MA 322-021
2nd Summer Session
Review Questions for Final Exam

The following list is not exhaustive. Be sure to study old exams and homework problems too. In particular, I did not include many computation problems in the list below. Be sure to practice row-reduction, matrix algebra, and inner product algebra including length of vectors.

1. Find the general form of the solution of the following system of linear equations. Write your answer in parametric vector form.

$$\begin{cases} x_1 - 5x_2 - 9x_3 + 8x_4 = -7 \\ 2x_2 + 6x_3 - 8x_4 = 4 \\ 6x_2 + 18x_3 - 24x_4 = 12 \end{cases}$$

2. Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$T(x_1, x_2) = (5x_1 + 7x_2, -3x_1 - 6x_2).$$

   Determine whether T is invertible. If so, find its inverse.

3. Given the system of equations

$$\begin{cases} -3x_1 + hx_2 = 0 \\ 4x_1 + 2x_2 = 3k \end{cases}$$

   Find h and k such that the solution set of the system does not exist, is unique, and is infinite.

4. Let A be an $m \times n$ matrix. Explain why Col A is a subspace of $\mathbb{R}^m$ and Nul A is a subspace of $\mathbb{R}^n$. That is, actually show they are subspaces via the definition.

5. Let $\boldsymbol{v_1} = (1, 4, -6), \boldsymbol{v_2} = (4, -1, -7)$, and $\boldsymbol{v_3} = (2, 2, 0)$. Does the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ form a basis for $\mathbb{R}^3$?

6. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 0 & 0 \\ 6 & 1 & 1 \end{bmatrix}$. Find eigenvalues and bases for the corresponding eigenspaces.

7. Let $\boldsymbol{u}, \boldsymbol{v}$ be vectors in $\mathbb{R}^n$. Prove that Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a subspace of $\mathbb{R}^n$.

8. Determine whether the following matrix is invertible. If it is, find its inverse.

$$A = \begin{bmatrix} 1 & -14 & 0 \\ -9 & -21 & 0 \\ 16 & 10 & -2 \end{bmatrix}$$

9. Let T be the shear transformation of $\mathbb{R}^n$ which sends $\boldsymbol{e_1}$ to $\boldsymbol{e_2}$ and sends $\boldsymbol{e_2}$ to $2\boldsymbol{e_1}+\boldsymbol{e_2}$, and let S be the linear transformation which rotates $\mathbb{R}^2$ counterclockwise through an angle of $\pi/4$. Determine whether the composite transformations

$$T \circ S \text{ and } S \circ T$$

are the same transformation.

10. Let $A = \begin{bmatrix} 4 & 8 & -4 & 0 & 14 \\ -6 & -9 & 3 & -3 & -15 \\ -12 & -16 & 0 & -8 & -12 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$. Find bases for Col A and Nul A.

11. Let $A = \begin{bmatrix} 2 & 4 \\ 10 & 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} a \\ b \end{bmatrix}$ where $a, b \in \mathbb{R}$. Find the inverse of A and use it to find an expression for the solution to $A\boldsymbol{x} = \boldsymbol{b}$.

12. Let $\boldsymbol{u} = (1,2)$ and $\boldsymbol{y} = (1,1)$ be vectors in $\mathbb{R}^2$. Find the orthogonal projection of $\boldsymbol{y}$ onto $\boldsymbol{u}$. That is, find $\hat{\boldsymbol{y}} = \text{proj}_{\boldsymbol{u}}\boldsymbol{y}$ What is the distance between $\boldsymbol{y}$ and Span$\{\boldsymbol{u}\}$?

13. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$. What is A^T, the transpose of A?

14. Let A be an $n \times n$ matrix and let $a_{i,j}$ denote the (i,j)-entry of A. What is the (i,j)-entry of A^T?

15. Let A, B be invertible $n \times n$ matrices. Explain why $(AB)^{-1}$ exists and why $(AB)^{-1} = B^{-1}A^{-1}$. (You cannot use the fact that the product of invertible matrices is invertible. That is what you must explain).

16. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 10 & 8 \\ 0 & 0 & 5 \end{bmatrix}$ Diagonalize A.

17. Give an example that shows $AB \neq BA$ in general.

18. Give a specific example where $AB = BA$.

19. Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$ be a set of vectors in a vector space V and let H =Span$\{S\}$. If one of the vectors, say $\boldsymbol{v_k}$ is a linear combination of the others, why does the set formed by removing $\boldsymbol{v_k}$ from S still span H?

20. Let $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$. Perform a single row operation on A and determine whether or not $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b} \in \mathbb{R}^2$.

21. Let $H = \left\{ \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} : x, y \in \mathbb{R} \right\}$. That is, H is the subset of $\mathbb{R}^3$ such that the third coordinate is always 3. Is H a subspace of $\mathbb{R}^3$?

22. Let A be a 12×5 matrix. You may assume that Nul $(A^T A)$ =Nul A. (This is true of any matrix.) What is the size of $A^T A$? Suppose the columns of A are linearly independent. Explain why $A^T A$ is invertible.

23. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ be functions that satisfy

$$T(S(\boldsymbol{x})) = \boldsymbol{x} \text{ and } S(T(\boldsymbol{x})) = \boldsymbol{x},$$

and suppose that **only** T is a linear transformation. Show directly that S is also a linear transformation.

24. Let $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. What is A^2? A^3? A^{2137}? What is A^k, for $k = 0, 1, 2, 3, \ldots$?

25. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 2 \\ 5 & 3 \end{bmatrix}$. What is the rank of A? Use The Rank Theorem to determine the dimension of the null space of A.

26. Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ be a set of four vectors in $\mathbb{R}^3$. Suppose that the span of S is all of $\mathbb{R}^3$. Explain why there is a set of three linearly independent vectors in S.

27. Calculate the determinant of $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & -4 \\ 3 & -2 & 5 \end{bmatrix}$ two different ways.

28. Suppose U is a matrix such that $U^T U = I$. Show that $\det(U) = \pm 1$.

29. Let $T(\boldsymbol{x}) = A\boldsymbol{x}$ be a linear transformation. Determine whether $\boldsymbol{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ is in the range of T where

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}.$$

30. Let $\boldsymbol{u_1} = (1, 0, 1), \boldsymbol{u_2} = (-1, 4, 1), \boldsymbol{u_3} = (2, 1, -2)$, and $\boldsymbol{x} = (8, -4, -3)$. Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is an orgthogonal basis for $\mathbb{R}^3$. Express $\boldsymbol{x}$ as a linear combination of $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ two different ways.