A TINY BIT OF HOMOLOGICAL ALGEBRA

Due April 24, 2019

Exercise 0.1. In an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

of homomorphisms of abelian groups, the following are equivalent

(i) f is surjective.
(ii) g is the trivial homomorphism.
(iii) h is injective.

Exercise 0.2. In an exact sequence

$$A \xrightarrow{d} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E \xrightarrow{k} F$$

of homomorphisms of abelian groups the following are equivalent.

(i) g is an isomorphism.
(ii) f and h are trivial homomorphisms.
(iii) d is surjective and k is injective.

Exercise 0.3 (Five lemma). In the commutative diagram of abelian groups below, if the two rows are exact and α, β, δ, and ϵ are isomorphisms then γ is also an isomorphism.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

Exercise 0.4. For a short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of abelian groups the following are equivalent.

(i) There exists a homomorphism $p\colon B \to A$ such that $p \circ \alpha = \mathrm{id}\colon A \to A$.
(ii) There exists a homomorphism $s\colon C \to B$ such that $\beta \circ s = \mathrm{id}\colon C \to C$.
(iii) There exists an isomorphism $\theta\colon B \to A \oplus C$ such that the following diagram commutes.

$$\begin{tikzcd}
 & & B \arrow[dd] \arrow[dr, "\beta"] & & \\
0 \arrow[r] & A \arrow[ur, "\alpha"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

The map $A \to A \oplus C$ is $a \mapsto (a, 0)$ and $A \oplus C \to C$ is given by $(a, c) \mapsto c$.

A short exact sequence is said to be **split exact** if it satisfies any of these equivalent conditions.