HOMOLOGICAL ALGEBRA CHEAT SHEET

1. Tensor Product

Given abelian groups A, B, define $A \otimes B$ to be the group generated by symbols $a \otimes b$ mod the relations $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. So, in particular, $0 \otimes 0 = a \otimes 0 = 0 \otimes b$ and $-(a \otimes b) = -a \otimes b = a \otimes -b$.

Tensor products satisfy the following:

(1) $A \otimes B = B \otimes A$
(2) $(\bigoplus_i A_i) \otimes B = \bigoplus_i (A_i \otimes B)$
(3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
(4) $\mathbb{Z} \otimes A = A$
(5) $\mathbb{Z}_n \otimes A = A/nA$
(6) $A \otimes B$ is universal wrt bilinear maps $A \times B \to C$

More generally, we define $A \otimes_R B := A \otimes B/(ra \otimes b = a \otimes rb)$

2. Tor$(-, -)$

The Tor functors are the derived functors of the tensor product. The Tor functor satisfies the following:

(1) $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$
(2) $\mathrm{Tor}(\bigoplus_i A_i, B) = \bigoplus_i (A_i, B)$
(3) $\mathrm{Tor}(A, B) = 0$ if A or B are (torsion) free
(4) $\mathrm{Tor}(A, B) = \mathrm{Tor}(T(A), B)$, where $T(A)$ is the torsion subgroup of A
(5) $\mathrm{Tor}(\mathbb{Z}_n, A) = \mathrm{Ker}(A \xrightarrow{n} A)$
(6) $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_q$, where $q = \gcd(m, n)$
(7) If A, B are finitely generated abelian groups, then $\mathrm{Tor}(A, B) = T(A) \otimes T(B)$
(8) If $0 \to B \to C \to D \to 0$ is exact, then $A \otimes B \to A \otimes C \to A \otimes D \to 0$. And there is the natural exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

3. Ext(-,-)

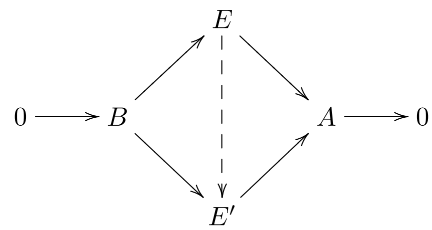
The Ext functors are defined to be the derived functors of Hom. Ext has a nice formulation in terms of group extensions:

$\mathrm{Ext}(A, B) :=$ the (group) set of isomorphism classes of extensions of B by A.

That is, $E \in \mathrm{Ext}(A, B)$ if E fits into an exact sequence

$$0 \to B \to E \to A \to 0$$

and $E = E'$ in $\text{Ext}(A, B)$ if



commutes.

If H is finitely generated, then:

(1) $\text{Ext}(H \oplus H', G) = \text{Ext}(H, G) \oplus \text{Ext}(H', G)$
(2) $\text{Ext}(H, G) = 0$ if H is free
(3) $\text{Ext}(\mathbb{Z}_n, G) = G/nG$
(4) If $0 \to A \to B \to C \to 0$ is exact, then
$$0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G)$$
is exact. $\text{Ext}(-, G)$ is such that
$$0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to \text{Ext}(C, G) \to \text{Ext}(B, G) \to \text{Ext}(A, G) \to 0$$
is exact.

4. Universal Coefficient Theorems

Given a chain complex $C.$, there are the (non-natural) split-exact sequences
$$0 \to H_n(C.) \otimes A \to H_n(C., A) \to \text{Tor}(H_{n-1}(C.), A) \to 0, \text{ and}$$
$$0 \to \text{Ext}(H_{n-1}(C.), G) \to H^n(C., G) \to \text{Hom}(H_n(C.), G) \to 0.$$

5. Kunneth Formulas

5.1. **Algebraic.** Let R be a PID, $C., C'.$ be chain complex of free R-modules. There is the (non-splitting) exact sequence
$$0 \to \bigoplus_i (H_i(C.) \otimes_R H_{n-1}(C'.)) \to H_n(C. \otimes_R C'.) \to \bigoplus_i (\text{Tor}(H_i(C.), H_{n-i-1}(C'.)) \to 0.$$

5.2. **Topological.** Let X, Y be topological spaces and let k be a field. Then
$$\bigoplus_{i+j=k} H_i(X; k) \otimes H_j(Y; k) \cong H_k(X \times Y; k).$$
More (and less) generally, let X, Y be CW-comples and R a PID. There is the natural exact sequence
$$0 \to \bigoplus_i (H_i(X; R) \otimes_R H_{n-1}(Y; R) \to H_n(X \times Y; R) \to \bigoplus_i \text{Tor}_R(H_i(X; R), H_{n-i-1}(Y; R) \to 0.$$

(In all of the above, we can replace the abelian groups by R-modules and do things over the ground ring R instead of $\mathbb{Z}$.)