Exam 2
Take Home Component

Math 322-022 **Name:** ____________________

7/17/09 **Four Digit Code:** ____________________

Directions KEY

This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all 8 of the pages. Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct). Circle or otherwise indicate your final answers. If you need more space, write on the front side of blank looseleaf paper, number and initial each page, and staple or paper clip the papers to the back of the exam.

Advice

Look through the entire paper before you begin *and* after you are finished. Check your work when you are finished. If you get stuck on a problem, put your work aside for awhile and do something else. Then come back and begin working.

Constraints

This is the take home portion of Exam 2; as such you are allowed to use your book and *your* notes from class. You are not allowed to use any other resources, including the Internet and other people (including classmates, friends, other teachers). You are not allowed to use a calculator on this exam.

Due

This test is due Monday, July 14, at the *beginning* of class. Late tests will not be accepted, unless your absence or tardiness is excused, per university rules.

Agreement

I acknowledge the above conditions and agree to conduct myself honorably during the course of this exam.

Signed: ____________________

Question	Score	Maximum
1		20
2		10
3		15
4		15
5		30
6		10
Total		100

1. (*20 points*) Let H and K be two subspaces of $\mathbb{R}^n$. We define the **intersection of H and K**, denoted $H \cap K$ to be the set of all vectors that are in both H and K. That is,

$$H \cap K = \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v} \in H \text{ and } \boldsymbol{v} \in K\}.$$

Prove that $H \cap K$ is also a subspace of $\mathbb{R}^n$.

Pf. Let $\vec{u}, \vec{v} \in H \cap K$. So $\vec{u}, \vec{v} \in H$ and $\vec{u}, \vec{v} \in K$. Since both H + K are subspaces of $\mathbb{R}^n$, $\vec{u} + \vec{v} \in H$ and $\vec{u} + \vec{v} \in K$. Thus $\vec{u} + \vec{v} \in H \cap K$.

Now let $c \in \mathbb{R}$, then $c\vec{u} \in H$ and $c\vec{u} \in K$, so $c\vec{u} \in H \cap K$.

Finally, since H + K are subspaces $\vec{0} \in H$ and $\vec{0} \in K$ which gives that $\vec{0} \in H \cap K$.

Therefore $H \cap K$ is a subspace of $\mathbb{R}^n$. //

2. (*10 points*) Determine whether the following matrix is invertible. If it is, find its inverse.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}.$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ -3 & -7 & 0 & 0 & 1 & 0 \\ 8 & 5 & -1 & 0 & 0 & 1 \end{array}\right] \overset{\frac{1}{5}R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ -3 & -7 & 0 & 0 & 1 & 0 \\ 8 & 5 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} 3R_1+R_2 \\ \sim \\ -8R_1+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & -7 & 0 & 3/5 & 1 & 0 \\ 0 & 5 & -1 & -8/5 & 0 & 1 \end{array}\right] \overset{-1/7 R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & -3/35 & \frac{-1}{7} & 0 \\ 0 & 5 & -1 & -8/5 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} -5R_2+R_3 \\ \sim \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & -3/35 & -1/7 & 0 \\ 0 & 0 & -1 & \frac{-41}{35} & \frac{5}{7} & 1 \end{array}\right] \overset{-R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & -3/35 & \frac{-1}{7} & 0 \\ 0 & 0 & 1 & \frac{41}{35} & \frac{-5}{7} & -1 \end{array}\right]$$

$$\left(-5\left(\frac{-3}{35}\right) + \left(-\frac{8}{5}\right) = \frac{3}{7} - \frac{8}{5} = \frac{15}{35} - \frac{56}{35} = -\frac{41}{35}\right)$$

Since $A \sim I_3$, A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{-3}{35} & \frac{-1}{7} & 0 \\ \frac{41}{35} & \frac{-5}{7} & -1 \end{bmatrix}$$

3. (*15 points*) Let T be the shear transformation of $\mathbb{R}^n$ which sends $\mathbf{e}_1$ to $\mathbf{e}_2$ and sends $\mathbf{e}_2$ to $2\mathbf{e}_1 + \mathbf{e}_2$, and let S be the linear transformation which rotates $\mathbb{R}^2$ counterclockwise through an angle of $\pi/4$. Determine whether the composite transformations

$$T \circ S \text{ and } S \circ T$$

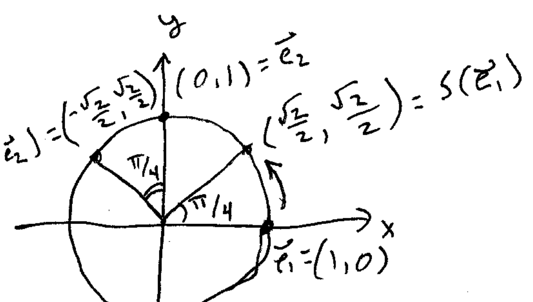
are the same transformation.

First we find the standard matrices of T and S. Then $T \circ S$ and $S \circ T$ will be the product of the matrices.

$$T(\vec{e}_1) = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } T(\vec{e}_2) = 2\vec{e}_1 + \vec{e}_2 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So the standard matrix for T is $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

For S, note the following diagram



So $S(\vec{e}_1) = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, $S(\vec{e}_2) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, and

the standard matrix for S is

$$B = \begin{bmatrix} \sqrt{2}/2 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

$$AB = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \sqrt{2} - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \sqrt{2} + \frac{\sqrt{2}}{2} \end{bmatrix}$$

Since $AB \neq BA$ we have that $T \circ S$ and $S \circ T$ are not the same transformations.

[redacted]

4. (*15 points*) Compute the following if possible. If it is not possible explain why.

a. (*5 pts*) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + 2 \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 18 & 16 & 14 \\ 12 & 10 & 8 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 18 & 17 \\ 16 & 15 & 14 \\ 13 & 12 & 11 \end{bmatrix}$$

b. (*5 pts*) $\underset{A}{\begin{bmatrix} -3 & 7 & -6 \\ 4 & -9 & 4 \\ 2 & 10 & 2 \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} -1 & 3 & 2 \\ 2 & -3 & -4 \end{bmatrix}}$

Thus multiplication is undefined since the # of columns of A ≠ # of rows of B.

c. (*5 pts*) $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{3} + \frac{4}{6} & -\frac{2}{3} + \frac{4}{6} \\ -\frac{1}{3} + \frac{2}{6} & \frac{2}{3} + \frac{2}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{2}{3} & -\frac{2}{3} + \frac{2}{3} \\ -\frac{1}{3} + \frac{1}{3} & \frac{2}{3} + \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. *(30 points)* Let $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$.

a. *(10 pts)* Find a basis for Col A.

A basis for Col A is the set of pivot columns, so we put A into echelon form.

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix} \underset{\substack{3R_1+R_3 \\ -3R_1+R_4}}{\overset{2R_1+R_2}{\sim}} \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 2 & -6 & -2 & 18 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix}$$

$$\underset{-2R_2+R_3}{\sim} \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -20 \end{bmatrix} \underset{R_3 \leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ 0 & 1 & -3 & -1 & 9 \\ 0 & 0 & 0 & 5 & -20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is echelon form and the pivot columns are columns 1, 2, 4. So a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}.$$

b. (*10 pts*) Find a basis for Nul A.

Basis for Nul A is a parametric form of the solutions to $A\vec{x}=\vec{0}$, so finish reducing A to reduced echelon form (in order to reduce $[A \mid \vec{0}]$.

$$[A \mid \vec{0}] \sim \begin{bmatrix} 1 & 2 & -2 & 0 & 7 & 0 \\ 0 & 1 & -3 & -1 & 9 & 0 \\ 0 & 0 & 0 & 5 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{-2R_2+R_1}{\sim} \begin{bmatrix} 1 & 0 & 4 & 2 & -11 & 0 \\ 0 & 1 & -3 & -1 & 9 & 0 \\ 0 & 0 & 0 & 5 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{\frac{1}{5}R_3}{\sim} \begin{bmatrix} 1 & 0 & 4 & 2 & -11 & 0 \\ 0 & 1 & -3 & -1 & 9 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{\substack{-2R_3+R_1 \\ R_3+R_2}}{\sim} \begin{bmatrix} 1 & 0 & 4 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So
$$\begin{aligned} x_1 + 4x_3 - 3x_5 &= 0 \\ x_2 - 3x_3 + 5x_5 &= 0 \\ x_4 - 4x_5 &= 0 \\ 0 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -4x_3 + 3x_5 \\ x_2 &= 3x_3 - 5x_5 \\ x_3 &\text{ is free} \\ x_4 &= 4x_5 \\ x_5 &\text{ is free} \end{aligned} \qquad \vec{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

c. (*5 pts*) For the matrix A given above, Col A is a subspace of $\mathbb{R}^n$ for what n? Why?

Since Col A is the set of linear comb of the columns of A and each column has 4 entries, Col A is a subspace of $\mathbb{R}^4$

d. (*5 pts*) Nul A is a subspace of $\mathbb{R}^k$ for what k? Why?

If $\vec{x} \in$ Nul A, $A\vec{x}=\vec{0}$. Since A has 5 columns, $\vec{x}$ must have 5 entries, so Nul A is a subspace of $\mathbb{R}^5$.

6. (*10 points*) Give an example of a linear transformation that is:

a. (*5 pts*) One-to-one but not onto.

Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ w/ standard matrix A, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $A\vec{x} = \vec{0}$ will have no free variable, thus ~~there~~ only the trivial solution and T is 1-1.

But since A does not have a pivot in every row $A\vec{x} = \vec{b}$ does not have a solution for every $\vec{b} \in \mathbb{R}^3$ and so T is not onto.

b. (*5 pts*) Onto but not one-to-one.

Let S be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ w/ standard matrix B, where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^2$ since there is a pivot in each row, and S is onto.

But $A\vec{x} = \vec{0}$ will have free variable x_3. Therefore $A\vec{x} = \vec{0}$ has non-trivial solutions and S is not 1-1.