MA 261 - Induction!

Proof by Induction: To prove a statement $P(n)$ is true for all natural numbers n, we can do the following:

i. Prove it is true when $n = 1$. (Base Case)

ii. Prove that if $k \in \mathbb{Z}$ and $P(k)$ is true then $P(k+1)$ is also true. (the hypothesis of this statement is called the 'inductive hypothesis')

[To complete step 2, remind yourself what you would need to assume and what you would need to show.]

Prove the following theorems using mathematical induction:

1. **Theorem A.10:** If n is a natural number then
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$
2. **Theorem A.18:** If n is a natural number then $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$.
3. If n is a natural number then
$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$
4. **Theorem A.19:** For every natural number n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
5. **Theorem A.21:** For every natural number n, $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \ldots + n)^2$.
6. **Theorem 1.18:** Let a, b, k, and n be integers with $n > 0$ and $k > 0$. If $a \equiv b \pmod{n}$ then $a^k \equiv b^k \pmod{n}$.
7. **Theorem A.20:** For every natural number $n > 3$, $2^n < n!$.
8. For every natural number n, $8 | (3^{2n} - 1)$.
9. For every natural number n, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.
10. Let $r \neq 1$ be a real number. For every natural number n, $1 + r + r^2 + \ldots + r^{n-1} = \dfrac{r^n - 1}{r - 1}$
11. **Binomial theorem.** For integers x, y, and n, $(x+y)^n = \displaystyle\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k$.
12. For every natural number $n \geq 3$, if n distinct points on a circle are connected in consecutive order with straight lines, then the interior angles of the resulting polygon add up to $(n-2)180$ degrees.

Definition: The *fibonacci numbers* are a sequence of numbers, denoted f_n, defined as follows:

$$f_1 = 1, \qquad f_2 = 1, \qquad \text{and} \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 3$$

Exercise Find the first 10 fibonacci numbers.

13. Let f_n be the nth fibonacci number. For every natural number n, $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$.
14. For every natural number n, $(f_n, f_{n-1}) = 1$.

Strong Mathematical Induction To prove a statement using *strong* induction (or the second principle of induction):

i. Prove the base case.

ii. Let $k \in \mathbb{N}$ and assume $P(1), P(2), \ldots, P(k)$ are all true.

iii. Prove $P(k+1)$ is true.

Use strong mathematical induction to prove the following theorems.

15. Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For every natural number n, $f_n = \dfrac{a^n - b^n}{a-b}$.
16. For every natural number n, $f_n < (5/3)^n$.
17. **Theorem 1.38:** Let a and b be integers. If $(a, b) = 1$, then there exists integers x and y so that $ax + by = 1$.
18. **Theorem A.31:** Every natural number greater than 7 can be written as a sum of 3's and 5's where the coefficients of 3 and 5 are nonnegative.
19. Every natural number n can be written as $n = 2^k l$ where k is a non-negative integer and l is an odd integer.
20. **Theorem A.30:** Every natural number can be written as the sum of distinct powers of 2. (i.e. in the form $n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_m}$ where $0 \leq i_1 < \cdots i_m$ are all integers)

    Hint: handle the cases where n is even and n is odd separately.
21. A country has n cities. Any two cities are connected by a one-way road. Show that there is a route that passes through every city.

Finally...

The Well-Ordering Axiom for the Natural Numbers states that if S is any non-empty subset of natural numbers then S has a smallest element.

22. Show that the principal of mathematical induction follows from the well-ordering axiom.