A bijective answer to a question of Zvonkin*

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Abstract

The purpose of this note is to give a bijective proof of the identity $E\left[\prod_{1\leq i<j\leq n}(X_j - X_i)^2\right] = 0!\cdot 1!\cdots n!$, where $X_1,\ldots,X_n$ are independent, identically distributed normal random variables with mean 0 and variance 1. The bijection is obtained by combining a bijection of Gessel and a bijection of Ehrenborg with the interpretation that the moments of the normal distribution count the number of matchings.

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Theorem 1 *Let $X_1,\ldots,X_n$ be independent, identically distributed normal random variables with mean* 0 *and variance* 1*. Then we have*

$$E\left[\prod_{1\leq i<j\leq n}(X_j - X_i)^2\right] = 0!\cdot 1!\cdots n!. \tag{1}$$

Zvonkin [3] asked for a combinatorial proof of this identity. The purpose of this note is to provide such a proof. Recall that the nth moment of a normally distributed random variable with mean 0 and variance 1 has the combinatorial interpretation as the number of matchings on an n element set. It is straightforward to see that this number, which we denote by $m(n)$, is given by $(n-1)\cdot(n-3)\cdots 1$ if n is even and zero if n is odd.

We begin by recalling Gessel's bijective proof [2] of the Vandermonde identity

$$\prod_{1\leq i<j\leq n}(a_j - a_i) = \sum_{\sigma}(-1)^{\sigma}\cdot\prod_{i=1}^{n} a_i^{\sigma(i)}, \tag{2}$$

where we view σ as a bijection from $\{1,2,\ldots,n\}$ to $\{0,1,\ldots,n-1\}$. Multiplying out the left-hand side of (2) we obtain $2^{\binom{n}{2}}$ terms. To each of these terms associate a tournament T on the vertices $1,2,\ldots,n$ in the following manner. If we choose the term a_j from the factor $(a_j - a_i)$, let the tournament T have the directed arc $i \longrightarrow j$. Otherwise let T have the arc in the opposite direction. Let the sign of the

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tournament, denoted $(-1)^T$, be -1 to the number of arcs that goes from a node to a smaller labeled node. Hence the left-hand side of the Vandermonde identity (2) is given by

$$\prod_{1 \leq i < j \leq n} (a_j - a_i) = \sum_T (-1)^T \prod_{i=1}^{n} a_i^{\text{indeg}}, \tag{3}$$

where T ranges over all tournaments on n elements.

In the summation appearing in equation (3) all the terms corresponding to intransitive tournaments will cancel. To see this, let T be an intransitive tournament with a directed 3-cycle $i \longrightarrow j \longrightarrow k \longrightarrow i$. By reversing this cycle, we change the sign of the tournament but not any of the indegrees. Hence it is enough to consider transitive tournaments. Note a transitive tournament T corresponds to a permutation σ by letting $\sigma(i)$ be one more than the indegree of the node i. Finally observe that the sign of the permutation σ is the same as the sign associated to the tournament.

Now consider the left-hand side of (1) in Theorem 1. By expanding the square of the Vandermonde product, we obtain a sum over pairs of tournaments. The exponent d_i in a monomial $X_1^{d_1} \cdot X_2^{d_2} \cdots X_n^{d_n}$ in the expansion is the total indegree at the node i. The expected value of this monomial is $m(d_1) \cdot m(d_2) \cdots m(d_n)$. Hence the combinatorial structure of the terms in the expansion of the left-hand side of (1) is as follows. Let W be $(T_1, T_2, M_1, \ldots, M_n)$ where T_1 and T_2 are tournaments on the nodes $1, \ldots, n$ and M_i is a matching on all the arcs arriving at node i. Finally, let the sign of such a structure W, that is, $(-1)^W$, be the product of the signs of the two tournaments T_1 and T_2.

Thus, Theorem 1 claims that $\sum_W (-1)^W = 0! \cdot 1! \cdots n!$, where W ranges over all such structures. We prove this in two steps. First apply the Gessel involution to each tournament separately. Hence, it is enough to consider the transitive tournaments which correspond to permutations. The expression then becomes,

$$\begin{aligned} \sum_\pi \sum_\tau (-1)^\pi \cdot (-1)^\tau \cdot \prod_{i=1}^{n} m(\pi(i) + \tau(i)) &= \sum_\pi \sum_\tau (-1)^{\pi^{-1} \circ \tau} \cdot \prod_{i=0}^{n-1} m(i + \pi^{-1}(\tau(i))) \\ &= n! \cdot \sum_\sigma (-1)^\sigma \cdot \prod_{i=0}^{n-1} m(i + \sigma(i)) \\ &= n! \cdot \det\left(m(i+j)\right)_{0 \leq i,j \leq n-1}. \end{aligned}$$

Here the permutations π and τ come respectively from the transitive tournaments T_1 and T_2, and $\sigma = \pi^{-1} \circ \tau$ is a permutation on the set $\{0, 1, \ldots, n-1\}$. The factor $n!$ is due to the fact that the resulting sum does not depend on π. The determinant obtained is a Hankel determinant with value $0! \cdot 1! \cdots (n-1)!$. This proves Theorem 1. It remains to evaluate the determinant with a bijection.

In [1] Ehrenborg gives a bijective evaluation of a different Hankel determinant, namely $\det\left(e_{i+j}(x)\right)$, where $e_n(x)$ is the nth exponential polynomial. That bijection works with partitions. By restricting the bijection to matchings, we obtain the desired bijection. For completeness, we present this bijection. Let $R_0, \ldots, R_{n-1}, C_0, \ldots, C_{n-1}$ be disjoint sets such that the cardinality of R_i and C_i is each i. Moreover, assume that the set R_i possesses a linear order. Let S be the disjoint union of all the R_i's and C_i's. A matching M of the set S and a permutation σ are called *compatible* if for all pairs P in M there exists an index i such that $P \subseteq R_i \cup C_{\sigma(i)}$. Thus our determinant can be expressed as

$$\det\left(m(i+j)\right)_{0 \leq i,j \leq n-1} = \sum_{(\sigma, M)} (-1)^\sigma,$$

where the sum ranges over permutations σ and compatible matchings M.

For σ and M compatible, define the *crossing numbers* a_i to be the number of pairs P that intersect both R_i and $C_{\sigma(i)}$. Observe that $a_i \leq |R_i| = i$ and $a_i \leq |C_{\sigma(i)}| = \sigma(i)$. We claim that

$$\sum_{(\sigma,M)} (-1)^{\sigma} = \prod_{i=0}^{n-1} i!,$$

where the sum ranges over permutations σ and matchings M so that their crossing numbers are distinct. In this case we have $a_i = i$ for all indices i and we conclude that σ is the identity permutation. Since there are $i!$ ways to choose a matching between R_i and C_i, the claim follows.

It remains to show that there is a sign reversing involution on the pairs of permutations and matchings that do not have distinct crossing numbers. For such a pair (σ, M) let (j, k) be the lexicographically least pair such that $a_j = a_k$. Let σ' be the permutation σ composed with the transposition (j, k). Clearly, $(-1)^{\sigma'} = -(-1)^{\sigma}$.

Now we construct a matching M' compatible with σ'. Let $a = a_j = a_k$. Let $\{r_1, c_1\}, \ldots, \{r_a, c_a\}$ be the pairs of $R_j \cup C_{\sigma(j)}$ that intersect both R_j and $C_{\sigma(j)}$. We can order these pairs according to the elements $r_1, \ldots, r_a$. Similarly, let $\{r'_1, c'_1\}, \ldots, \{r'_a, c'_a\}$ be the pairs of $R_k \cup C_{\sigma(k)}$ intersecting both R_k and $C_{\sigma(k)}$. Let M' be the matching obtained from M by replacing these pairs with:

$$\{r_1, c'_1\}, \ldots, \{r_a, c'_a\}, \qquad \{r'_1, c_1\}, \ldots, \{r'_a, c_a\}.$$

Observe that the M' is compatible with σ'.

We claim that the map $(\sigma, M) \longmapsto (\sigma', M')$ is an involution. Observe that both the pairs (σ, M) and (σ', M') have the same sequence of crossing numbers. Hence the map, when applied again, chooses the same indices j and k and switches pairs back. This proves that the map is an involution.

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