

TOPOLOGICAL BUNDLES

1. FIBER BUNDLES

A fiber bundle consists of the data (E, B, π, F) , where E, B , and F are topological spaces and $\pi : E \rightarrow B$ is a continuous surjection satisfying the local triviality condition below. B is called the **base space**, E the **total space**, and F the **fiber**. The map π is called the **projection map**. (Assume B connected)

For any $x \in B$ there is an open neighborhood U of x (called a trivializing neighborhood) such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times F$, in such a way that π carries over to the projection onto the first factor. That is, the following commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\ \pi \downarrow & \swarrow \text{proj}_1 & \\ U & & \end{array}$$

where $\text{proj}_1 : U \times F \rightarrow U$ is the natural projection and $\phi : \pi^{-1}(U) \rightarrow U \times F$ is a homeomorphism. The set of all $\{(U_i, \phi_i)\}$ is called a **local trivialization** of the bundle.

For any $x \in B$, the preimage $\pi^{-1}(x)$ is homeomorphic to F and is called the **fiber over x** . A fiber bundle (E, B, π, F) is often denoted

$$F \rightarrow E \rightarrow B$$

to indicate a “short exact sequence” of spaces. Note that every bundle $\pi : E \rightarrow B$ is an open map, since projections are open maps. Therefore B has the quotient topology determined by π .

1.1. Covering Map. A **covering space** is a fiber bundle such that the bundle projection is a local homeomorphism. It follows that the fiber is discrete.

1.2. Vector and principal bundles. A special class of fiber bundles, called **vector bundles**, are those whose fibers are vector spaces. Important examples are the tangent bundle and the cotangent bundles of a smooth manifold.

Another special class of fiber bundles, called **principal bundles**. A principal G -bundle is a fiber bundle $\pi : P \rightarrow X$ together with a continuous right action $P \times G \rightarrow P$ by a group G such that G preserves the fibers of P and acts freely and transitively on them. The abstract fiber is taken to be G itself.

It follows that the orbits of the G -action are precisely the fibers of $\pi : P \rightarrow X$ and the orbit space P/G is homeomorphic to the base space X .

2. PULLBACK BUNDLE

Let $\pi : E \rightarrow B$ be a fiber bundle with fiber F and let $f : B' \rightarrow B$ be a continuous map. Define the **pullback bundle** by

$$f^*E = \{(x, e) \in B' \times E \mid f(x) = \pi(e)\} \subset B' \times E$$

and equip it the subspace topology and the projection map $f^*\pi : E \rightarrow B'$ onto the first factor, i.e.,

$$f^*\pi(x, e) = x.$$

The projection onto the second factor gives a map $\tilde{f} : f^*E \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ f^*\pi \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B \end{array}$$

If (U, ϕ) is a local trivialization of E then $(f^{-1}U, \psi)$ is a local trivialization of f^*E , where

$$\psi(x, e) = (x, \text{proj}_2(\phi(e))).$$

Therefore f^*E is a fiber bundle over B' with fiber F . The bundle f^*E is called the **pullback of E by f** or the **bundle induced by f** . The map \tilde{f} is then a bundle morphism covering f .

2.1. Properties. Any section of E over B induces a section of f^*E , called the **pullback section** f^*s , by defining $f^*s = s \circ f$.

If the bundle $E \rightarrow B$ is a vector bundle or a principal bundle then so is the pullback f^*E . In the case of a principal G -bundle the right action of G on f^*E is given by

$$(x, e)g = (x, eg).$$

Therefore \tilde{f} is equivariant (i.e. \tilde{f} commutes with the group action) and so defines a morphism of principal bundles.

Finally, the pullback bundle is a categorical pullback in the category of fiber bundles, i.e.,

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ & & f^*E & & \\ & \swarrow & & \searrow & \\ B' & & & & E \\ & \swarrow f^*\pi & & \searrow \tilde{f} & \\ & & B & & \\ & \swarrow f & & \searrow \pi & \end{array}$$

is universal in the obvious sense.