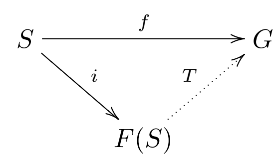
THE FREES

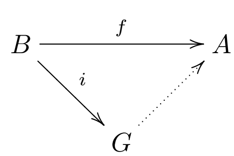
Definition 1. *A group F is* ***free group*** *if there is a subset $S \subset F$ such that any element of G can be written uniquely as a product of elements of S (and their inverses).*

The free group on S is characterized by the following universal property: if G is a any group and $f : S \to G$ is any function (where $F(S)$ is the free with generators in S), then there exists a unique group homomorphism $T : F(S) \to G$ such that $T(s) = f(s)$ for all $s \in S$.



Definition 2. *A* ***free abelian group*** *is an abelian group G with a basis. That is, every element of G can be written uniquely as a finite linear combination of elements of the basis, with integer coeffecients.*

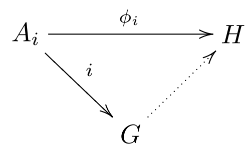
If G is a free abelian group with basis B, then we have the following universal property: for every function $f : B \to A$, where A is an abelian group, there is a unique group homomorphism from G to A which extends f.



Definition 3. *Let G be a group and let $(A_i)_{i \in I}$ be a family of subgroups of G. Then G is said to be a* ***free product*** *of the subgroups A_i if, given any group H and a homomorphism $f_i : A_i \to H$ for each $i \in I$, there is a unique homomophism $f : G \to H$ such that $f|_{A_i} = f_i$ for all $i \in I$. The subgroups A_i are called the* ***free factors*** *of G.*

For example, given groups G and H, the free product $G * H$ can be constructed as follows: given presentations for G and H, take the generators of G and of H, take the disjoint union of those, and adjoin the corresponding relations for G and for H. This is a presentation of $G * H$, the point being that there is no interaction between G and H in the free product.

The free product is the coproduct in the category of groups: given a group H and group homomorphisms $\phi_i : A_i \to H$ there is a unique homomorphism such that the following commutes:



The more general construction is the **free product with amalgamation**. Suppose G and H are groups and $\phi : F \to G$ and $\psi : F \to H$ are homomorphisms from a group F. Take the free group $G * H$ and adjoin relations $\phi(f) = \psi(f)$ for all $f \in F$. In other words, denote normal closure in $G * H$ of the group generated by elements $\phi(f)\psi(f)^{-1} = e$ by N. Then the free product with amalgamation of G and H with respect to ϕ and ψ is $(G * H)/N$. This construction is the pushout in the category of groups: