Riemann Sum

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(P_k) \Delta x_k$$

where $f(x)$ is continuous on $[a, b]$, partitioned into $n$ subintervals. The $k^{th}$ interval contains point $P_k$ and has width $\Delta x_k$. 
Ex: Suppose we want to estimate \( \int_1^5 8^x \, dx \) by evaluating \( \sum_{k=1}^{n} 8^{1+k \Delta x} \cdot \Delta x \).

If \( \Delta x = 0.2 \), what is \( n \)?

\( \Delta x \) is width of subintervals

\([1, 5]\) has \( n \) subintervals of length 0.2

\[ \Delta x = \frac{5 - 1}{n} = 0.2 \]

\[ \frac{4}{n} = 0.2 \]

\[ 4 = 0.2n \]

\[ \frac{4}{0.2} = n \]

\[ n = 20 \]
Ex: Estimate \[ \int_{4}^{10} x^2 \, dx = \sum_{k=1}^{n} (2+k \cdot \Delta x)^2 \cdot \Delta x \].

If \( n = 10 \), what is \( \Delta x \)?

[4,10] is split into 10 subintervals

\[ \Delta x = \frac{10-4}{10} = \frac{6}{10} = 0.6 \]
Ex: We will estimate \( \int_{-6}^{0} x^2 \, dx \) by the sum
\[
\sum_{k=1}^{n} \left[ A + B(k\Delta x) + C(k\Delta x)^2 \right] \cdot \Delta x
\]
where \( n = 30 \) and \( \Delta x = 0.2 \)
find \( A, B, C \).

\( \int_{-6}^{0} x^2 \, dx \) tells us \( a = -6, b = 0, f(x) = x^2 \)
then our Riemann sum is...

\[
\sum_{k=1}^{n} f(x_k) \Delta x = \sum_{k=1}^{n} f(-6 + k\Delta x) \cdot \Delta x
\]
\[
= \sum_{k=1}^{n} (-6 + k\Delta x)^2 \cdot \Delta x
\]
\[
= \sum_{k=1}^{n} (36 - 12k\Delta x + (k\Delta x)^2) \cdot \Delta x
\]
then we'll play the matching game!

\[
A = 36, \quad B = -12, \quad C = 1
\]
Ex: You estimate \( \int_3^{15} f(x) \, dx = \)
\[
\sum_{k=1}^{n} f(3 + k \cdot \frac{A}{n}) \cdot \frac{A}{n} = \Delta x
\]

What is \( A \)?

\( a = 3, \ b = 15 \)

\( \Delta x = \frac{15 - 3}{n} = \frac{12}{n} \)

then \( x_k = a + k \Delta x \)

\( = 3 + k \cdot \frac{12}{n} \)

thus \( A = 12 \)
Ex: Estimate the area under the graph of \( f(x) = x^3 \) from \( x = 6 \) to \( x = 36 \) by partitioning \([6, 36]\) into 30 subintervals, using right endpoints. What is the area of the 20th rectangle?

\[
\Delta x = \frac{36 - 6}{30} = \frac{30}{30} = 1
\]

\[
x_k = a + k \Delta x = 6 + k
\]

\[
A(\square) = f(x_{20}) \cdot \Delta x_{20} = f(6+20) \cdot 1 = 26^3 \cdot 1 = 17,576
\]
Ex: Suppose we are given data points for a function $f(x)$:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

If $f$ is a linear function between the given points, find $\int_{1}^{4} f(x) \, dx$.

\[
\int_{1}^{4} f(x) \, dx = T_1 + T_2 + T_3
\]

\[
= \frac{(2+5)}{2} + \frac{(5+8)}{2} + \frac{(8+12)}{2}
\]

\[
= 3.5 + 6.5 + 10
\]

\[
= 20
\]
Ex: Let \( f(x) \) be the greatest integer function.

(Thus \( f(2.3) = 2 \), \( f(4) = 4 \), \( f(6.9) = 6 \))

Find \( \int_{6}^{10} f(x) \, dx \)

\[
\int_{6}^{10} f(x) \, dx = R_1 + R_2 + R_3 + R_4 \\
= f(6) \cdot 1 + f(7) \cdot 1 + f(8) \cdot 1 + f(9) \cdot 1 \\
= 6 + 7 + 8 + 9 \\
= 30
\]

\[
\int_{6}^{10} f(x) \, dx = 30
\]