A TINY BIT OF HOMOLOGICAL ALGEBRA

A sequence of abelian groups C_i and homomorphisms $f_i\colon C_i \to C_{i-1}$ is **exact** if the image of f_{i+1} is equal to the kernel of f_i. (This is a chain complex where all homology groups are trivial.)

Exercise 0.1. In an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

of homomorphisms of abelian groups show the following are equivalent

(i) f is surjective.
(ii) g is the trivial homomorphism.
(iii) h is injective.

Exercise 0.2. In an exact sequence

$$A \xrightarrow{d} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E \xrightarrow{k} F$$

of homomorphisms of abelian groups show the following are equivalent.

(i) g is an isomorphism.
(ii) f and h are trivial homomorphisms.
(iii) d is surjective and k is injective.

Exercise 0.3 (Five lemma). In the commutative diagram of abelian groups below, if the two rows are exact and α, β, δ, and ϵ are isomorphisms show γ is also an isomorphism.

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$