Inequalities for **cd**-indices of joins and products of polytopes*

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Abstract

The **cd**-index is a polynomial which encodes the flag f-vector of a convex polytope. For polytopes U and V, we determine explicit recurrences for computing the **cd**-index of the free join $U \circledvee V$ and the **cd**-index of the Cartesian product $U \times V$. As an application of these recurrences, we prove the inequality $\Psi(U \circledvee (V \times W)) \leq \Psi((U \circledvee V) \times W)$ involving the **cd**-indices of three polytopes.

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1 Introduction

The flag f-vector displays the enumerative data of flags of faces of a convex polytope. This vector includes all the information of the f-vector or face vector, that is, the number of faces of each dimension of a polytope. For an n-dimensional polytope the flag f-vector has 2^n entries while the f-vector has n entries. Recall the entries of the f-vector are not independent. The Euler-Poincaré relation holds between the f-vector entries, and is in fact the only linear relation among these entries. Motivated by the f-vector case, Bayer and Billera discovered all the linear relations holding between the flag f-vector entries, known as the generalized Dehn-Sommerville relations [1]. Hence when studying the flag f-vectors of polytopes, it is enough to consider the vectors in the subspace satisfying the generalized Dehn-Sommerville relations. The dimension of this subspace is the nth Fibonacci number which is exponentially smaller than 2^n.

The **cd**-index of a polytope, a polynomial in the non-commuting variables **c** and **d**, is an invariant that compactly encodes the flag f-vector of the polytope. One way to view the **cd**-index is that it gives an explicit basis for the subspace cut out by the generalized Dehn-Sommerville relations. Of course, there are many bases that one could choose for this subspace. However, many reasons have emerged to show that the **cd**-index is the right basis. One important reason to study the **cd**-index is its non-negativity property. For polytopes, an entry of the flag f-vector is a non-negative linear combination of the coefficients of the **cd**-index of the polytope. Hence inequalities for the **cd**-index translate into inequalities for the flag f-vector.

For example, the most well-known open inequality for the **cd**-index is the conjecture of Stanley [24]. He conjectured that the **cd**-index of any Gorenstein* lattice of rank $n + 1$ is coefficient-wise greater

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than or equal to the **cd**-index of the n-dimensional simplex. In the case of polytopes this conjecture was proved by Billera and Ehrenborg [5].

A second reason to consider the **cd**-index is that it is a powerful invariant for computations. There are explicit expressions for how the **cd**-index changes after applying a geometric operation to the underlying polytope. The operations which have been studied so far are prism, pyramid, cutting off a face and Minkowski sum with a line segment in general direction [11, 13]. Also the **cd**-index of zonotopes, a special class of polytopes, is fairly well-understood [6, 7].

In this paper we continue this line of work. We first consider the problem of computing the **cd**-index of the free join of two polytopes knowing the **cd**-indices of each polytope. We then study the related question of the Cartesian product of polytopes.

These problems were studied in [13], but without a satisfying answer in the following sense. The authors constructed two bilinear operators M and N such that

$$\begin{aligned}\Psi(V \circledvee W) &= M(\Psi(V), \Psi(W)),\\ \Psi(V \times W) &= N(\Psi(V), \Psi(W)),\end{aligned}$$

where $V \circledvee W$ denotes the free join of the polytopes V and W, $V \times W$ denotes the Cartesian product, and Ψ denotes the **cd**-index. The expressions given for these two bilinear operators were cumbersome since rather than expressing all the computations in $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ they involved the auxiliary variables **a** and **b**.

In this paper we develop recursions for the operators M and N that only involve the variables **c** and **d**. That is, with these recursions one can compute the **cd**-index of a free join of polytopes completely in terms of **cd**-polynomials.

The main technique that we use is to work with the underlying coalgebra structure of posets and the **cd**-index. This method was introduced in [13] and has revealed a rich variety of results; for example, see [2, 5, 7]. In order to obtain more succinct expressions for the recursions, we were forced to include a new element of degree minus one. Moreover, since this extended algebra does not satisfy the associative law, care must be taken when working with it.

In Section 8 we prove our main inequality which relates the **cd**-indices between the free join of polytopes and the Cartesian product. It is:

$$\Psi(U \circledvee (V \times W)) \leq \Psi((U \circledvee V) \times W),$$

for U, V and W three polytopes; see Theorem 8.2. The proof of this theorem relies on the coalgebra techniques and the recursions which we develop in the earlier sections of this paper. In the concluding remarks, we give a corollary of the inequality which provides evidence for Stanley's conjecture on Gorenstein* lattices.

2 Posets and the cd-index

For terminology on partially ordered sets (posets), we refer the reader to [22]. Unless otherwise stated, throughout this paper we assume that all posets are graded with unique minimal element $\hat{0}$, unique maximal element $\hat{1}$ and rank function ρ.

For two elements x and y in a poset P such that $x \le y$, define the *interval* $[x, y]$ to be the set $\{z \in P \ : \ x \le z \le y\}$. The interval $[x, y]$ inherits the same order relations as the poset P and hence is also a graded poset of rank $\rho(x, y) = \rho(y) - \rho(x)$.

The main example of graded posets the reader should keep in mind is the face lattice of a convex polytope. For a polytope V we denote the face lattice by $\mathcal{L}(V)$. The minimal element of the face lattice is the empty face, the maximal element is the entire polytope, and the rank of a face F is given by $\rho(F) = \dim(F) + 1$.

A *chain* c in a poset P is a linearly ordered subset of P. A chain is maximal (saturated) if one cannot add one more element without breaking the condition that the chain is linearly ordered. For a graded poset P all the chains we consider will contain the minimal element $\hat{0}$ and the maximal element $\hat{1}$. That is, we write $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ for a chain c.

Let P be a graded poset of rank $n + 1$ and let S be a subset of the set $\{1, 2, \ldots, n\}$. Define $f_S(P) = f_S$ to be the number of chains (flags) of the poset P whose ranks are exactly given by the set S. The 2^n values given by f_S constitute the *flag f-vector* of a poset. The *flag h-vector* is defined by the identity

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

This is equivalent to the relation

$$f_S = \sum_{T \subseteq S} h_T.$$

Hence the flag f-vector and the flag h-vector contain the same information.

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commuting variables. For S a subset of $\{1, \ldots, n\}$ define the $\mathbf{ab}$-monomial $u_S = u_1 \cdots u_n$ by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The $\mathbf{ab}$-*index* $\Psi(P)$ of a poset of rank $n + 1$ is the $\mathbf{ab}$-polynomial

$$\Psi(P) = \sum_{S} h_S \cdot u_S,$$

where S ranges over all subsets of the set $\{1, \ldots, n\}$.

An alternative definition of the $\mathbf{ab}$-index of a poset P is as follows. For each chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ associate to it the *weight* $w(c) = w_1 \cdots w_n$, where

$$w_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_{k-1})\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

The $\mathbf{ab}$-index of the poset P is then given by

$$\Psi(P) = \sum_{c} w(c),$$

where the sum is over all chains c in the poset P.

The Möbius function μ of a poset P is defined by the relations $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \le z < y} \mu(x, z)$ for $x < y$. A poset P is *Eulerian* if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$

for all intervals $[x, y]$ in the poset P. This is equivalent to saying that each interval of the poset satisfies the Euler-Poincaré relation. For instance, face lattices of convex polytopes are Eulerian. Fine conjectured and Bayer and Klapper [4] proved the following important result for Eulerian posets.

Theorem 2.1 (Fine, Bayer-Klapper) *The* **ab***-index* $\Psi(P)$ *of an Eulerian poset* P *is a polynomial in* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When $\Psi(P)$ is written as a polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$, it is called the **cd**-*index* of the poset P. For a proof of this theorem, see [23, Theorem 1.1]. See also the discussion in [5, Section 3]. The flag f-vector of an Eulerian poset satisfies linear relations known as the generalized Dehn-Sommerville relations [1]. These relations are equivalent to the existence of the **cd**-index [4]. Thus the **cd**-index gives a basis for the subspace that satisfies the generalized Dehn-Sommerville relations.

Explicit formulas for the coefficients of the **cd**-index have been given by Bayer and Ehrenborg in [2] and by Billera, Ehrenborg and Readdy in [5]. The **cd**-coefficients were computed directly through rank 9 by Meisinger [18, Appendix D]. See also [10]. Unfortunately, these formulas have not been fruitful in proving equalities and inequalities for the **cd**-index.

Stanley [23] proved that the **cd**-indices of S-shellable complexes have non-negative coefficients. In particular, since polytopes are S-shellable, the **cd**-index of a polytope is non-negative. Stanley conjectured the **cd**-index of any Gorenstein* lattice of rank $n + 1$ is coefficient-wise greater than or equal to the **cd**-index of the n-dimensional simplex [24, Conjecture 2.7]. This conjecture was proved in the case of polytopes by Billera and Ehrenborg [5] by a shelling argument, where shellings were simultaneously used in different dimensions.

3 Coalgebra techniques

We now discuss coalgebras. For general references, see [19, 25]. For a $\mathbb{Z}$-module W, a coproduct is a linear map $\Delta : W \longrightarrow W \otimes W$. To denote the coproduct of an element $w \in W$, we use the Sweedler notation $\Delta(w) = \sum_w w_{(1)} \otimes w_{(2)}$. This is a sum over finitely many pairs $w_{(1)}$ and $w_{(2)}$, and the notation suppresses the number of terms in the sum. A coproduct Δ is *coassociative* if it satisfies the identity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

Definition 3.1 *A* $\mathbb{Z}$*-module* W *with product and coproduct* Δ *is called a Newtonian coalgebra if it satisfies the following identity.*

$$\Delta(u \cdot v) = \sum_u u_{(1)} \otimes u_{(2)} \cdot v + \sum_v u \cdot v_{(1)} \otimes v_{(2)}. \tag{3.1}$$

Newtonian coalgebras were first defined in [15, 17] in the context of divided differences. The three Newtonian coalgebras $\mathcal{P}$, $\mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle$ and $\mathbb{Z}\langle\mathbf{c}, \mathbf{d}\rangle$ that we will work with were first studied in [13] and used in [2, 5, 7].

In order to use more compact notation in what follows, we introduce a different coproduct on our three coalgebras. To do this, first adjoin a new element ϵ to the coalgebra W, and denote this space by $\overline{W}$. Define a new coproduct Δ^* on $\overline{W}$ by

$$\Delta^*(u) = \Delta(u) + u \otimes \epsilon + \epsilon \otimes u. \tag{3.2}$$

To avoid confusion we mark the summation sign in the Sweedler notation for the coproduct Δ^* with a star, that is, we write $\Delta^*(u) = \sum_u^* u_{(1)} \otimes u_{(2)}$. The coproduct Δ^* satisfies the following product rule.

$$\Delta^*(u \cdot v) = \sum_u{}^{*} u_{(1)} \otimes u_{(2)} \cdot v + \sum_v{}^{*} u \cdot v_{(1)} \otimes v_{(2)} - u \cdot \epsilon \otimes v - u \otimes \epsilon \cdot v. \tag{3.3}$$

As a small warning to the reader, the adjoined element ϵ will in general not satisfy the associative law. Hence care must be taken when this element appears in products.

We will now review the three Newtonian coalgebras $\mathcal{P}$, $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ and $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ and introduce their extended versions $\overline{\mathcal{P}}$, $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$ and $\overline{\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle}$.

Let $\mathcal{P}$ be the free $\mathbb{Z}$-module generated by all types (isomorphism classes) of graded posets of rank greater than or equal to one. Define a coproduct Δ on $\mathcal{P}$ by

$$\Delta(P) = \sum_{\hat{0} < x < \hat{1}} [\hat{0}, x] \otimes [x, \hat{1}],$$

and extend this coproduct to $\mathcal{P}$ by linearity. Observe that this coproduct is coassociative. To define a product on $\mathcal{P}$ it is enough to define the product between two posets P and Q and extend by linearity to $\mathcal{P}$. Define the star product $P * Q$ to be the poset on the set $(P - \{\hat{1}\}) \cup (Q - \{\hat{0}\})$. The order relation is given by $x \leq_{P*Q} y$ if one of the following three conditions is satisfied: (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$, (iii) $x \in P$ and $y \in Q$. It is now straightforward to see that $\mathcal{P}$ is a Newtonian coalgebra and that the star product is associative on $\mathcal{P}$.

To define $\overline{\mathcal{P}}$, let $\bullet$ denote the unique graded poset of rank 0, that is, the one element poset and adjoin $\bullet$ to $\mathcal{P}$ to obtain $\overline{\mathcal{P}}$. Observe $\overline{\mathcal{P}}$ is the free $\mathbb{Z}$-module generated by all types (isomorphism classes) of graded posets. The coproduct Δ^* defined by $\Delta^*(P) = \Delta(P) + P \otimes \bullet + \bullet \otimes P$ can be expressed by

$$\Delta^*(P) = \sum_{\hat{0} \leq x \leq \hat{1}} [\hat{0}, x] \otimes [x, \hat{1}].$$

This is the coalgebra structure on graded posets that was studied in [9, 21]. To extend the star product on $\mathcal{P}$ to $\overline{\mathcal{P}}$, define $P * \bullet$ to be the poset P with the coatoms removed. Similarly, define $\bullet * P$ to be the poset P with the atoms removed. Finally, let $\bullet * \bullet$ be the zero element in $\overline{\mathcal{P}}$. Observe that the star product on $\overline{\mathcal{P}}$ is not associative, since $(P * \bullet) * Q$ is not necessarily equal to $P * (\bullet * Q)$.

Let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ denote the ring of non-commutative polynomials in $\mathbf{a}$ and $\mathbf{b}$, where the degrees of $\mathbf{a}$ and $\mathbf{b}$ are each defined to be one. For an $\mathbf{ab}$-monomial $u = u_1 \cdots u_n$, define

$$\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n,$$

and extend linearly to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. For instance, $\Delta(1) = 0$ and $\Delta(\mathbf{a}) = 1 \otimes 1$. This is a coassociative coproduct on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. It is straightforward to verify that $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is a Newtonian coalgebra.

We now extend the Newtonian coalgebra $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\overline{\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle}$ by adjoining the element ϵ. Let $\overline{\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle}$ be the $\mathbb{Z}$-module $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle \oplus \mathbb{Z}\epsilon$. The element ϵ should be considered as the **ab**-index of the one element poset $\bullet$. Hence the degree of ϵ is -1. Extend the product on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\overline{\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle}$ by the following rules

$$\begin{array}{rclclcrcl} \epsilon \cdot \epsilon & = & 0, & & & \qquad & \epsilon \cdot (u_1 \cdots u_n) & = & u_2 \cdots u_n, \\ \epsilon \cdot 1 & = & 1 \cdot \epsilon & = & \epsilon, & \qquad & (u_1 \cdots u_n) \cdot \epsilon & = & u_1 \cdots u_{n-1}. \end{array}$$

Observe this product is non-associative. For instance $(\mathbf{a} \cdot \epsilon) \cdot \mathbf{b} = \mathbf{b}$, but $\mathbf{a} \cdot (\epsilon \cdot \mathbf{b}) = \mathbf{a}$. The linear operation $w \longmapsto w \cdot \epsilon$ was used in [7, Section 8]. When the element ϵ occurs at the end of a product, there is no ambiguity about the interpretation of the associative law. However, when the element ϵ occurs in the middle of a product, we have the following restricted associative law.

> Let u and v be two **ab**-polynomials such that the last letter of u equals the first letter of v. Then
>
> $$(u \cdot \epsilon) \cdot v = u \cdot (\epsilon \cdot v). \tag{3.4}$$

In this paper we will remark when this situation occurs, for example, in the proofs of Theorems 5.1 and 7.1.

Define the coproduct Δ^* on $\overline{\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle}$ by $\Delta^*(\epsilon) = \epsilon \otimes \epsilon$ and for an **ab**-monomial $u = u_1 \cdots u_n$, define $\Delta^*(u) = u \otimes \epsilon + \epsilon \otimes u + \Delta^*(u)$. To avoid needless bookkeeping, we recommend using equations (3.1) and (3.2) for computations involving the coproduct Δ^*.

Let $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ be the subring of $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ generated by $\mathbf{c}$ and $\mathbf{d}$ and let $\overline{\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle}$ be the submodule $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle \oplus \mathbb{Z}\epsilon$ of $\overline{\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle}$. Observe that the product is closed on $\overline{\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle}$ since $\epsilon \cdot \mathbf{c} = \mathbf{c} \cdot \epsilon = 2$ and $\epsilon \cdot \mathbf{d} = \mathbf{d} \cdot \epsilon = \mathbf{c}$. Moreover, $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ is a Newtonian subalgebra of $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$. To see this, it is enough to observe that $\Delta(\mathbf{c}) = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{d}) = 1 \otimes \mathbf{c} + \mathbf{c} \otimes 1$ and use the Newtonian condition (3.1). Hence the coproduct Δ on $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ can computed without reference to the variables $\mathbf{a}$ and $\mathbf{b}$.

Example 3.2 The coproduct $\Delta(\mathbf{dc})$ is computed as follows.

$$\begin{array}{rcl} \Delta(\mathbf{dc}) & = & 1 \otimes \mathbf{c} \cdot \mathbf{c} + \mathbf{c} \otimes 1 \cdot \mathbf{c} + 2 \cdot \mathbf{d} \cdot 1 \otimes 1 \\ & = & 1 \otimes \mathbf{c}^2 + \mathbf{c} \otimes \mathbf{c} + 2 \cdot \mathbf{d} \otimes 1. \end{array}$$

Hence we also have

$$\Delta^*(\mathbf{dc}) \quad = \quad \epsilon \otimes \mathbf{dc} + 1 \otimes \mathbf{c}^2 + \mathbf{c} \otimes \mathbf{c} + 2 \cdot \mathbf{d} \otimes 1 + \mathbf{dc} \otimes \epsilon.$$

Extend the **ab**-index Ψ of a graded poset to a linear map from $\overline{\mathcal{P}}$ to $\overline{\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle}$ by adding the extra condition that $\Psi(\bullet) = \epsilon$. The essential result for the **ab**-index, and hence also for the **cd**-index, is the following theorem of [13].

Theorem 3.3 (Ehrenborg-Readdy) *The linear map $\Psi : \mathcal{P} \longrightarrow \mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ is a Newtonian coalgebra homomorphism.*

This theorem implies that the map $\Psi : \overline{\mathcal{P}} \longrightarrow \overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$ preserves the coproduct Δ^*. The two conclusions from this theorem that we will use are:

$$\Psi(P * Q) = \Psi(P) \cdot \Psi(Q), \tag{3.5}$$
$$\Delta^*(\Psi(P)) = \sum_{\hat{0} \leq x \leq \hat{1}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}]). \tag{3.6}$$

Identity (3.5) for posets of rank greater than or equal to one is due to Stanley [23, Lemma 1.1].

Identity (3.6) is important for the following reason. Let f and g be two linear maps from the $\mathbb{Z}$-module $\overline{\mathcal{P}}$ into a ring. Consider the expression $\sum_x f(\Psi([\hat{0}, x])) \cdot g(\Psi([x, \hat{1}]))$, where x ranges over elements in a poset P such that $\hat{0} \leq x \leq \hat{1}$. Since the map Ψ is a coalgebra homomorphism, this expression is equal to the much easier to compute expression $\sum_w^* f(w_{(1)}) \cdot g(w_{(2)})$ where w is $\Psi(P)$. Hence the fact that Ψ is a coalgebra homomorphism allows us to translate relations about the **ab**-index of posets into relations between **ab**-polynomials.

Let A_r be the unique graded poset of rank 2 with r atoms. The **ab**-index of A_r is given by $\Psi(A_r) = \mathbf{a} + (r-1) \cdot \mathbf{b}$.

Lemma 3.4 *The linear map $\Psi : \mathcal{P} \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is surjective.*

Proof: This proof mimics the proof of Proposition 2.1 in [23]. The ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is spanned by monomials v in the letters $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$. For such a monomial $v = v_1 \cdots v_n$, let $P = P_1 * \cdots * P_n$ where $P_i = A_1$ if $v_i = \mathbf{a}$ and $P_i = A_2$ if $v_i = \mathbf{a}+\mathbf{b}$. It follows that $\Psi(P) = v$ and hence Ψ is surjective. $\Box$

For an alternative statement and proof of this lemma, see Billera and Liu [8, Proposition 1.1]. By Lemma 3.4, to show a linear identity holds for all **ab**-polynomials, it is enough to verify it for the **ab**-indices of posets. Furthermore, by recalling that $\Psi(\bullet) = \epsilon$, we have that the map $\Psi : \overline{\mathcal{P}} \longrightarrow \overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$ is also surjective. Hence linear identities involving the **ab**-index of the poset $\bullet$ also hold in $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$.

There are two useful linear involutions on the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. The first is the star involution. It is defined for an **ab**-monomial $u = u_1 u_2 \cdots u_n$ by $u^* = u_n \cdots u_2 u_1$. This involution reads a polynomial backwards and on the poset level corresponds to turning a poset upside-down. That is, for a poset P define P^* to be the poset with the order relation $x \leq_{P^*} y$ if and only if $y \leq_P x$. By linearity we extend this map to an involution of $\mathcal{P}$. We then have $\Psi(P^*) = \Psi(P)^*$. Observe that this involution is an anti-isomorphism of the Newtonian coalgebra, that is, it satisfies

$$(u \cdot v)^* = v^* \cdot u^* \quad \text{and} \quad \Delta(u^*) = \sum_u {u_{(2)}}^* \otimes {u_{(1)}}^*.$$

The second involution $\overline{\cdot}$ is defined on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by replacing **a**'s with **b**'s and vice-versa. For instance, $\overline{\mathbf{aab} + 3 \cdot \mathbf{bab}} = \mathbf{bba} + 3 \cdot \mathbf{aba}$. This is a Newtonian coalgebra isomorphism, that is,

$$\overline{u \cdot v} = \overline{u} \cdot \overline{v} \quad \text{and} \quad \Delta(\overline{u}) = \sum_u \overline{u_{(1)}} \otimes \overline{u_{(2)}}.$$

Note this involution has no effect on the ring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. We linearly extend both these involutions to $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$ by letting $\epsilon^* = \epsilon$ and $\overline{\epsilon} = \epsilon$.

4 The free join of polytopes

Let V be an m-dimensional polytope and W an n-dimensional polytope. Embed the polytope V in $\mathbb{R}^{m+n+1}$ by

$$V' = \{(x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n}, 0) \in \mathbb{R}^{m+n+1} \ : \ (x_1, \ldots, x_m) \in V\}.$$

Similarly embed W by

$$W' = \{(\underbrace{0, \ldots, 0}_{m}, x_1, \ldots, x_n, 1) \in \mathbb{R}^{m+n+1} \ : \ (x_1, \ldots, x_n) \in W\}.$$

The *free join* $V \circledvee W$ is the $(m + n + 1)$-dimensional polytope defined to be the convex hull of V' and W'.

As an example, the free join of two line segments is the convex hull of two line segments in general position, that is, a tetrahedron. In general, the free join of two simplices is a simplex. The pyramid of a polytope V is the free join of V with a point (the zero-dimensional polytope). This corresponds to taking a point outside the affine span of the polytope and then taking the convex hull.

For two (not necessarily graded) posets P and Q, define the *Cartesian product* $P \times Q$ to be the set $P \times Q = \{(x, y) \ : \ x \in P, y \in Q\}$ with the order relation $(x, z) \leq_{P \times Q} (y, w)$ if $x \leq_P y$ and $z \leq_Q w$. Observe that when P and Q are graded posets then so is the Cartesian product $P \times Q$. The face lattice of the free join of two polytopes is the Cartesian product of the two face lattices, that is, for two polytopes V and W we have $\mathcal{L}(V \circledvee W) = \mathcal{L}(V) \times \mathcal{L}(W)$; see Kalai [16].

The mixing operator M was developed by Ehrenborg and Readdy [13] to study the **ab**-index of the Cartesian products of posets. To define it, let I be the index set

$$I = \{(r, s, n) \ : \ r, s \in \{1, 2\}, n \geq 2, n \equiv r + s + 1 \bmod 2\}.$$

For $(r, s, n) \in I$ define the bilinear operator $M_{r,s,n} : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \times \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by the following recursions:

$$\begin{aligned} M_{1,2,2}(u, v) &= u \cdot \mathbf{a} \cdot v, \\ M_{2,1,2}(u, v) &= v \cdot \mathbf{b} \cdot u, \\ M_{1,s,n+1}(u, v) &= \sum_u u_{(1)} \cdot \mathbf{a} \cdot M_{2,s,n}(u_{(2)}, v), \\ M_{2,s,n+1}(u, v) &= \sum_v v_{(1)} \cdot \mathbf{b} \cdot M_{1,s,n}(u, v_{(2)}). \end{aligned}$$

Definition 4.1 *The* mixing operator *is given by the sum*

$$M(u, v) = \sum_{(r,s,n) \in I} M_{r,s,n}(u, v).$$

Observe that this sum is well-defined since only finitely many of the terms are non-zero.

Using the mixing operator we can now state how the **ab**-index of a Cartesian product of posets depends on the **ab**-index of each factor and how the **cd**-index of a free join depends on the **cd**-index of each factor.

Proposition 4.2 (Ehrenborg-Readdy) *Let P and Q be two posets. Then the* **ab**-*index of the Cartesian product of P and Q is given by*

$$\Psi(P \times Q) = M(\Psi(P), \Psi(Q)).$$

Thus for two polytopes V and W the **cd**-*index of the free join of V and W is given by*

$$\Psi(V \circledvee W) = M(\Psi(V), \Psi(W)).$$

The disadvantage with the mixing operator definition is that it does not permit computations in the ring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. That is, given two **cd**-polynomials u and v, the expression given for $M(u, v)$ contains the auxiliary variables **a** and **b**. As mentioned in the introduction, this is the motivation for developing Theorem 5.1.

The Cartesian product of posets is commutative and associative and satisfies $(P \times Q)^* = P^* \times Q^*$. Since the linear map $\Psi : \mathcal{P} \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is surjective, the mixing operator inherits these properties.

Corollary 4.3 *The mixing operator satisfies the following three identities:*

$$\begin{aligned} M(u^*, v^*) &= M(u, v)^*, \\ M(u, v) &= M(v, u), \\ M(u, M(v, w)) &= M(M(u, v), w), \end{aligned}$$

for any three **ab**-*polynomials u, v and w. That is, the mixing operator is commutative, associative, and is preserved under the involution star.*

Recall that a derivation G is a linear operator which satisfies the Leibniz rule (or product rule) $G(u \cdot v) = G(u) \cdot v + u \cdot G(v)$. Let $G : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ be the derivation given by $G(\mathbf{a}) = \mathbf{ba}$ and $G(\mathbf{b}) = \mathbf{ab}$. Let $\mathrm{Pyr} : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ be the linear operator defined by $\mathrm{Pyr}(w) = w \cdot \mathbf{c} + G(w)$. Since $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$ observe that G and Pyr also restrict to become linear operators from $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ to $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. In [13] it was proved that

$$M(u, 1) = \mathrm{Pyr}(u), \tag{4.1}$$

for any **ab**-polynomial u. Since the **cd**-index of a point is 1, identity (4.1) shows how to compute the **cd**-index of the pyramid of a polytope knowing the **cd**-index of the original polytope.

Example 4.4 The mixing operator $M(\mathbf{dc}, 1)$ is computed as follows:

$$\begin{aligned} M(\mathbf{dc}, 1) = \mathrm{Pyr}(\mathbf{dc}) &= \mathbf{dc} \cdot \mathbf{c} + G(\mathbf{dc}) \\ &= \mathbf{dc}^2 + G(\mathbf{d}) \cdot \mathbf{c} + \mathbf{d} \cdot G(\mathbf{c}) \\ &= \mathbf{dc}^2 + \mathbf{cdc} + \mathbf{d}^2. \end{aligned}$$

We extend the mixing operator M to $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle}$ by letting $M(u, \epsilon) = M(\epsilon, u) = u$. This is motivated by the fact that the one element poset is the unit for the Cartesian product of posets. Similarly, we extend Pyr to $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle}$ by setting $\mathrm{Pyr}(\epsilon) = 1$.

We end this section with two lemmas.

Lemma 4.5 *For an* **ab***-polynomial u, we have*

$$\mathrm{Pyr}(u) = \sum_u^{*} u_{(1)} \cdot \mathbf{ba} \cdot u_{(2)}.$$

Proof: By expanding the mixing operator $M(1, u)$ via Definition 4.1, we have $\mathrm{Pyr}(u) = u \cdot \mathbf{b} + \mathbf{a} \cdot u + \sum_u u_{(1)} \cdot \mathbf{ba} \cdot u_{(2)}$, which is the desired equality. $\square$

Lemma 4.6 *For two* **ab***-polynomials u and v, we have*

$$\Delta^*(M(u, v)) = \sum_u^{*} \sum_v^{*} M(u_{(1)}, v_{(1)}) \otimes M(u_{(2)}, v_{(2)}).$$

It is straightforward to see that Lemma 4.6 holds for $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle}$. Namely, the product M and the coproduct Δ^* on $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle}$ form a bialgebra, and hence also a Hopf algebra structure. In fact, this Hopf algebra is isomorphic to the Hopf algebra of quasi-symmetric functions [9].

5 Recursions for the mixing operator

We will now state and prove recursions for the mixing operator. The proof relies upon expressing the **ab**-index of the poset $P \times (Q * A_1)$ in terms of the **ab**-indices of the poset Q and the intervals $[\hat{0}, x]$ and $[x, \hat{1}]$, where $x \in P$. By applying coalgebra techniques, the desired recursions are obtained.

Theorem 5.1 *For two* **cd***-polynomials u and v,*

$$M(u, v \cdot \mathbf{c}) = \sum_u^{*} M(u_{(1)}, v) \cdot \mathbf{d} \cdot u_{(2)}, \tag{5.1}$$

$$M(u, v \cdot \mathbf{d}) = \sum_u^{*} M(u_{(1)}, v) \cdot \mathbf{d} \cdot \mathrm{Pyr}(u_{(2)}). \tag{5.2}$$

The first few values of the mixing operator M are displayed in Table 1.

u	v	$M(u,v)$
1	1	$\mathbf{c}$
1	$\mathbf{c}$	$\mathbf{c}^2+\mathbf{d}$
1	$\mathbf{c}^2$	$\mathbf{c}^3+\mathbf{cd}+\mathbf{dc}$
1	$\mathbf{d}$	$\mathbf{cd}+\mathbf{dc}$
$\mathbf{c}$	$\mathbf{c}$	$\mathbf{c}^3+2\cdot\mathbf{cd}+2\cdot\mathbf{dc}$
$\mathbf{c}$	$\mathbf{c}^2$	$\mathbf{c}^4+2\cdot\mathbf{c}^2\mathbf{d}+3\cdot\mathbf{cdc}+2\cdot\mathbf{dc}^2+2\cdot\mathbf{d}^2$
$\mathbf{c}$	$\mathbf{d}$	$\mathbf{c}^2\mathbf{d}+2\cdot\mathbf{cdc}+\mathbf{dc}^2+2\cdot\mathbf{d}^2$
$\mathbf{c}^2$	$\mathbf{c}^2$	$\mathbf{c}^5+2\cdot\mathbf{c}^3\mathbf{d}+4\cdot\mathbf{c}^2\mathbf{dc}+4\cdot\mathbf{cdc}^2+2\cdot\mathbf{dc}^3+4\cdot\mathbf{cd}^2+4\cdot\mathbf{dcd}+4\cdot\mathbf{d}^2\mathbf{c}$
$\mathbf{c}^2$	$\mathbf{d}$	$\mathbf{c}^3\mathbf{d}+2\cdot\mathbf{c}^2\mathbf{dc}+2\cdot\mathbf{cdc}^2+\mathbf{dc}^3+3\cdot\mathbf{cd}^2+2\cdot\mathbf{dcd}+3\cdot\mathbf{d}^2\mathbf{c}$
$\mathbf{d}$	$\mathbf{d}$	$\mathbf{c}^2\mathbf{dc}+\mathbf{cdc}^2+2\cdot\mathbf{cd}^2+2\cdot\mathbf{dcd}+2\cdot\mathbf{d}^2\mathbf{c}$

Table 1: The mixing operator $M(u,v)$ computed for **cd**-monomials of low degree.

Example 5.2 As an example, we compute the mixing operators $M(\mathbf{dc},\mathbf{c})$ and $M(\mathbf{dc},\mathbf{d})$. Recall that $\Delta^*(\mathbf{dc}) = \epsilon\otimes\mathbf{dc}+1\otimes\mathbf{c}^2+\mathbf{c}\otimes\mathbf{c}+2\cdot\mathbf{d}\otimes 1+\mathbf{dc}\otimes\epsilon$. Then

$$\begin{aligned}
M(\mathbf{dc},1\cdot\mathbf{c}) &= M(\epsilon,1)\cdot\mathbf{d}\cdot\mathbf{dc}+M(1,1)\cdot\mathbf{d}\cdot\mathbf{c}^2+M(\mathbf{c},1)\cdot\mathbf{d}\cdot\mathbf{c}\\
&\quad +2\cdot M(\mathbf{d},1)\cdot\mathbf{d}\cdot 1+M(\mathbf{dc},1)\cdot\mathbf{d}\cdot\epsilon\\
&= \mathbf{c}^2\mathbf{dc}+2\cdot\mathbf{cdc}^2+\mathbf{dc}^3+\mathbf{cd}^2+\mathbf{dcd}+3\cdot\mathbf{d}^2\mathbf{c},\\
M(\mathbf{dc},1\cdot\mathbf{d}) &= M(\epsilon,1)\cdot\mathbf{d}\cdot\mathrm{Pyr}(\mathbf{dc})+M(1,1)\cdot\mathbf{d}\cdot\mathrm{Pyr}(\mathbf{c}^2)+M(\mathbf{c},1)\cdot\mathbf{d}\cdot\mathrm{Pyr}(\mathbf{c})\\
&\quad +2\cdot M(\mathbf{d},1)\cdot\mathbf{d}\cdot\mathrm{Pyr}(1)+M(\mathbf{dc},1)\cdot\mathbf{d}\cdot\mathrm{Pyr}(\epsilon)\\
&= \mathbf{c}^2\mathbf{dc}^2+\mathbf{cdc}^3+\mathbf{c}^2\mathbf{d}^2+2\cdot\mathbf{cdcd}+3\cdot\mathbf{cd}^2\mathbf{c}+\mathbf{dc}^2\mathbf{d}\\
&\quad +3\cdot\mathbf{dcdc}+2\cdot\mathbf{d}^2\mathbf{c}^2+3\cdot\mathbf{d}^3.
\end{aligned}$$

Example 5.3 The **cd**-index of the free join of an m-gon with an n-gon is calculated as follows.

$$\begin{aligned}
\Psi(P_m \circledvee P_n) &= M(\mathbf{c}^2+(m-2)\cdot\mathbf{d},\mathbf{c}^2+(n-2)\cdot\mathbf{d})\\
&= M(\mathbf{c}^2,\mathbf{c}^2)+(m+n-4)\cdot M(\mathbf{c}^2,\mathbf{d})+(n-2)\cdot(m-2)\cdot M(\mathbf{d},\mathbf{d})\\
&= \mathbf{c}^5+2\cdot\mathbf{c}^3\mathbf{d}+4\cdot\mathbf{c}^2\mathbf{dc}+4\cdot\mathbf{cdc}^2+2\cdot\mathbf{dc}^3+4\cdot\mathbf{cd}^2+4\cdot\mathbf{dcd}+4\cdot\mathbf{d}^2\mathbf{c}\\
&\quad +(m+n-4)\cdot(\mathbf{c}^3\mathbf{d}+2\cdot\mathbf{c}^2\mathbf{dc}+2\cdot\mathbf{cdc}^2+\mathbf{dc}^3+3\cdot\mathbf{cd}^2+2\cdot\mathbf{dcd}+3\cdot\mathbf{d}^2\mathbf{c})\\
&\quad +(n-2)\cdot(m-2)\cdot(\mathbf{c}^2\mathbf{dc}+\mathbf{cdc}^2+2\cdot\mathbf{cd}^2+2\cdot\mathbf{dcd}+2\cdot\mathbf{d}^2\mathbf{c}).
\end{aligned}$$

From the recursions in Theorem 5.1, we obtain results about the coefficients of the **cd**-polynomial $M(u,v)$.

Proposition 5.4 *For two* **cd***-monomials u and v, let m be the number of* **d***'s in u and n be the number of* **d***'s in v. Then every* **cd***-monomial in the expansion of $M(u,v)$ has a non-negative coefficient and contains at least* $\max(m,n)$ *number of* **d***'s.*

Proof: The **cd**-polynomial $M(u,v)$ can be computed using the operation Pyr and the recursions in Theorem 5.1. In all of these expressions there are only positive terms. Hence all the coefficients are non-negative.

For the second result, by symmetry of the mixing operator it is enough to prove that any monomial in $M(u,v)$ contains at least n $\mathbf{d}$'s. We prove this by induction on the degree of v. In the base case $v = 1$ there is nothing to prove. Assume that the statement holds for v, and we will prove it for $v \cdot \mathbf{c}$ and $v \cdot \mathbf{d}$. In identity (5.1) every term contains at least n number of $\mathbf{d}$'s, which completes the case $v \cdot \mathbf{c}$. Similarly, every term in (5.2) contains at least $n+1$ number of $\mathbf{d}$'s, which completes the second case. □

To prove Theorem 5.1, we begin by observing that the mixing operator commutes with the involution $\overline{\cdot}$.

Lemma 5.5 *For two* **ab**-*polynomials* u *and* v,

$$\overline{M(u,v)} = M(\overline{u}, \overline{v}).$$

Proof: For $r \in \{1,2\}$ define $\overline{r} = 3 - r$. We claim that $\overline{M_{r,s,n}(u,v)} = M_{\overline{r},\overline{s},n}(\overline{v},\overline{u})$. This is straightforward to prove by induction on n. By summing over all (r,s,n) in the index set I, we obtain $\overline{M(u,v)} = M(\overline{v},\overline{u})$, which in turn equals $M(\overline{u},\overline{v})$. □

Recall that A_r denotes the graded poset of rank 2 having r atoms. We denote these r atoms by $a_1, \ldots, a_r$. In the following proposition, we work with the special case $r = 1$.

Proposition 5.6 *For two graded posets* P *and* Q, *we have*

$$\Psi(P \times (Q * A_1)) = \Psi(P \times Q) \cdot \mathbf{a} + \sum_{x \in P,\, \hat{0} \leq x < \hat{1}} \Psi([\hat{0},x] \times Q) \cdot \mathbf{ab} \cdot \Psi([x,\hat{1}]).$$

Proof: We consider the poset $Q * A_1$ which is the poset Q with a new rank added consisting solely of the element a_1. This is now the only coatom of $Q * A_1$. Observe that the interval $[\hat{0}, a_1]$ in $Q * A_1$ is isomorphic to the poset Q.

For a chain c in the poset $P \times (Q * A_1)$, we write it as $c = \{(\hat{0},\hat{0}) = (x_0,y_0) < (x_1,y_1) < \cdots < (x_k,y_k) = (\hat{1},\hat{1})\}$. Let i be the smallest index such that $y_i = \hat{1}$. Let $x = x_i$. Two cases now occur.

(1) The case $\hat{0} \leq x < \hat{1}$. Consider the element (x, a_1) in $P \times (Q * A_1)$. It may or may not be in the chain c. Let c' be the chain $c - \{(x,a_1)\}$ and let c'' be the chain $c \cup \{(x,a_1)\}$. The weights of the chains c' and c'' are

$$\begin{aligned} w(c') &= w_{[\hat{0},x] \times Q}(c_1) \cdot (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c'') &= w_{[\hat{0},x] \times Q}(c_1) \cdot \mathbf{b} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c') + w(c'') &= w_{[\hat{0},x] \times Q}(c_1) \cdot \mathbf{a} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \end{aligned}$$

where c_1 is the chain c restricted to $[\hat{0}, x] \times Q$ and c_2 is the chain c restricted to the interval $[(x, \hat{1}_Q), (\hat{1}_P, \hat{1}_Q)] \cong [x, \hat{1}]$. Thus summing over all chains c such that $\hat{0} \leq x < \hat{1}$, we obtain

$$\sum_{c \,:\, \hat{0} \leq x < \hat{1}} w(c) = \sum_{\hat{0} \leq x < \hat{1}} \Psi([\hat{0}, x] \times Q) \cdot \mathbf{a} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]). \tag{5.3}$$

(2) The case $x = \hat{1}$. Consider the element $(\hat{1}, a_1)$ in $P \times (Q * A_1)$. It may or may not be in the chain c. Let c' be the chain $c - \{(\hat{1}, a_1)\}$ and let c'' be the chain $c \cup \{(\hat{1}, a_1)\}$. In this case the weights of the chains c' and c'' are

$$\begin{aligned} w(c') &= w_{P \times Q}(c_1) \cdot (\mathbf{a} - \mathbf{b}), \\ w(c'') &= w_{P \times Q}(c_1) \cdot \mathbf{b}, \\ w(c') + w(c'') &= w_{P \times Q}(c_1) \cdot \mathbf{a}, \end{aligned}$$

where c_1 is the chain c restricted to the poset $P \times Q$. Summing over all chains c with $x = \hat{1}$, we obtain

$$\sum_{c \,:\, x = \hat{1}} w(c) = \Psi(P \times Q) \cdot \mathbf{a}. \tag{5.4}$$

Adding equations (5.3) and (5.4), we obtain the desired identity. $\square$

The more general case of Proposition 5.6 we simply state without proof as the result is not needed in this section. Let $\mathbf{c}_r$ denote $\mathbf{a} + (r-1) \cdot \mathbf{b}$ and $\mathbf{d}_r$ denote $\mathbf{ab} + (r-1) \cdot \mathbf{ba}$. Via a more careful chain argument, one may prove the following proposition.

Proposition 5.7 *For two graded posets P and Q, we have*

$$\Psi(P \times (Q * A_r)) = \Psi(P \times Q) \cdot \mathbf{c}_r + \sum_{\hat{0} \leq x < \hat{1}} \Psi([\hat{0}, x] \times Q) \cdot \mathbf{d}_r \cdot \Psi([x, \hat{1}]).$$

Proposition 5.8 *For two* **ab***-polynomials u and v,*

$$\begin{aligned} M(u, v \cdot \mathbf{a}) &= \sum_u{}^{*} M(u_{(1)}, v) \cdot \mathbf{ab} \cdot u_{(2)}, & (5.5) \\ M(u, v \cdot \mathbf{b}) &= \sum_u{}^{*} M(u_{(1)}, v) \cdot \mathbf{ba} \cdot u_{(2)}. & (5.6) \end{aligned}$$

Proof: Rewrite the identity in Proposition 5.6 as

$$\begin{aligned} \Psi(P \times (Q * A_1)) &= \sum_{x \in P,\, \hat{0} \leq x \leq \hat{1}} \Psi([\hat{0}, x] \times Q) \cdot \mathbf{ab} \cdot \Psi([x, \hat{1}]) \\ &= \sum_P{}^{*} \Psi(P_{(1)} \times Q) \cdot \mathbf{ab} \cdot \Psi(P_{(2)}). \end{aligned}$$

Letting $u = \Psi(P)$ and $v = \Psi(Q)$ and using the fact that Ψ is a coalgebra homomorphism, we obtain identity (5.5). Since Ψ is surjective, we know this identity holds for all **ab**-polynomials u and v. Identity (5.6) follows by applying the involution $\overline{\cdot}$ to (5.5) and using Lemma 5.5. $\square$

Proof of Theorem 5.1: By adding the two identities (5.5) and (5.6) in Proposition 5.8, we obtain (5.1). By applying Proposition 5.8 twice to the expression $M(u, v \cdot \mathbf{ab})$, and using the coassociativity of Δ^* and Lemma 4.5, we have:

$$\begin{aligned} M(u, v \cdot \mathbf{ab}) &= \sum_u^* M(u_{(1)}, v \cdot \mathbf{a}) \cdot \mathbf{ba} \cdot u_{(2)} \\ &= \sum_u^* M(u_{(1)}, v) \cdot \mathbf{ab} \cdot u_{(2)} \cdot \mathbf{ba} \cdot u_{(3)} \\ &= \sum_u^* M(u_{(1)}, v) \cdot \mathbf{ab} \cdot \mathrm{Pyr}(u_{(2)}). \end{aligned} \tag{5.7}$$

Observe that $u_{(2)}$ may be the element ϵ. However, since it is adjacent to two **b**'s, the associative law (3.4) holds. By applying the involution $\overline{\cdot}$, we obtain

$$M(u, v \cdot \mathbf{ba}) = \sum_u^* M(u_{(1)}, v) \cdot \mathbf{ba} \cdot \mathrm{Pyr}(u_{(2)}). \tag{5.8}$$

Identity (5.2) in Theorem 5.1 follows by adding identities (5.7) and (5.8). $\square$

6 The Cartesian product of polytopes

In this section and the next we show how to compute the **cd**-index of the Cartesian product of two polytopes given the **cd**-index of each polytope.

Let V be an m-dimensional polytope and W an n-dimensional polytope. The *Cartesian product* of the polytopes V and W is the $(m+n)$-dimensional polytope

$$V \times W = \{(x_1, \ldots, x_{m+n}) \in \mathbb{R}^{m+n} \; : \; (x_1, \ldots, x_m) \in V, \, (x_{m+1}, \ldots, x_{m+n}) \in W\}.$$

Let P and Q be two graded posets. The *diamond product* $P \diamond Q$ is the graded poset $(P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$. This product on posets corresponds to the Cartesian product of polytopes, that is, $\mathcal{L}(V \times W) = \mathcal{L}(V) \diamond \mathcal{L}(W)$; see Kalai [16]. Observe that B_1, the unique graded poset of rank 1, is the identity under the diamond product, that is, $P \diamond B_1 = P$ for all posets P. This corresponds to the fact that the zero-dimensional polytope consisting of a single point is the identity under the Cartesian product of polytopes.

Let κ be an algebra map from the algebra $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to itself defined by $\kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}$ and $\kappa(\mathbf{b}) = 0$. Observe that κ applied to an **ab**-monomial u is equal to zero if u contains the letter **b**. Also we have $\kappa(\mathbf{a}^n) = (\mathbf{a} - \mathbf{b})^n$. Hence for a poset P of rank $n+1$, we obtain $\kappa(\Psi(P)) = (\mathbf{a} - \mathbf{b})^n$. Extend κ to $\overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$ by $\kappa(\epsilon) = 0$.

The next definition is from [13, Section 10].

Definition 6.1 *Define the bilinear operator* $N : \overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle} \times \overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle} \longrightarrow \overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle}$ *by the identity*

$$N(u, v) = \sum_u^* \sum_v^* \kappa(u_{(1)}) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot M(u_{(2)}, v_{(2)}).$$

Proposition 6.2 (Ehrenborg-Readdy) *Let P and Q be two graded posets. Then the* **ab***-index of the diamond product $P \diamond Q$ is given by*

$$\Psi(P \diamond Q) = N(\Psi(P), \Psi(Q)).$$

Hence for two polytopes V and W, the **cd***-index of the Cartesian product $V \times W$ is given by*

$$\Psi(V \times W) = N(\Psi(V), \Psi(W)).$$

Similar to Corollary 4.3 and its proof, we obtain the following result.

Corollary 6.3 *The bilinear operator N has 1 as its identity element, and is commutative and associative, that is,*

$$\begin{aligned} N(u, 1) &= u, \\ N(u, v) &= N(v, u), \\ N(u, N(v, w)) &= N(N(u, v), w). \end{aligned}$$

for any three **ab***-polynomials u, v and w. Also it satisfies $N(u, \epsilon) = 0$.*

7 Recursion for the Cartesian product

We will now state and prove recursions for the bilinear operator N. The proof follows the same outline as the proof of Theorem 5.1. However, this time we have to study the more complicated poset $P \diamond (Q * A_r)$ when r is not necessarily equal to 1.

Theorem 7.1 *For two* **cd***-polynomials u and v, the bilinear operator N satisfies the following recursions:*

$$N(u, v \cdot \mathbf{c}) = \sum_u^* N(u_{(1)}, v) \cdot \mathbf{d} \cdot u_{(2)}, \quad (7.1)$$

$$N(u, v \cdot \mathbf{d}) = \sum_u^* N(u_{(1)}, v) \cdot \mathbf{d} \cdot \mathrm{Pyr}(u_{(2)}). \quad (7.2)$$

The first few values of the operator N are displayed in Table 2.

u	v	$N(u,v)$
$\mathbf{c}$	$\mathbf{c}$	$\mathbf{c}^2 + 2 \cdot \mathbf{d}$
$\mathbf{c}$	$\mathbf{c}^2$	$\mathbf{c}^3 + 2 \cdot \mathbf{cd} + 2 \cdot \mathbf{dc}$
$\mathbf{c}$	$\mathbf{d}$	$\mathbf{cd} + 2 \cdot \mathbf{dc}$
$\mathbf{c}^2$	$\mathbf{c}^2$	$\mathbf{c}^4 + 2 \cdot \mathbf{c}^2\mathbf{d} + 4 \cdot \mathbf{cdc} + 2 \cdot \mathbf{dc}^2 + 4 \cdot \mathbf{d}^2$
$\mathbf{c}^2$	$\mathbf{d}$	$\mathbf{c}^2\mathbf{d} + 2 \cdot \mathbf{cdc} + 2 \cdot \mathbf{dc}^2 + 2 \cdot \mathbf{d}^2$
$\mathbf{d}$	$\mathbf{d}$	$\mathbf{cdc} + \mathbf{dc}^2 + 2 \cdot \mathbf{d}^2$

Table 2: The operator $N(u,v)$ computed for **cd**-monomials of low degree.

Example 7.2 The expressions $N(\mathbf{dc},\mathbf{c})$ and $N(\mathbf{dc},\mathbf{d})$ are computed as follows.

$$\begin{aligned}
N(\mathbf{dc}, 1 \cdot \mathbf{c}) &= N(1,1) \cdot \mathbf{d} \cdot \mathbf{c}^2 + N(\mathbf{c},1) \cdot \mathbf{d} \cdot \mathbf{c} + 2 \cdot N(\mathbf{d},1) \cdot \mathbf{d} \cdot 1 + N(\mathbf{dc},1) \cdot \mathbf{d} \cdot \epsilon \\
&= \mathbf{cdc} + 2 \cdot \mathbf{dc}^2 + 2 \cdot \mathbf{d}^2, \\
N(\mathbf{dc}, 1 \cdot \mathbf{d}) &= N(1,1) \cdot \mathbf{d} \cdot \mathrm{Pyr}(\mathbf{c}^2) + N(\mathbf{c},1) \cdot \mathbf{d} \cdot \mathrm{Pyr}(\mathbf{c}) \\
&\quad + 2 \cdot N(\mathbf{d},1) \cdot \mathbf{d} \cdot \mathrm{Pyr}(1) + N(\mathbf{dc},1) \cdot \mathbf{d} \cdot \mathrm{Pyr}(\epsilon) \\
&= \mathbf{cdc}^2 + \mathbf{dc}^3 + \mathbf{cd}^2 + 2 \cdot \mathbf{dcd} + 3 \cdot \mathbf{d}^2\mathbf{c}.
\end{aligned}$$

Example 7.3 The **cd**-index of the Cartesian product of an n-gon with an m-gon is calculated as follows.

$$\begin{aligned}
N(\mathbf{c}^2 + (n-2) \cdot \mathbf{d}, \mathbf{c}^2 + (m-2) \cdot \mathbf{d}) &= \mathbf{c}^4 + 2 \cdot \mathbf{c}^2\mathbf{d} + 4 \cdot \mathbf{cdc} + 2 \cdot \mathbf{dc}^2 + 4 \cdot \mathbf{d}^2 \\
&\quad + (n+m-4) \cdot (\mathbf{c}^2\mathbf{d} + 2 \cdot \mathbf{cdc} + 2 \cdot \mathbf{dc}^2 + 2 \cdot \mathbf{d}^2) \\
&\quad + (n-2) \cdot (m-2) \cdot (\mathbf{cdc} + \mathbf{dc}^2 + 2 \cdot \mathbf{d}^2).
\end{aligned}$$

Similar to Proposition 5.4, we obtain a statement giving a lower bound on the number of **d**'s appearing in $N(u,v)$.

Proposition 7.4 *For two* **cd***-monomials u and v, let m be the number of* **d***'s in u and n be the number of* **d***'s in v. Then every monomial in the expansion of $N(u,v)$ has a non-negative coefficient and contains at least* $\max(m,n)$ *number of* **d***'s.*

In order to obtain a recursion for the operator N, we will generalize [12, Proposition 5.3] and [13, Proposition 4.3]. This will be done in the spirit of Proposition 5.7.

Proposition 7.5 *For two graded posets P and Q, we have that*

$$\Psi(P \diamond (Q * A_r)) = \Psi(P \diamond Q) \cdot \mathbf{c}_r + \sum_{\hat{0} < x < \hat{1}} \Psi([\hat{0},x] \diamond Q) \cdot \mathbf{d}_r \cdot \Psi([x,\hat{1}]).$$

Proof: Consider the poset $Q * A_r$. Recall this is the poset Q together with a new rank consisting of the elements $a_1, \ldots, a_r$ forming the r coatoms of the poset $Q * A_r$.

For c a chain in the poset $P \diamond (Q * A_r)$, we write it as $c = \{\hat{0} < (x_1, y_1) < \cdots < (x_k, y_k) = (\hat{1}, \hat{1})\}$. Let i be the smallest index such that $y_i = \hat{1}$. Two cases occur, each having two subcases.

(1) Assume that $i \geq 2$, that is, y_{i-1} exists, and $y_{i-1} \in \{a_1, \ldots, a_{r-1}\}$. Let $x = x_{i-1}$. The first subcase is $\hat{0} < x < \hat{1}$. The element $(x, \hat{1})$ may or may not be in the chain c. Let c' be the chain without this element, that is, $c' = c - \{(x, \hat{1})\}$ and let c'' be the chain with this element, that is, $c'' = c \cup \{(x, \hat{1})\}$. The weights of chains c' and c'' are then

$$\begin{aligned} w(c') &= w_{[\hat{0},x]\diamond Q}(c_1) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot w_{[x,\hat{1}]}(c_2), \\ w(c'') &= w_{[\hat{0},x]\diamond Q}(c_1) \cdot \mathbf{b} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c') + w(c'') &= w_{[\hat{0},x]\diamond Q}(c_1) \cdot \mathbf{b} \cdot \mathbf{a} \cdot w_{[x,\hat{1}]}(c_2), \end{aligned}$$

where c_1 and c_2 are the restrictions of c to the posets $[\hat{0}, x] \diamond Q$ and $[(x, \hat{1}_Q), (\hat{1}_P, \hat{1}_Q)] \cong [x, \hat{1}]$, respectively. The second subcase is $x = \hat{1}$. The weight of the chain c is then given by

$$w(c) = w_{P\diamond Q}(c_1) \cdot \mathbf{b},$$

where c_1 is the restriction of the chain c to the poset $P \diamond Q$.

(2) The second case is $i = 1$ or $y_{i-1} \notin \{a_1, \ldots, a_{r-1}\}$, that is, y_{i-1} does not exist, $y_{i-1} = a_r$ or $y_{i-1} \in Q - \{\hat{0}, \hat{1}\}$. Let $x = x_i$. Consider the element (x, a_r). It may or may not be in the chain c. Let c' be the chain $c - \{(x, a_r)\}$ and let c'' be $c \cup \{(x, a_r)\}$. The first subcase is $\hat{0} < x < \hat{1}$. The weights of the chains c' and c'' are then given by

$$\begin{aligned} w(c') &= w_{[\hat{0},x]\diamond Q}(c_1) \cdot (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c'') &= w_{[\hat{0},x]\diamond Q}(c_1) \cdot \mathbf{b} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c') + w(c'') &= w_{[\hat{0},x]\diamond Q}(c_1) \cdot \mathbf{a} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2). \end{aligned}$$

As in the first case, c_1 and c_2 are the restrictions to the posets $[\hat{0}, x] \diamond Q$ and $[x, \hat{1}]$. The second subcase is $x = \hat{1}$. The weights of the chains c' and c'' are then given by

$$\begin{aligned} w(c') &= w_{P\diamond Q}(c_1) \cdot (\mathbf{a} - \mathbf{b}), \\ w(c'') &= w_{P\diamond Q}(c_1) \cdot \mathbf{b}, \\ w(c') + w(c'') &= w_{P\diamond Q}(c_1) \cdot \mathbf{a}, \end{aligned}$$

where c_1 is the restrictions to $P \diamond Q$.

Summing over all chains c in $P \diamond (Q * A_r)$, we obtain

$$\begin{aligned} \Psi(P \diamond (Q * A_r)) &= \Psi(P \diamond Q) \cdot (\mathbf{a} + (r-1) \cdot \mathbf{b}) \\ &\quad + \sum_{\hat{0}<x<\hat{1}} \Psi([\hat{0}, x] \diamond Q) \cdot (\mathbf{ab} + (r-1) \cdot \mathbf{ba}) \cdot \Psi([x, \hat{1}]). \end{aligned}$$

Observe that the contribution from the first case has a factor of $r - 1$ since there are $r - 1$ choices for the element y_{i-1}. □

Proposition 7.6 *For* **ab***-polynomials* u *and* v*,*

$$\begin{aligned} N(u, v \cdot \mathbf{a}) &= \sum_u^* N(u_{(1)}, v) \cdot \mathbf{ab} \cdot u_{(2)}, \\ N(u, v \cdot \mathbf{b}) &= \sum_u^* N(u_{(1)}, v) \cdot \mathbf{ba} \cdot u_{(2)}. \end{aligned}$$

Proof: The proof is similar to the proof of Proposition 5.8. Let $u = \Psi(P)$ and $v = \Psi(Q)$ in Proposition 7.5. The first identity follows by setting $r = 1$. By observing that both sides in Proposition 7.5 are polynomials in the variable r, the second identity follows by extracting the linear term in r. Since Ψ is surjective, we know that the identities will hold for all **ab**-polynomials u and v. □

An important corollary to Proposition 7.6 is the following.

Corollary 7.7 *For two* **ab***-polynomials* u *and* v*,*

$$\overline{N(u, v)} = N(\overline{u}, \overline{v}).$$

Proof: This is a straightforward induction argument on the degree of the variable v. The induction basis is $v = 1$ which follows from Corollary 6.3. The induction steps $v = w \cdot \mathbf{a}$ and $v = w \cdot \mathbf{b}$ follow from Proposition 7.6. □

Proof of Theorem 7.1: By adding the two identities in Proposition 7.6, we obtain the recursion (7.1). Now by applying Proposition 7.6 twice, we have

$$\begin{aligned} N(u, v \cdot \mathbf{ab}) &= \sum_u^* N(u_{(1)}, v \cdot \mathbf{a}) \cdot \mathbf{ba} \cdot u_{(2)} \\ &= \sum_u^* N(u_{(1)}, v) \cdot \mathbf{ab} \cdot u_{(2)} \cdot \mathbf{ba} \cdot u_{(3)} \\ &= \sum_u^* N(u_{(1)}, v) \cdot \mathbf{ab} \cdot \mathrm{Pyr}(u_{(2)}). \end{aligned} \tag{7.3}$$

Observe that $u_{(2)}$ could be the element ϵ. However, the associative law (3.4) holds since $u_{(2)}$ is between two **b**'s.

Apply the involution $\overline{\cdot}$ to give

$$N(u, v \cdot \mathbf{ba}) = \sum_u^* N(u_{(1)}, v) \cdot \mathbf{ba} \cdot \mathrm{Pyr}(u_{(2)}). \tag{7.4}$$

By summing the two identities (7.3) and (7.4), we obtain the second recursion (7.2). □

We end this section by listing a number of other identities that hold for the two bilinear operators M and N.

Proposition 7.8 *For two* **ab***-polynomials* u *and* v*, we have*

$$\begin{array}{rcl} N(\mathbf{a}\cdot u, \mathbf{a}\cdot v) & = & \mathbf{a}\cdot M(u,v), \\ N(\mathbf{b}\cdot u, \mathbf{b}\cdot v) & = & \mathbf{b}\cdot M(u,v), \\ N((\mathbf{b}-\mathbf{a})\cdot u, v) & = & (\mathbf{b}-\mathbf{a})\cdot N(u,v), \\ N(\mathbf{a}\cdot u, \mathbf{b}\cdot v) & = & \mathbf{a}\cdot N(u, \mathbf{b}\cdot v) + \mathbf{b}\cdot N(\mathbf{a}\cdot u, v), \\ M(u,v) & = & N(\mathbf{a}\cdot u, v) + N(u, \mathbf{b}\cdot v). \end{array}$$

The first and third identity have direct combinatorial proofs. The others can be obtained by applying the involution $\overline{\cdot}$ and Definition 6.1.

8 Inequalities

Many of the interesting problems for the **cd**-index involve determining inequalities among its coefficients. The Stanley conjecture is one such problem of this type. To make this more precise, define a partial order $\leq$ on $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ by comparing coefficients, that is, for two **cd**-polynomials u and v, we say that $u \leq v$ if for all **cd**-monomials w the coefficient of w in u is less than or equal to the coefficient of w in v. We say that a **cd**-polynomial u is non-negative if $0 \leq u$. Observe that this order behaves well with respect to addition, multiplication, coproduct and the pyramid operation. In other words, if u and v are two non-negative **cd**-polynomials then we have that $u+v$, $u\cdot v$ and $\text{Pyr}(u)$ are also non-negative. Also when considering the coproduct $\Delta(u) = \sum_u u_{(1)} \otimes u_{(2)}$, each of the $u_{(1)}$ and $u_{(2)}$ is non-negative. Moreover, Propositions 5.4 and 7.4 show that the two bilinear operations M and N preserve non-negativity.

One may similarly define a partial order $\leq'$ on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ by comparing coefficients of **ab**-monomials. It is important to notice that the order $\leq'$ restricted to the subring $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ is weaker than the order $\leq$. For instance, $0 \leq' \mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2 - \mathbf{d}$, whereas $0 \not\leq \mathbf{c}^2 - \mathbf{d}$.

Theorem 8.1 *Let* u*,* v *and* w *be three non-negative* **cd***-polynomials. Then these* **cd***-polynomials satisfy the inequality*

$$M(u, N(v,w)) \leq N(M(u,v), w). \tag{8.1}$$

Since the **cd**-index of a polytope is non-negative, we conclude the following polytopal inequality.

Theorem 8.2 *Let* U*,* V *and* W *be three polytopes. Then we have the inequality*

$$\Psi(U \circledvee (V \times W)) \leq \Psi((U \circledvee V) \times W). \tag{8.2}$$

In order to prove Theorem 8.1, we must first generalize equation (5.2).

Proposition 8.3 *For three* **ab***-polynomials u, v and w, we have*

$$M(u, v \cdot \mathbf{d} \cdot w) = \sum_{u}^{*} M(u_{(1)}, v) \cdot \mathbf{d} \cdot M(u_{(2)}, w). \tag{8.3}$$

Moreover, this equation also holds when $u, v, w \in \overline{\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle}$.

Proof: The argument is by induction on the degree of w. When $w = \epsilon$ this is equation (5.1), while when $w = 1$ this is equation (5.2). This completes the base cases. Consider now the case when $w = w' \cdot \mathbf{a}$. We have

$$\begin{aligned} M(u, v \cdot \mathbf{d} \cdot w' \cdot \mathbf{a}) &= \sum_{u}^{*} M(u_{(1)}, v \cdot \mathbf{d} \cdot w') \cdot \mathbf{ab} \cdot u_{(2)} \\ &= \sum_{u}^{*} M(u_{(1)}, v) \cdot \mathbf{d} \cdot M(u_{(2)}, w') \cdot \mathbf{ab} \cdot u_{(3)} \\ &= \sum_{u}^{*} M(u_{(1)}, v) \cdot \mathbf{d} \cdot M(u_{(2)}, w' \cdot \mathbf{a}), \end{aligned}$$

where the second step is by the induction hypothesis. This completes the case $w = w' \cdot \mathbf{a}$. By applying the involution $\overline{\cdot}$, the case $w = w' \cdot \mathbf{b}$ follows. □

Similar to Proposition 8.3, we have the identity

$$N(u, v \cdot \mathbf{d} \cdot w) = \sum_{u}^{*} N(u_{(1)}, v) \cdot \mathbf{d} \cdot M(u_{(2)}, w). \tag{8.4}$$

In equations (8.3) and (8.4) one may replace the $\mathbf{d}$ with either $\mathbf{ab}$ or $\mathbf{ba}$ and the identities still hold.

Proof of Theorem 8.1: We begin by observing that the inequality $M(u, N(v, w)) \leq N(M(u, v), w)$ holds if any of the variables u, v and w are equal to the element ϵ. We now proceed to prove the inequality by induction on the degree of w. For the base case $w = 1$, both sides of inequality (8.1) are equal, and there is nothing to prove.

Let $w = w' \cdot \mathbf{c}$. Then we have that

$$\begin{aligned} M(u, N(v, w' \cdot \mathbf{c})) &= \sum_{v}^{*} M(u, N(v_{(1)}, w') \cdot \mathbf{d} \cdot v_{(2)}) \\ &= \sum_{v}^{*} \sum_{u}^{*} M(u_{(1)}, N(v_{(1)}, w')) \cdot \mathbf{d} \cdot M(u_{(2)}, v_{(2)}) \\ &\leq \sum_{v}^{*} \sum_{u}^{*} N(M(u_{(1)}, v_{(1)}), w') \cdot \mathbf{d} \cdot M(u_{(2)}, v_{(2)}) \\ &= N(M(u, v), w' \cdot \mathbf{c}). \end{aligned}$$

The steps used here are Theorem 7.1, Proposition 8.3, the induction hypothesis and finally Theorem 7.1 with Lemma 4.6. Hence this completes the case $w = w' \cdot \mathbf{c}$.

Now consider the case when $w = w' \cdot \mathbf{d}$. The left-hand side of inequality (8.1) can be expanded as

$$M(u, N(v, w' \cdot \mathbf{d})) = \sum_{v}^{*} M(u, N(v_{(1)}, w') \cdot \mathbf{d} \cdot \mathrm{Pyr}(v_{(2)}))$$

$$
\begin{aligned}
&= \sum_v{}^* \sum_u{}^* M(u_{(1)}, N(v_{(1)}, w')) \cdot \mathbf{d} \cdot M(u_{(2)}, \mathrm{Pyr}(v_{(2)})) \\
&\leq \sum_v{}^* \sum_u{}^* N(M(u_{(1)}, v_{(1)}), w') \cdot \mathbf{d} \cdot M(u_{(2)}, \mathrm{Pyr}(v_{(2)})) \\
&= \sum_u{}^* \sum_v{}^* N(M(u_{(1)}, v_{(1)}), w') \cdot \mathbf{d} \cdot \mathrm{Pyr}(M(u_{(2)}, v_{(2)})) \\
&= N(M(u, v), w' \cdot \mathbf{d}).
\end{aligned}
$$

The steps are the same as in the previous case with the added ingredient of the associativity of the mixing operator. This yields the desired inequality and completes the induction. □

Expanding the **cd**-indices in Theorem 8.2 in terms of the flag h- and flag f-vectors, we obtain the following corollary. The first inequality should be straightforward to prove, whereas we expect a proof of the second stronger inequality using elementary methods to be much more difficult.

Corollary 8.4 *Let U, V and W be three polytopes with the sum of their dimensions equal to $n-1$. Then we have the two inequalities*

$$
\begin{aligned}
f_S(U \circledvee (V \times W)) &\leq f_S((U \circledvee V) \times W), \\
h_S(U \circledvee (V \times W)) &\leq h_S((U \circledvee V) \times W),
\end{aligned}
$$

where S is a subset of $\{1, \ldots, n\}$.

9 Concluding remarks

Billera and Ehrenborg [5] recently proved that the **cd**-index of the n-dimensional simplex is coefficient-wise smaller than the **cd**-index of any n-dimensional polytope. This was conjectured by Stanley. Let B_n denote the Boolean algebra of rank n, that is, the face lattice of the $(n-1)$-dimensional simplex. Stanley's complete conjecture is that the Boolean algebra B_n minimizes the **cd**-index over all Gorenstein* lattices of rank n [24]. The following corollary of our results offers some evidence supporting this conjecture.

Corollary 9.1 *Let P and Q be two Eulerian posets of ranks m and n. Assume that the respective* **cd***-indices are greater than or equal to the* **cd***-indices of the Boolean algebra of the same rank, that is, $\Psi(P) \geq \Psi(B_m)$ and $\Psi(Q) \geq \Psi(B_n)$. Then we have*

$$
\Psi(P \times Q) \geq \Psi(B_{m+n}) \quad \text{and} \quad \Psi(P \diamond Q) \geq \Psi(B_{m+n-1}).
$$

The first inequality follows directly from Proposition 5.4. To prove the second, apply Theorem 8.1.

It worth noting that the recursions for the mixing operator M in Theorem 5.1 and the operator N in Theorem 7.1 are identical. This suggests that there should be a unified theory of the free join and the Cartesian product.

Other recent work in proving inequalities includes Novik [20], who gives lower bounds on the coefficients of the **cd**-index of Eulerian Buchsbaum complexes. Bayer and Hetyei [3] obtain linear inequalities for the flag vectors of Eulerian posets.

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