q \cdot \mathbf{c}^k$. (Observe that the super index $k$ is superfluous since it is given by the degree of the monomial $q$.) For instance, inequality (3.3) in Theorem 3.7 can be written as $u \cdot \sigma^k(q) \cdot v \geq 0$. Also note that the two inequalities $\sigma^n(\mathbf{dc}^{n-2}) \geq 0$ and $\sigma^n(\mathbf{c}^{n-2}\mathbf{d}) \geq 0$ are just the classical statements that an $n$-dimensional polytope has at least $n+1$ vertices, respectively $n+1$ facets.

Before we consider 5 through 8-dimensional polytopes, let us briefly review the lower dimensional cases. (Also observe that we omit Theorem 5.1 in order to keep the numbering consistent with the dimensions.)

**Theorem 5.2** *The* **cd***-index (equivalently the $f$-vector) of a polygon $P$ satisfies the inequality*

$$\sigma^2(\mathbf{d}) \geq 0 \tag{5.2.1}$$

**Theorem 5.3** *The* **cd***-index (equivalently the $f$-vector) of a 3-dimensional polytope $P$ satisfies the following two inequalities*

$$1 * \sigma^2(\mathbf{d}) \geq 0 \qquad \sigma^2(\mathbf{d}) * 1 \geq 0 \qquad (5.3.1) \quad (5.3.1^*)$$

Theorem 5.3 is due to Steinitz [24]. As mentioned in the introduction, the converse of this theorem is the more interesting part. The best known result for 4-dimensional polytopes is due to Bayer [1]:

**Theorem 5.4** *The* **cd***-index (equivalently the flag $f$-vector) of a 4-dimensional polytope $P$ satisfies the following list of six inequalities.*

$$\begin{array}{lcllcl}
\sigma^4(\mathbf{dc}^2) & \geq & 0 & \sigma^4(\mathbf{c}^2\mathbf{d}) & \geq & 0 \qquad (5.4.1) \quad (5.4.1^*) \\
g_2^4 & \geq & 0 & & & \qquad (5.4.2) \\
1 * \sigma^2(\mathbf{d}) * 1 & \geq & 0 & & & \qquad (5.4.3) \\
\mathbf{c} * \sigma^2(\mathbf{d}) & \geq & 0 & \sigma^2(\mathbf{d}) * \mathbf{c} & \geq & 0 \qquad (5.4.4) \quad (5.4.4^*)
\end{array}$$

We now list the currently best inequalities for 5-dimensional polytopes.

**Theorem 5.5** *The* **cd***-index of a* 5*-dimensional polytope* $P$ *satisfies the following list of* 13 *inequalities.*

| | | | |
|---|---|---|---|
| $\sigma^5(\mathbf{dc}^3) \geq 0$ | $\sigma^5(\mathbf{c}^3\mathbf{d}) \geq 0$ | (5.5.1) | (5.5.1*) |
| $1 * \sigma^4(\mathbf{dc}^2) \geq 0$ | $\sigma^4(\mathbf{c}^2\mathbf{d}) * 1 \geq 0$ | (5.5.2) | (5.5.2*) |
| $1 * \sigma^4(\mathbf{c}^2\mathbf{d}) \geq 0$ | $\sigma^4(\mathbf{dc}^2) * 1 \geq 0$ | (5.5.3) | (5.5.3*) |
| $1 * g_2^4 \geq 0$ | $g_2^4 * 1 \geq 0$ | (5.5.4) | (5.5.4*) |
| $\mathbf{c} * \sigma^2(\mathbf{d}) * 1 \geq 0$ | $1 * \sigma^2(\mathbf{d}) * \mathbf{c} \geq 0$ | (5.5.5) | (5.5.5*) |
| $\mathbf{c}^2 * \sigma^2(\mathbf{d}) \geq 0$ | $\sigma^2(\mathbf{d}) * \mathbf{c}^2 \geq 0$ | (5.5.6) | (5.5.6*) |
| $\sigma^2(\mathbf{d}) * \sigma^2(\mathbf{d}) \geq 0$ | | (5.5.7) | |

Before continuing with dimension 6 two observations are needed. First, so far the inequalities have described a cone. From now on, the inequalities we present determines a polyhedron. Second, the number of facets of the polyhedron grows rapidly. Hence we will only list the irreducible inequalities in dimensions 6 through 8, that is, inequalities that cannot be factored using the Kalai convolution.

**Theorem 5.6** *The* **cd***-index of a* 6*-dimensional polytope* $P$ *satisfies the following list of irreducible inequalities.*

| | | | |
|---|---|---|---|
| $\sigma^6(\mathbf{dc}^4) \geq 0$ | $\sigma^6(\mathbf{c}^4\mathbf{d}) \geq 0$ | (5.6.1) | (5.6.1*) |
| $\sigma^6(\mathbf{c}^2\mathbf{dc}^2) \geq 0$ | | (5.6.2) | |
| $\sigma^4(\mathbf{dc}^2) \cdot \mathbf{d} \geq 0$ | $\mathbf{d} \cdot \sigma^4(\mathbf{c}^2\mathbf{d}) \geq 0$ | (5.6.3) | (5.6.3*) |
| $g_2^6 \geq 0$ | $g_2^{6*} \geq 0$ | (5.6.4) | (5.6.4*) |
| $g_3^6 \geq 0$ | | (5.6.5) | |

**Theorem 5.7** *The* **cd***-index of a* 7*-dimensional polytope* $P$ *satisfies the following list of eight irreducible inequalities.*

| | | | |
|---|---|---|---|
| $\sigma^7(\mathbf{dc}^5) \geq 0$ | $\sigma^7(\mathbf{c}^5\mathbf{d}) \geq 0$ | (5.7.1) | (5.7.1*) |
| $\sigma^7(\mathbf{c}^2\mathbf{dc}^3) \geq 0$ | $\sigma^7(\mathbf{c}^3\mathbf{dc}^2) \geq 0$ | (5.7.2) | (5.7.2*) |
| $\sigma^5(\mathbf{dc}^3) \cdot \mathbf{d} \geq 0$ | $\mathbf{d} \cdot \sigma^5(\mathbf{c}^3\mathbf{d}) \geq 0$ | (5.7.3) | (5.7.3*) |
| $g_2^7 \geq 0$ | $g_2^{7*} \geq 0$ | (5.7.4) | (5.7.4*) |

**Theorem 5.8** *The* **cd***-index of an* 8*-dimensional polytope* $P$ *satisfies the following list of irreducible inequalities.*

$$
\begin{array}{llllll}
\sigma^8(\mathbf{dc}^6) \geq 0 & \sigma^8(\mathbf{c}^6\mathbf{d}) \geq 0 & (5.8.1) & (5.8.1^*) \\
\sigma^8(\mathbf{c}^2\mathbf{dc}^4) \geq 0 & \sigma^8(\mathbf{c}^4\mathbf{dc}^2) \geq 0 & (5.8.2) & (5.8.2^*) \\
\sigma^8(\mathbf{c}^3\mathbf{dc}^3) \geq 0 & & (5.8.3) & \\
\sigma^8(\mathbf{dc}^2\mathbf{dc}^2) \geq 0 & \sigma^8(\mathbf{c}^2\mathbf{dc}^2\mathbf{d}) \geq 0 & (5.8.4) & (5.8.4^*) \\
\sigma^6(\mathbf{dc}^4)\cdot\mathbf{d} \geq 0 & \mathbf{d}\cdot\sigma^6(\mathbf{c}^4\mathbf{d}) \geq 0 & (5.8.5) & (5.8.5^*) \\
\sigma^4(\mathbf{dc}^2)\cdot\mathbf{dc}^2 \geq 0 & \mathbf{c}^2\mathbf{d}\cdot\sigma^4(\mathbf{c}^2\mathbf{d}) \geq 0 & (5.8.6) & (5.8.6^*) \\
\sigma^4(\mathbf{dc}^2)\cdot\mathbf{d}^2 \geq 0 & \mathbf{d}^2\cdot\sigma^4(\mathbf{c}^2\mathbf{d}) \geq 0 & (5.8.7) & (5.8.7^*) \\
\sigma^4(\mathbf{c}^2\mathbf{d})\cdot\mathbf{dc}^2 \geq 0 & \mathbf{c}^2\mathbf{d}\cdot\sigma^4(\mathbf{dc}^2) \geq 0 & (5.8.8) & (5.8.8^*) \\
g_2^8 \geq 0 & g_2^{8*} \geq 0 & (5.8.9) & (5.8.9^*) \\
g_2^6\cdot\mathbf{d} \geq 0 & \mathbf{d}\cdot g_2^{6*} \geq 0 & (5.8.10) & (5.8.10^*) \\
g_3^8 \geq 0 & g_3^{8*} \geq 0 & (5.8.11) & (5.8.11^*) \\
g_4^8 \geq 0 & & (5.8.12) &
\end{array}
$$

The calculations in Theorems 5.5 through 5.8 were carried out in Maple. We end this section by summarizing some data on these polyhedra. Recall that the Fibonacci number minus one is the number of **cd**-monomials of degree $n$ excluding the monomial $\mathbf{c}^n$. Hence $F_n - 1$ is the dimension of the $n$th polyhedron.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $F_n - 1$ | 1 | 2 | 4 | 7 | 12 | 20 | 33 |
| # facets of the polyhedron | 1 | 2 | 6 | 13 | 29 | 60 | 119 |
| # irreducible facets of the polyhedron | 1 | 0 | 3 | 2 | 8 | 8 | 22 |

## 6 Concluding remarks

Theorem 3.1 produces many new inequalities for us to consider. However, these lifted inequalities do not give an equality when applied to the simplex. Thus it is natural to consider the following generalization of Theorem 3.1.

**Conjecture 6.1** *Let* $H$ *be a* **cd***-polynomial such that the inequality* $\langle H|\Psi(L)\rangle \geq 0$ *holds for all Gorenstein* lattices* $L$*. Moreover, let* $u$ *and* $v$ *be two* **cd***-monomials such that* $u$ *does not end in* **c**, $v$ *does not begin with* **c** *and they are not both equal to* 1*. Then the following inequality holds for all Gorenstein* lattices* $L$ *of rank* $n+1$*:*

$$\langle u \cdot H \cdot v|\Psi(L) - \Psi(\Sigma_n)\rangle \geq 0.$$

This conjecture extends Conjecture 2.7 of Stanley [23].

One possible method to prove this conjecture for polytopes is to use the following proposition and conjecture.

**Proposition 6.2** *If the inequality $\Psi(\Sigma_n) \preceq'_H \Psi(P)$ holds for all $n$-dimensional polytopes $P$ then for all $n$-dimensional polytopes $P$ we have $\Psi(\Sigma_n) \preceq_H \Psi(P)$.*

The proof of this proposition follows the exact same lines as the argument given for the implication $(b) \Longrightarrow (a)$ in the proof of Theorem 3.1.

**Conjecture 6.3** *Assume that $H$ is a **cd**-polynomial homogeneous of degree $k$ such that the inequality $\langle H|\Psi(Q)\rangle \geq 0$ holds for all $k$-dimensional polytopes $Q$. Let $P$ be an $n$-dimensional polytope where $n > k$. Let $F$ be a face of dimension $m$ of $P$ and let $F_1, \ldots, F_r$ be the facets of $P$ that contain the face $F$. Similarly, let $G_1, \ldots, G_{n-m}$ be the facets of the simplex $\Sigma_n$ containing an $m$-dimensional face $G$ of $\Sigma_n$. Then*

$$\Psi((G_1 \cup \cdots \cup G_{n-m})') \preceq'_H \Psi((F_1 \cup \cdots \cup F_r)').$$

When $m = 0$ this conjecture states that $\Psi(\Sigma_n) \preceq'_H \Psi((F_1 \cup \cdots \cup F_r)')$. Thus Conjecture 6.1 follows from Proposition 6.2 and Conjecture 6.3.

It is straightforward to verify Conjecture 6.1 for polytopes in the case when $u = 1$, $v = \mathbf{dc}^{n-4}$ and $H = \mathbf{d} - \mathbf{c}^2$ and dually in the case $u = \mathbf{c}^{n-4}\mathbf{d}$, $v = 1$ and $H = \mathbf{d} - \mathbf{c}^2$. Namely, the inequality $g_2^n(P) \geq 0$ can be expressed as:

$$\left\langle \mathbf{d}^2\mathbf{c}^{n-4} - \mathbf{c}^2\mathbf{dc}^{n-4} + (3-n) \cdot \mathbf{dc}^{n-2} \middle| \Psi(P) - \Psi(\Sigma_n) \right\rangle \geq 0.$$

To this inequality add $n-3$ times the inequality $\left\langle \mathbf{dc}^{n-2} \middle| \Psi(P) - \Psi(\Sigma_n) \right\rangle \geq 0$ and these cases follow.

Two questions deserve a deeper study. First, when is a new inequality new? That is, when is an inequality not implied by non-negative linear combinations of known inequalities? For instance, we conjecture that in the case $u = 1$, $H = \sigma^{n-2}\big(\mathbf{dc}^{n-4}\big) = \mathbf{dc}^{n-4} - (n-3) \cdot \mathbf{c}^{n-2}$ and $v = \mathbf{d}$ for $n \geq 6$ that the associated inequality is not implied by the non-negativity of the toric $g$-vector, the minimization inequalities offered by the simplex or the Kalai convolutions of these inequalities. Second, when do we stop trying to find linear inequalities? In other words, how do we recognize that we have the smallest polyhedron containing all flag $f$-vectors of polytopes?

Recall the two inequalities that an $n$-dimensional polytope has at least $n+1$ vertices and at least $n+1$ facets. In terms of the **cd**-monomial basis they are expressed as $\sigma^n\big(\mathbf{dc}^{n-2}\big) \geq 0$ and $\sigma^n\big(\mathbf{c}^{n-2}\mathbf{d}\big) \geq 0$. Observe that in dimensions 4 through 8 these two inequalities appear as facets of the polyhedra. However, there is only one polytope appearing on these facets, namely the simplex. Hence it is a challenging problem to determine if these inequalities are sharp, or if it is possible to sharpen them.

Also when studying the irreducible facet inequalities in Theorems 5.5 and 5.7 one might suspect that the two inequalities $\left\langle g_2^5 \middle| \Psi(P) \right\rangle \geq 0$ and $\left\langle g_3^7 \middle| \Psi(P) \right\rangle \geq 0$ are missing. These inequalities are not facet inequalities. This fact follows from an identity due to Stenson [25], namely

$$(k+2) \cdot g_k^{2k+1} = \sum_{i=0}^{k} (i+1) \cdot g_i^{2i} * g_{k-i}^{2(k-i)}.$$

Moreover, Stenson proved that the inequalities $\left\langle \mathbf{c}^i\mathbf{d}\mathbf{c}^j | \Psi(P) - \Psi(\Sigma_n) \right\rangle \geq 0$, where $i, j \geq 2$ and $i + j + 2 = n$, are not implied by the Kalai convolutions of the non-negativity of the toric $g$-vector. These inequalities are expressed as $\sigma^n\big(\mathbf{c}^i\mathbf{d}\mathbf{c}^j\big) \geq 0$ in Theorems 5.6 through 5.8.

Meisinger, Kleinschmidt and Kalai proved that a 9-dimensional rational polytope has a three-dimensional face that has less than 78 vertices or less than 78 faces [16]. However, with the recent proof that the entries in the toric $g$-vector are non-negative [14], their result now extends to all polytopes. Their proof uses the following observation. Assume that $P$ is a 9-dimensional polytope with every three-dimensional face having at least $m$ vertices and at least $m$ faces. If the inequality $\langle L | \Psi(Q) \rangle \geq 0$ holds for all 5-dimensional polytopes then the two inequalities

$$\left\langle (\mathbf{dc} - (m-2) \cdot \mathbf{c}^3) * L | \Psi(P) \right\rangle \geq 0 \quad \text{and} \quad \left\langle (\mathbf{cd} - (m-2) \cdot \mathbf{c}^3) * L | \Psi(P) \right\rangle \geq 0$$

also hold. Hence consider the system of linear inequalities

$$\begin{cases} \left\langle (\mathbf{dc} - 76\mathbf{c}^3) * L | z \right\rangle & \geq & 0, \\ \left\langle (\mathbf{cd} - 76\mathbf{c}^3) * L | z \right\rangle & \geq & 0, \\ \langle K | z \rangle & \geq & 0, \end{cases}$$

where $L$ ranges over linear inequalities for 5-dimensional polytopes and $K$ ranges over linear inequalities for 9-dimensional polytopes. They showed that this system is infeasible which implies that there is no 9-dimensional polytope with all its three-dimensional faces having at least 78 vertices and at least 78 faces. Using this technique and the inequalities derived from Theorem 3.1, we were able to improve upon the constant 78.

**Theorem 6.4** *A 9-dimensional polytope has a three-dimensional face that has less than 72 vertices or less than 72 faces.*

There are quadratic inequalities known on the entries of the flag $f$-vector. Two large classes of quadratic inequalities are given by Braden and MacPherson [8] and Billera and Ehrenborg [5]. However, quadratic inequalities are not as fundamental as linear inequalities. That is, the set of flag $f$-vectors of convex polytopes seems to have as a first good approximation the polyhedron determined by linear inequalities. Very little is known about this issue and it deserves a deeper study.

It would interesting to continue the work of Readdy [17], who studied the question of determining the largest coefficient of the **ab**-index of certain polytopes. Thus to continue Corollary 3.9 it would be interesting to determine which coefficient of the **cd**-index is the largest for different polytopes. In a recent preprint [15] Mahajan proved that in the **cd**-index of the simplex $\Sigma_n$ the monomials with the largest coefficient are given by

$$\begin{cases} \mathbf{cd}^{(n-2)/2}\mathbf{c} & \text{if } n \text{ is even} \\ \mathbf{cdcd}^{(n-5)/2}\mathbf{c} \text{ and } \mathbf{cd}^{(n-5)/2}\mathbf{cdc} & \text{if } n \text{ is odd} \end{cases}$$

for $n$ sufficiently large.

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