Exam 1

Math 322-022
6/29/09

Name: ______________________
Four Digit Code: ______________________

Solutions

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- You may use any non-graphing calculator except cell phone calculators.
- The use of any electronic equipment such as mp3 players, iPods, cell phones, etc., is forbidden (except calculators). You may NOT use calculators on cell phones.
- Justify your answers algebraically whenever possible to ensure full credit. When you do use your calculator, show and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- This test has 7 problems and is worth 100 points. It is your responsibility to make sure that you have all 7 of the pages!
- The use of any outside assistance, including notes, books, or other people's exams is considered cheating and will result (at a minimum) in a score of 0 on this exam.
- Good luck!

Question	Score	Maximum
1		10
2		20
3		10
4		10
5		15
6		15
7		20
Total		100

1. (*10 points*) Let B be the matrix

$$\begin{bmatrix} 3 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & -1 & -2 \\ 2 & -1 & 3 & 0 & 4 \end{bmatrix}$$

Using as few elementary row operations as you can, find a matrix in echelon form which is row-equivalent to B.

$$\begin{bmatrix} 3 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & -1 & -2 \\ 2 & -1 & 3 & 0 & 4 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 1 & -1 & -1 & -2 \\ 3 & -2 & 1 & 2 & 1 \\ 2 & -1 & 3 & 0 & 4 \end{bmatrix}$$

$$\begin{array}{c} \\ -3R_1+R_2 \\ \sim \\ -2R_1+R_3 \end{array} \begin{bmatrix} 1 & 1 & -1 & -1 & -2 \\ 0 & -5 & 4 & 5 & 7 \\ 0 & -3 & 5 & 2 & 8 \end{bmatrix}$$

$$\begin{array}{c} \\ \sim \\ -\frac{3}{5}R_2+R_3 \end{array} \begin{bmatrix} 1 & 1 & -1 & -1 & -2 \\ 0 & -5 & 4 & 5 & 7 \\ 0 & 0 & \frac{8}{5} & -1 & \frac{19}{5} \end{bmatrix}$$

This is in echelon form since each row has a pivot, there are zeros below each pivot and each pivot is below and to the right of the on to the left & above.

2. (*20 points*) Find the general form of the solution of the following system of linear equations. Write your answer in parametric vector form.

$$\begin{cases} x_1 - 5x_2 - 9x_3 + 8x_4 = -7 \\ x_2 + 3x_3 - 4x_4 = 2 \\ 2x_2 + 6x_3 - 8x_4 = 4 \end{cases}$$

$$\begin{bmatrix} 1 & -5 & -9 & 8 & -7 \\ 0 & 1 & 3 & -4 & 2 \\ 0 & 2 & 6 & -8 & 4 \end{bmatrix} \begin{array}{c} 5R_2+R_1 \\ \sim \\ -2R_2+R_3 \end{array} \begin{bmatrix} 1 & 0 & 6 & -12 & 3 \\ 0 & 1 & 3 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in reduced echelon form. The associated linear system is:

$$\left.\begin{array}{r} x_1 + 6x_3 - 12x_4 = 3 \\ x_2 + 3x_3 - 4x_4 = 2 \\ 0 = 0 \end{array}\right\} \quad \begin{array}{l} x_1 = -6x_3 + 12x_4 + 3 \\ x_2 = -3x_3 + 4x_4 + 2 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{array}$$

As a vector, the solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 + 12x_4 + 3 \\ -3x_3 + 4x_4 + 2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 12 \\ 4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

3. (*10 points*) Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule

$$T(\boldsymbol{x}) = 3\boldsymbol{x} + \boldsymbol{e_1}$$

where $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Is this a linear transformation? Why or why not?

Note that $T(\vec{0}) = 3(\vec{0}) + \vec{e}_1 = \vec{e}_1 = (1,0) \neq \vec{0}$.

But $T(\vec{0}) = \vec{0}$ is true for all linear transformations.

So T is not a linear transformation.

4. (*10 points*)

a. (*5 pts*) Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ be a set of vectors. What does it mean for this set to be linearly independent?

S is linear dependent if the vector equation

$$c_1\vec{u} + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

has only the trivial solution. That is, the above equation is true iff $c_1 = c_2 = \cdots = c_n = 0$.

b. (*5 pts*) What does it mean for $\boldsymbol{u} \in \text{Span}(S)$?

$\vec{u} \in \text{Span}(S)$ means $\vec{u}$ is a linear comb of the vectors of S. That is,

$$\vec{u} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$$

for some scalars $d_1, d_2, \ldots, d_n \in \mathbb{R}$.

5. (*15 points*)
Given the system of equations

$$\begin{cases} -2x_1 + hx_2 = 1 \\ 6x_1 + x_2 = k \end{cases} \Rightarrow \begin{bmatrix} -2 & h & 1 \\ 6 & 1 & k \end{bmatrix} \overset{3R_1+R_2}{\sim} \begin{bmatrix} -2 & h & 1 \\ 0 & 3h+1 & 3+k \end{bmatrix}$$

Find h and k such that the solution set of the system

a. (*5 pts*) does not exist.

A solution DNE if there is a pivot in the last column.

So we need ~~3h+1≠0~~ and $3+k \neq 0$

$3h+1=0$ $\quad k \neq -3$

$h = \frac{-1}{3}$

So choose $h = \frac{-1}{3}$ and $k = 0$

b. (*5 pts*) is unique.

There is a unique solution if there is a pivot in each row of the coefficient matrix.

So $3h+1 \neq 0$ and k can take on any value.

$h \neq -\frac{1}{3}$

So choose $h, k = 0$.

c. (*5 pts*) is infinite.

There ~~is~~ are infinitely many solutions if there is a free variable. So we need $3h+1=0$, $3+k=0$

$h = -\frac{1}{3}$ $\quad k = -3$

7. (*20 points*) Mark each statement either **True** or **False**. If the answer is false, explain why.

a. (*5 pts*) If a system of equations has two different solutions, then it has infinitely many solutions.

True. A system has 0, 1, or infinitely many

b. (*5 pts*) Every matrix is row equivalent to a unique matrix in echelon form.

False. A matrix is row equiv to a unique matrix in reduced echlon form.

-OR-

Echlon forms are not unique.

c. (*5 pts*) If a set of vectors contains the zero vector, then the set is linearly dependent.

True

d. (*5 pts*) If a matrix A is $m \times n$ and if the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for some $\boldsymbol{b}$ then the columns of A span $\mathbb{R}^m$.

False. If $A\vec{x} = \vec{b}$ has a solution for <u>each</u> $\vec{b} \in \mathbb{R}^m$ then the columns of A span $\mathbb{R}^m$.