Exam 2
In Class Component

Math 322-022 **Name:** ______________________
7/17/09 **Four Digit Code:** ______________________

Solutions

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- You may use any non-graphing calculator except cell phone calculators.
- The use of any electronic equipment such as mp3 players, iPods, cell phones, etc., is forbidden (except calculators). You may NOT use calculators on cell phones.
- Justify your answers algebraically whenever possible to ensure full credit. When you do use your calculator, show and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- This test has 5 problems and is worth 50 points. It is your responsibility to make sure that you have all 6 of the pages!
- The use of any outside assistance, including notes, books, or other people's exams is considered cheating and will result (at a minimum) in a score of 0 on this exam.
- Good luck!

Question	Score	Maximum
1		10
2		10
3		10
4		15
5		5
Total		50

1. (*10 points*) The Invertible Matrix Theorems ties many different ideas from the class together. Given that A is an $n \times n$ invertible matrix, what does the Invertible Matrix Theorem tell you about:

a. (*2 pts*) The columns of A

Span $\mathbb{R}^n$

Lin. Ind

Each have a pivot

b. (*2 pts*) The linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$

Onto

1-1

c. (*2 pts*) The reduced echelon form of A

Is the $n \times n$ identity matrix

d. (*2 pts*) The equation $A\boldsymbol{x} = \mathbf{0}$

Has only the trivial solution

e. (*2 pts*) The equation $A\boldsymbol{x} = \boldsymbol{b}$, for $\boldsymbol{b} \in \mathbb{R}^n$

Has a solution for each $\vec{b} \in \mathbb{R}^n$

2. (*10 points*)
a. (*3 pts*) Let S be a subset of $\mathbb{R}^m$. What three properties does S have to satisfy to be a subspace of $\mathbb{R}^m$?

1. $\vec{0} \in S$

2. If $\vec{u}, \vec{v} \in S$ then $\vec{u} + \vec{v} \in S$

3. If $\vec{u} \in S$, $c \in \mathbb{R}$, then $c\vec{u} \in S$

b. (*3 pts*) Let A be an $n \times m$ matrix. What is the definition of null space of A, Nul A?

The set of all solutions to $A\vec{x} = \vec{0}$

c. (*4 pts*) Show that Nul A is a subspace of $\mathbb{R}^m$ by verifying the three properties from part (a).

Since $A\vec{0} = \vec{0}$, $\vec{0} \in$ Nul A.

Spse $\vec{u}, \vec{v} \in$ Nul A. Then $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$.

Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$, so $\vec{u} + \vec{v} \in$ Nul A.

If $c \in \mathbb{R}$, $A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$, so $c\vec{u} \in$ Nul A.

Thus Nul A is a subspace of $\mathbb{R}^m$.

3. (*10 points*)
a. (*3 pts*) What does it mean for a set $\mathbb{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_p\}$ to be a basis for a subspace H of $\mathbb{R}^n$?

$\mathbb{B}$ is a basis of H if it lin. ind and spans H.

b. (*7 pts*) Let $\boldsymbol{v}_1 = (3, 0, -6), \boldsymbol{v}_2 = (-4, 1, 7)$, and $\boldsymbol{v}_3 = (-2, 1, 5)$. Does the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ form a basis for $\mathbb{R}^3$? You must justify your answer.

Form $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$, then

$$A \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$2R_1 + R_3$ $R_2 + R_3$

Since there is a pivot in each column, A is invertible by IMT. Therefore its columns span $\mathbb{R}^3$, also by IMT.

4. (*15 points*) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$.

a. (*5 pts*) Find the characteristic polynomial of A.

$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 3 & -1-\lambda & 0 \\ -3 & 1 & -\lambda \end{bmatrix}$

Since $A - \lambda I$ is triangular,

$\det(A - \lambda I) = (3-\lambda)(-1-\lambda)(-\lambda)$

$= (-3 - 3\lambda + \lambda + \lambda^2)(-\lambda)$

$= -\lambda^3 - 2\lambda^2 + 3\lambda$

b. (*5 pts*) Find the eigenvalues of A.

Since A is triangular, the e-values are

$\lambda = 3,\ \lambda = -1,\ \lambda = 0$

c. (*5 pts*) Choose one eigenvalue λ from part (b) and find a basis for the eigenspace corresponding to λ.

$\lambda = 3$

$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -4 & 0 \\ -3 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & -3 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} -3 & 1 & -3 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 0 & 12 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3/5 \\ 0 & 0 & 0 \end{bmatrix}$, so for $(A - 3I)\vec{x} = \vec{0}$,

$\vec{x} = \begin{bmatrix} 4x_3 \\ 3/5 x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 3/5 \\ 1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 4 \\ 3/5 \\ 1 \end{bmatrix} \right\}$ is a basis

$\lambda = -1$

$A + I = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 0 & 0 \\ -3 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} -3 & 1 & 1 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & 1 \\ 0 & 1 & 1 \\ 4 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & 1 \\ 0 & 4/3 & 4/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

basis $\{(0,-1,1)\}$

$\lambda = 0$,

$A - 0I = A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Basis $\{(0,0,1)\}$

5. (*5 points*) If λ is an eigenvalue of an invertible matrix A, explain why $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}.

Spse $\vec{v}$ is an e-vector corresponding to λ, then

$$A\vec{v} = \lambda\vec{v}$$

$$A^{-1}(A\vec{v}) = A^{-1}(\lambda\vec{v})$$

$$\vec{v} = \lambda A^{-1}\vec{v}$$

$$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}, \text{ since } \lambda \neq 0$$