A GENTLE INTRODUCTION TO SPECTRAL SEQUENCES

1. Introduction

A spectral sequence is a generalization of the notion of a long exact sequence. Let (X, A) be a CW-pair. Then there is the familiar long exact sequence on homology

$$\cdots H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots$$

Now, instead of a pair, suppose there is a (finite) filtration of a space X:

$$\emptyset = X_0 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X_N = \cup_i X_i = X$$

Then there is a spectral sequence associated to the filtration that "approximates" the homology of X. Also, similar to the Mayer-Vietoris long exact sequence of a cover by two sets, there is a spectral sequence associated with an arbitrary cover of a space.

In the sequel, unless otherwise stated, all topological spaces are "nice" (path-connected CW-complexes) and homology calculations are over a field $\mathbb{F}$ (the latter is so that all homology groups will be vector spaces and messy algebraic extension problems can be ignored).

2. Algebraic Preliminaries

Definition 2.1. *A* ***bigraded module*** *is a family of modules* $E = \{E_{p,q}\}, p, q = 0, \pm 1, \pm 2, \ldots$. *A* ***differential*** $d : E \to E$ *of bidegree* $(-r, r-1)$ *is a family of linear maps*

$$d : E_{p,q} \to E_{p-r,q+r-1}$$

with $d^2 = 0$.

We define the homology $H(E, d)$ in the obvious way

$$H_{p,q} := \mathrm{Ker}(E_{p,q} \to E_{p-r,q+r-1})/\mathrm{Im}(E_{p+r,q-r+1} \to E_{p,q}).$$

Definition 2.2. *A (homological, first quadrant)* ***spectral sequence*** $E = \{E^r, d^r\}$ *is a sequence of bigraded modules* $E^1, E^2, \ldots$ *each with a differential* d^r *of degree* $(-r, r-1)$ *such that*

$$E^{r+1}_{p,q} \cong H_{p,q}(E^r, d^r).$$

Note two things:

(1) We assume that $E^r_{p,q} = 0$ if p or $q < 0$.
(2) E^{r+1} is completely determined by E^r and d^r, but d^{r+1}.

A useful to visualize a spectral sequence is as a book consisting of pages

Note that $E^3_{1,1}$ is such that any other differential $d^3, d^4, \ldots$ will not affect it. For fixed (p, q), $E^r_{p,q}$ eventually stabilizes in this way. This limiting group is denoted by $E^\infty_{p,q}$. On the E^∞ page, take the sum of the entries along the diagonal line$p + q = n$,

$$\bigoplus_{p+q=n} E^\infty_{p,q}.$$

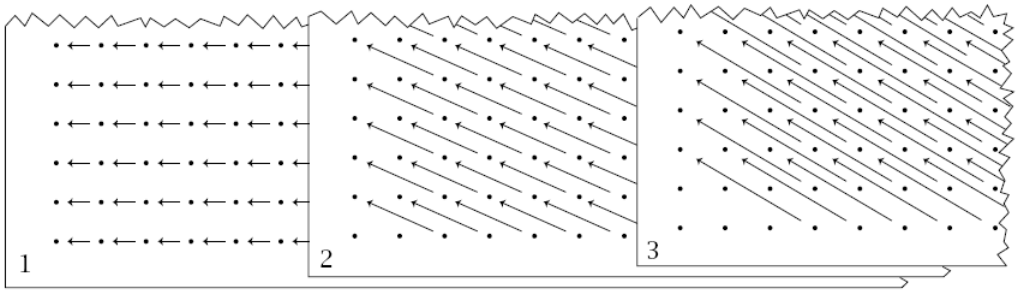


FIGURE 1. From Hatcher's Spectral Sequences book

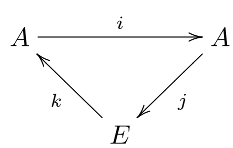
We say that the spectral sequence converges to this sum.

Many times, we can ignore the E^1 page. A general "theorem" about spectral sequences is like the following:

Theorem 2.3. *: $E^2_{p,q} \cong$ "something computable" $\Rightarrow$ "something desired".*

3. Building a Spectral Sequence

Definition 3.1. *A commutative triangle*

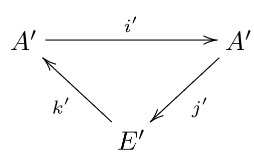


*is an **exact couple** if it is exact at the three corners (Ker j = Im i, etc).*

Lemma 3.2. *$d = jk$ is a differential.*

Proof. $d^2 = jkjk = j(0)k = 0$. □

The **derived couple** of an exact couple is

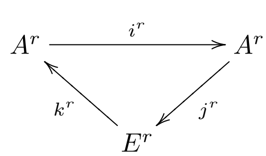


where $A' = i(A)$, $i' = i|_{i(A)}$, $E' = \text{Ker } d/\text{Im } d$, $j'(ia) = [ja]$, and $k'[e] = ke$. It can be easily shown that j' and k' are well-defined.

Theorem 3.3. *The derived couple of an exact couple is exact.*

The is an exercise in diagram chasing.

We can repeat this process to obtain a sequence of exact couples, where, at the rth iteration, we have the exact couple

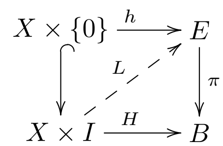


Now suppose that E, A are bimodules and deg $i = (1, -1)$, deg $j = (0, 0)$, deg $k = (-1, 0)$. Then at the rth stage for A^r, E^r the degree of i^r, j^r, k^r, respectively, are $(1, -1), (-r+1, r-1)$. Therefore we have that the degree of d^r is $(-r, r-1)$. Then since $E^{r+1} = H(E^r, d^r)$ we obtain a spectral sequence (E^r, d^r).

Theorem 3.4. *An exact couple gives rise to a spectral sequence.*

4. Serre-Leray Spectral Sequence

Recall that a map $E \xrightarrow{\pi} B$ is a fibration if the following diagram can always be filled in at L, i.e., if it satisfies the Homotopy Lifting Property.



$F = \pi^{-1}(b)$ is the fiber of the fibration (B connected $\Rightarrow$ the fibers of various b are homotopy equivalent). We can think of a fibration $F \to E \to B$ as a type of "short exact sequence" of spaces.

Example 4.1. *If the fiber is discrete we have a covering space.*

Example 4.2. *If a local trivialization is satisfied, we have a fiber bundle.*

Theorem 4.3 (Serre-Leray)**.** *Given a fibration $F \to E \to B$ with F connected and B such that $\pi_1(B)$ has trivial action on $H_*(F)$. Then there exists a spectral sequence such that*

$$E^2_{p,q} \cong H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).$$

The (very general) idea of the proof is to filter B by its n-skeletons. Since we have a fibration, we can lift at each step to obtain a filtration of E. It is an exercise that a filtration gives rise to an exact couple, which, in turn, gives rise to the desired spectral sequence.

5. Application of SLSS

Given a pointed topological space (X, x_0), the set of based paths forms a topological space which we denote PX. Note that $PX \simeq x_0$ by truncation of paths. The fiber above x_0 forms the loop space of X, denoted $\Omega(X)$. Then $\Omega(X) \to PX \to X$ is a fibration.

Consider the case that $X = S^k$ where $k \geq 2$. Then we have the fibration $\Omega(S^k) \to PS^k \to S^k$; we wish to analyze $\Omega(S^k)$. By Serre-Leray there is a spectral sequence

$$E^2_{p,q} = H_p(S^k; H_q(\Omega(S^k))) \Rightarrow H_{p+q}(PS^k).$$

Since $H_n(S^k)$ is nontrivial only for $n = 0, k$ the first non-zero differentials will be on the E^k page. Therefore

$$E^2 = E^3 = \cdots = E^k.$$

This implies that $H_q(\Omega S^k) = 0$ for $0 < q < k$ since $PS^3 \simeq \{pt\}$ and $H_p(S^k) = 0$ for $p \neq 0, k$.

Now on the E^k page, consider the differential

$$d : E^k_{k,0} \to E^k_{0,k-1} = H_k(S^k) \to H_{k-1}(\Omega S^k).$$

Which, on the next page becomes,

$$\ker\, d \to H_{k-1}(\Omega S^k)/\mathrm{Im}\, d.$$

Moreover, the differentials starting on the vertical line $p = k$ on the E^k page are the only possible non-zero differentials and therefore $E^{k+1} = E^\infty$. Therefore we have:

(1) $H_{k-1}(\Omega S^k)/\mathrm{Im}\, d \cong H_{k-1}(PS^k) \cong 0$
(2) $\ker\, d \oplus H_k(\Omega S^k) \cong H_k(PS^k) \cong 0$

Which, respectively, imply that d is surjective and injective. Thus d is an isomorphism. An induction argument gives that

$$H_n(\Omega S^k) \cong \begin{cases} \mathbb{F}, & \text{if } n \equiv (k-1) \\ 0, & \text{otherwise.} \end{cases}$$

6. References

There are many good references for spectral sequences. The ones below are simple the ones I found the most helpful (and order of helpfullness).

[1] MacLane, *Homology*
[2] Hatcher, *Spectral Sequences*, http://www.math.cornell.edu/ hatcher/SSAT/SSATpage.html
[3] McCleary, *A User's Guide to Spectral Sequences*