DERIVED CATEGORIES AND DERIVED FUNCTORS
MA 751 : TOPICS IN TOPOLOGY

KATE PONTO
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A category comes with a notion of isomorphism, but in many cases we want to be able to treat a broader collection of maps as if they are isomorphisms. Quasi-isomorphisms in homological algebra are a familiar example of this. These maps don't necessarily have inverses, but we would really like to be able to treat them as if they did. Another example is given by the weak homotopy equivalences in algebraic topology.

There are a range of approaches that will allow us to add in the "missing" inverses and the resulting category is called the *derived (or homotopy) category*. This is possibly most familiar from homological algebra, but those are just a few of many examples.

After we add inverses, we often have categories that are much better behaved and more computationally tractable. For example, if we can treat weak homotopy equivalences as isomorphisms, then each topological space is isomorphic to a CW complex. If we invert quasi-isomorphisms, bounded chain complexes are isomorphic to chain complexes of projectives.

Whenever we can do something to a category - like add inverses for some maps - we look for a corresponding operation on functors. If we start with a functor

$$F\colon \mathcal{C} \to \mathcal{D},$$

we want a functor (the *derived functor*) from the homotopy category of $\mathcal{C}$ to the homotopy category of $\mathcal{D}$. An obvious first condition to require is that F should take morphisms that become isomorphisms in the homotopy category of $\mathcal{C}$ to isomorphisms the homotopy category of $\mathcal{D}$. This leaves open the question of what else is required, how we can compute this kind of thing, and if there are interesting examples. (These all have really nice answers!)

What we are doing here is taking a functor and replacing it by one that is "homotopically" better behaved. This may be homotopy arising from topological or chain homotopy, or it may be weak homotopy equivalence or quasi-isomorphism. In those cases, this approach captures Tor and Ext as well as homotopy limits and homotopy colimits.

Practical Information:

- There are many ways of approaching this topic. I'm going to primarily be motivated by ideas from model category theory, but, time permitting, we will follow these ideas to other interpretations.
- It is important to be familiar with ideas from algebraic topology and homological algebra. Prerequisites could be MA 661 and MA 654/655, but talk to me if you are not sure if the course makes sense for you.
- There are many statements along the way to the main results that have very self contained proofs. I expect to allocate the responsibility for presenting these among the participants in the class.