Interpolation and Cubature at Geronimus Nodes Generated by Different Geronimus Polynomials

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Abstract. We extend the definition of Geronimus nodes to include pairs of real numbers where each coordinate consists of the alternation points of a possibly different Geronimus polynomial of the same degree. We give an explicit formula for the Lagrange polynomials for these nodes that involves the reproducing kernel for the product polynomials and deduce a cubature formula for polynomials in two variables with respect to a product measure.

1. Introduction

Geronimus polynomials are a normalization of the polynomials that can be defined by a three-term recurrence relation having constant coefficients. These include all four kinds of the Chebyshev polynomials and many other more subtle examples. Each Geronimus polynomial has alternation points, which reduce to the Chebshev points in the case of Chebyshev polynomials of the first kind. Our object is to extend the discussion of Geronimus nodes in [**6**] to allow pairs of alternation points for two different Geronimus polynomials of the same degree. This extends our cubature formula to integrals with respect to the product of the measures for each of the Geronimus polynomials. A similar extension for the Morrow-Patterson nodes is given in [**3**].

2. Geronimus polynomials

Let a, b, c and d be real constants with $a > 0$ and $c > 0$. The corresponding Geronimus polynomials are the terms of a sequence $\{p_n\}$ of polynomials defined recursively by

$$p_0(x) = 1, \quad p_1(x) = ax + b, \tag{1}$$

$$p_{n+1}(x) = (cx + d)p_n(x) - p_{n-1}(x), \quad n \geq 1.$$

These were first considered by Geronimus [**5**] in the case $c = 2$ and $d = 0$. An important property of the Geronimus polynomials given in [**6**] is that for each positive integer m there exists unique alternation points, i.e., numbers

$$h_0 > h_1 > \cdots > h_m \text{ such that}$$

$$p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad n, j = 0, 1, \ldots, m. \tag{2}$$

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In particular, $p_m(h_n) = (-1)^n$ for $n = 0, 1, \ldots, m$. The proof in [**6**] shows that $\{h_n\}_0^m$ is the set of roots of the polynomial $\pi_m = p_1 p_m - p_{m-1}$.

The main examples of the Geronimus polynomials are the four kinds of Chebyshev polynomials $\{T_n\}$, $\{U_n\}$, $\{V_n\}$ and $\{W_n\}$, which correspond to the cases where (a, b, c, d) is $(1, 0, 2, 0)$, $(2, 0, 2, 0)$, $(2, -1, 2, 0)$ and $(2, 1, 2, 0)$, respectively. (See Table 1, [**8**] and [**6**].)

Table 1: The four kinds of Chebyshev polynomials

Kind	Definition	Weight	$h_n = \cos\theta_n$
1st	$T_n(\cos\theta) = \cos n\theta$	$w_1(x) = \dfrac{2}{\pi\sqrt{1-x^2}}$	$\theta_n = \dfrac{n\pi}{m}$
2nd	$U_n(\cos\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$	$w_2(x) = \frac{2}{\pi}\sqrt{1-x^2}$	$\theta_n = \dfrac{(n+1)\pi}{m+2}$
3rd	$V_n(\cos\theta) = \dfrac{\cos(n+1/2)\theta}{\cos(\theta/2)}$	$w_3(x) = \frac{1}{\pi}\sqrt{\dfrac{1+x}{1-x}}$	$\theta_n = \dfrac{n\pi}{m+1}$
4th	$W_n(\cos\theta) = \dfrac{\sin(n+1/2)\theta}{\sin(\theta/2)}$	$w_4(x) = \frac{1}{\pi}\sqrt{\dfrac{1-x}{1+x}}$	$\theta_n = \dfrac{(n+1)\pi}{m+1}$

Many Bernstein-Szegö polynomials are Geronimus polynomials. (See [**2**, p. 204-206] and [**4**] .) It follows by induction that every Geronimus polynomial satisfies

$$p_n(x) = (ax+b)U_{n-1}\left(\frac{cx+d}{2}\right) - U_{n-2}\left(\frac{cx+d}{2}\right), \quad n \geq 1,$$

where $U_{-1} = 0$.

By Favard's theorem [**2**, p. 21], the Geronimus polynomials are orthogonal polynomials with respect to a moment functional ℓ satisfying

$$\ell(1) = \frac{c}{a}, \quad \ell(p_n^2) = 1, \quad n \geq 1.$$

Define $H_n = \ell(p_n^2)$ for all nonnegative integers n. Then $\{p_n/\sqrt{H_n}\}$ is an orthonormal sequence. Clearly $H_n = 1$ for all positive integers n and $H_0 = c/a$. For example, the moment functional for the Chebyshev polynomials of the ith kind is given by

$$\ell(p) = \int_{-1}^{1} p(x)w_i(x)\,dx,$$

where $w_i(x)$ is as given in Table 1.

3. Interpolation nodes

Let $\{p_n\}$ and $\{\tilde{p}_n\}$ be the Geronimus polynomials determined by the coefficients (a, b, c, d) and $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, respectively. Let m be a given positive integer and let $\{h_n\}_0^m$ and $\{\tilde{h}_n\}_0^m$ be the corresponding decreasing sequences of alternation points. We define the even Geronimus nodes $\mathcal{N}_0$ to be the set of ordered pairs $(h_n, \tilde{h}_q)$, $0 \leq n, q \leq m$, where n and q are both even or both odd and the odd Geronimus nodes $\mathcal{N}_1$ to be the set of ordered pairs $(h_n, \tilde{h}_q)$, $0 \leq n, q \leq m$, where n is even and q is odd or n is odd and q is even. Thus if $k = 0$ or $k = 1$, then the Geronimus nodes are given by

$$\mathcal{N}_k = \{(h_n, \tilde{h}_q) : \ (n, q) \in Q_k\},$$

where

$$Q_k = \{(n, q) : \ 0 \leq n, q \leq m, \ n - q = k \bmod 2\}.$$

The Geronimus nodes of [**6**] are the case where both sequences of Geronimus polynomials are the same. Recall that in this case, the Chebyshev points of [**10**] are the Geronimus nodes for the case $a = 1$, $b = 0$, $c = 2$, $d = 0$ in (1).

Note that since $\{p_n\}$ and $\{\tilde{p}_n\}$ are both orthogonal polynomials with respect to inner products, the set

$$\{p_{i-j}(s)\tilde{p}_j(t) : 0 \le j \le i,\ i = 0, 1, \ldots\}$$

is a set of orthogonal polynomials in two variables in the product space. Thus the reproducing kernel for the polynomials of degree at most n in two variables is given by

$$K_n(s,t,u,v) = \sum_{i=0}^{n}\sum_{j=0}^{i} \frac{p_{i-j}(s)\tilde{p}_j(t)p_{i-j}(u)\tilde{p}_j(v)}{H_{i-j}\tilde{H}_j},$$

where H_n is as given in Section 2 for $\{p_n\}$ and $\tilde{H}_n$ is defined similarly for $\{\tilde{p}_n\}$. By considering the first term in the above sum, we see that $K_n(s,t,s,t) > 0$.

4. Construction of Lagrange polynomials

Our construction follows a plan given in [**9**, §4.2]. Let m be a given positive integer, fixed throughout, and define

$$\begin{aligned} G_m(s,t,u,v) &= \frac{1}{2}[K_{m-1}(s,t,u,v) + K_m(s,t,u,v)] \\ &+ \frac{1}{2c\tilde{c}}[\tilde{a}(a-c)p_m(s)p_m(u) + a(\tilde{a}-\tilde{c})\tilde{p}_m(t)\tilde{p}_m(v)]. \end{aligned}$$

Clearly $G_m(s,t,u,v)$ is a polynomial of degree at most m in (s,t) and in (u,v). Also, we may write

$$G_m(s,t,u,v) = K_{m-1}(s,t,u,v) + \frac{1}{2}S_m(s,t,u,v),$$

where

$$S_m(s,t,u,v) = \sum_{i=1}^{m-1} p_{m-i}(s)\tilde{p}_i(t)p_{m-i}(u)\tilde{p}_i(v) + \frac{a\tilde{a}}{c\tilde{c}}[p_m(s)p_m(u) + \tilde{p}_m(t)\tilde{p}_m(v)].$$

Hence $G_m(s,t,s,t) > 0$ for all $(s,t) \in \mathbb{R}^2$. Given $(n,q) \in Q_k$, define

$$P_{n,q}(s,t) = \lambda_{n,q}G_m(s,t,h_n,\tilde{h}_q),\ \text{where}\ \lambda_{n,q} = \frac{1}{G_m(h_n,\tilde{h}_q,h_n,\tilde{h}_q)}.$$

THEOREM 1. *Let $k = 0$ or $k = 1$ and let $(n,q) \in Q_k$. Then $P_{n,q}$ is a polynomial of degree m satisfying $P_{n,q}(h_n,\tilde{h}_q) = 1$ and $P_{n,q}(x) = 0$ for all $x \in \mathcal{N}_k$ with $x \neq (h_n,\tilde{h}_q)$.*

Our proof of Theorem 1 follows easily from two Christoffel-Darboux identities for bivariate polynomials. (Compare [**9**, Theorem 4.2.1].) To state the identities, first define polynomials

$$\begin{aligned} X_i(s,t) &= p_{m-i}(s)\tilde{p}_i(t) - \epsilon\, p_i(s)\tilde{p}_{m-i}(t), \quad i = 0,1,\ldots,m, \\ Y_0(s,t) &= (as+b)p_m(s) - p_{m-1}(s) = \pi_m(s), \\ Y_i(s,t) &= p_{m-i+1}(s)\tilde{p}_i(t) - \epsilon\, p_{i-1}(s)\tilde{p}_{m-i}(t), \quad i = 1,2,\ldots,m, \\ Y_{m+1}(s,t) &= (\tilde{a}t+\tilde{b})\tilde{p}_m(t) - \tilde{p}_{m-1}(t) = \tilde{\pi}_m(t), \end{aligned}$$

where ϵ is a constant. It follows from (2) that all of these polynomials vanish at the nodes $\mathcal{N}_k$ when $\epsilon = (-1)^k$ and $k = 0, 1$.

PROPOSITION 2.

$$\begin{aligned} 2c(s-u)G_m(s,t,u,v) &= \sum_{i=0}^{m-1}{}' \left[X_i(s,t)p_{m-i-1}(u)\tilde{p}_i(v) - X_i(u,v)p_{m-i-1}(s)\tilde{p}_i(t)\right] \\ &\quad + \sum_{i=0}^{m}{}'' \left[Y_i(s,t)p_{m-i}(u)\tilde{p}_i(v) - Y_i(u,v)p_{m-i}(s)\tilde{p}_i(t)\right]. \end{aligned}$$

Here $'$ means that the term for $i = 0$ is multiplied by $\tilde{a}/\tilde{c}$ and $''$ means that the terms for $i = 0$ and $i = m$ are multiplied by $\tilde{a}/\tilde{c}$.

PROPOSITION 3.

$$\begin{aligned} 2\tilde{c}(t-v)G_m(s,t,u,v) &= \sum_{i=0}^{m-1}{}' \left[X_{m-i}(s,t)p_i(u)\tilde{p}_{m-i-1}(v) - X_{m-i}(u,v)p_i(s)\tilde{p}_{m-i-1}(t)\right] \\ &\quad + \sum_{i=0}^{m}{}'' \left[Y_{i+1}(s,t)p_{m-i}(u)\tilde{p}_i(v) - Y_{i+1}(u,v)p_{m-i}(s)\tilde{p}_i(t)\right]. \end{aligned}$$

Here $'$ means that the term for $i = 0$ is multiplied by a/c and $''$ means that the terms for $i = 0$ and $i = m$ are multiplied by a/c.

To deduce Theorem 1, note that $P_{n,q}(h_n, \tilde{h}_q) = 1$ by definition. If $x = (h_{n'}, \tilde{h}_{q'})$ is a node in $\mathcal{N}_k$, then

$$\begin{aligned} (h_{n'} - h_n)P_{n,q}(x) &= 0, \\ (\tilde{h}_{q'} - \tilde{h}_q)P_{n,q}(x) &= 0, \end{aligned}$$

by Propositions 2 and 3. Hence, if $x \neq (h_n, \tilde{h}_q)$ then $P_{n,q}(x) = 0$, as required.

The same arguments that established [**6**, Theorem 3.3] apply in this case to obtain a cubature theorem for our more general Geronimus nodes. (A still more general theorem is given in [**7**, Theorem 7]).

THEOREM 4. *Let $\{p_n\}$ and $\{\tilde{p}_n\}$ be two sequences of Geronimus polynomials and suppose their moment functionals are given by weight functions w and $\tilde{w}$. Let m be a positive integer and let $\{h_n\}_0^m$ and $\{\tilde{h}_q\}_0^m$ be corresponding alternation points. Then*

$$\iint_{\mathbb{R}^2} p(s,t)w(s)\tilde{w}(t)\,ds\,dt = \sum_{(n,q)\in Q_k} \lambda_{n,q}\, p(h_n, \tilde{h}_q)$$

for all bivariate polynomials p of degree at most $2m-1$ and for $k = 0, 1$.

5. Proof of Propositions 2 and 3

Our first step is to obtain the following identity for the reproducing kernel:

(3)

$$c(s-u)K_m(s,t,u,v) = \sum_{j=0}^{m} \frac{\tilde{p}_j(t)\tilde{p}_j(v)}{\tilde{H}_j}\left[p_{m-j+1}(s)p_{m-j}(u) - p_{m-j}(s)p_{m-j+1}(u)\right].$$

To verify this, define

$$a_j = \frac{\tilde{p}_j(t)\tilde{p}_j(v)}{\tilde{H}_j}, \quad b_j = \frac{p_j(s)p_j(u)}{H_j}, \quad j = 0, 1, \ldots, m,$$

and note that

$$K_m(s,t,u,v) = \sum_{i=0}^{m}\sum_{j=0}^{i} a_j b_{i-j} = \sum_{j=0}^{m} a_j \left(\sum_{i=0}^{m-j} b_i\right).$$

Thus (3) follows from the following Christoffel-Darboux formula

$$\sum_{i=0}^{n} b_i = \frac{p_{n+1}(s)p_n(u) - p_n(s)p_{n+1}(u)}{c(s-u)}$$

with $n = m - j$. (See [**1**, p. 246].)

Define

$$\begin{aligned} f_j(s,u) &= [p_{m-j+1}(s) - p_{m-j-1}(s)]p_{m-j}(u), \\ g_j(s,t,u,v) &= p_{m-j-1}(s)\tilde{p}_j(t)p_j(u)\tilde{p}_{m-j}(v) \end{aligned}$$

for $j = 0, 1, \ldots, m-1$. Then

$$c(s-u)[K_{m-1}(s,t,u,v) + K_m(s,t,u,v)] = A_0 + \sum_{j=1}^{m-1} a_j[f_j(s,u) - f_j(u,s)],$$

where

$$A_0 = a(s-u)\tilde{p}_m(t)\tilde{p}_m(v) + \frac{\tilde{a}}{\tilde{c}}[f_0(s,u) - f_0(u,s)].$$

By a computation,

$$\begin{aligned} &a_j f_j(s,u) + \epsilon[g_j(s,t,u,v) - g_{m-j}(s,t,u,v)] \\ &\quad= Y_j(s,t)p_{m-j}(u)\tilde{p}_j(v) - X_j(u,v)p_{m-j-1}(s)\tilde{p}_j(t) \end{aligned}$$

for $j = 1, 2, \ldots, m-1$ and a similar identity is obtained when (s,t) and (u,v) are interchanged. Since

$$\sum_{j=1}^{m-1}(g_j - g_{m-j}) = 0$$

by symmetry, it follows that

$$\begin{aligned} (4)\qquad & c(s-u)[K_{m-1}(s,t,u,v) + K_m(s,t,u,v)] \\ &= A_0 + \sum_{j=1}^{m-1}[X_j(s,t)p_{m-j-1}(u)\tilde{p}_j(v) - X_j(u,v)p_{m-j-1}(s)\tilde{p}_j(t)] \\ &\quad + \sum_{j=1}^{m-1}[Y_j(s,t)p_{m-j}(u)\tilde{p}_j(v) - Y_j(u,v)p_{m-j}(s)\tilde{p}_j(t)]. \end{aligned}$$

Let

$$h(s,t,u,v) = Y_0(s,t)p_m(u) + X_0(s,t)p_{m-1}(u) - Y_m(u,v)\tilde{p}_m(t) - f_0(s,u).$$

Then the right-hand side of the identity in Proposition 2 is just the sum of the right-hand side of the equation (4) and

$$(5)\qquad \frac{\tilde{a}}{\tilde{c}}[h(s,t,u,v) - h(u,v,s,t)] - a(s-u)\tilde{p}_m(t)\tilde{p}_m(v).$$

We simplify this expression and take its sum also with the left-hand side of (4). Applying (1) and the definitions of each of the terms of $h(s,t,u,v)$, we obtain

$$h(s,t,u,v) = \\ [p_1(s) - (cs+d)]p_m(s)p_m(u) + p_{m-1}(s)p_m(u) + p_m(s)p_{m-1}(u) - p_1(u)\tilde{p}_m(t)\tilde{p}_m(v).$$

Hence

$$h(s,t,u,v) - h(u,v,s,t) = (s-u)[(a-c)p_m(s)p_m(u) + a\tilde{p}_m(t)\tilde{p}_m(v)].$$

Thus (5) reduces to

$$\frac{s-u}{\tilde{c}}[\tilde{a}(a-c)p_m(s)p_m(u) + a(\tilde{a}-\tilde{c})\tilde{p}_m(t)\tilde{p}_m(v)],$$

which verifies Proposition 2. One can deduce Proposition 3 from Proposition 2 by interchanging p and $\tilde{p}$, s and t, and u and v.

6. Further comments

In [**6**] the author gave a rather complicated proof of the identity

$$\pi_m = \begin{cases} p_k^2 - p_{k-1}^2 & \text{if } m = 2k-1, \\ (p_{k+1} - p_{k-1})p_k & \text{if } m = 2k \end{cases} .$$

To deduce this more easily, we first observe that $Q_j = p_j p_{m-j+1} - p_{j-1}p_{m-j}$ is independent of j for $j = 1, \ldots, m-1$ since

$$Q_{j+1} - Q_j = (p_{j+1} + p_{j-1})p_{m-j} - p_j(p_{m-j+1} + p_{m-j-1}) = 0$$

by the second equation of (1). Since $Q_1 = \pi_m$, if $m = 2k-1$ then $\pi_m = Q_k = p_k^2 - p_{k-1}^2$ and if $m = 2k$ then $\pi_m = Q_k = p_k(p_{k+1} - p_{k-1})$.

These equations reduce finding the roots of π_m to solving polynomial equations of lower degree. For example, given real a, b, c and d with $a > 0$ and $c > 0$, explicit formulas for the alternation points $\{h_n\}_0^m$ can be obtained by solving a quadratic equation for $m = 1, 2, 3$. If also $ad - bc = 0$, then explicit formulas for the alternation points that involve a radical within a radical can be obtained for $m = 4, 6, 8$.

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References

1. G. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge, 1999.
2. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Math. and its Appl., Vol. 13, Gordon and Breach, New York–London–Paris, 1978.
3. S. Deterding and L. A. Harris, *Interpolation and Cubature at the Morrow-Patterson nodes generated by different Geronimus polynomials*, Math. Proc. R. Ir. Acad. **117**A (1), 5-12.
4. W. Gautschi and S. Notaris, *Gauss-Kronrod quadrature formulae for weight functions of Bernstein-Szegö type*, J. Comput. Appl. Math. 25 (1989) 199–224; erratum: J. Comput. Appl. Math. 27 (1989) 429.
5. J. Geronimus, *On a set of polynomials*, Ann. of Math. (2) **31** (1930), 681–686.
6. L. A. Harris, *Bivariate polynomial interpolation at the Geronimus nodes*, in Complex Analysis and Dynamical Systems V, Contemp. Math. 591, American Mathematical Society, Providence, RI, 2013, pp. 135–147.
7. L. A. Harris, *Lagrange polynomials, reproducing kernels and cubature in two dimensions*, J. Approx. Theory 195 (2015), 43-56.

8. J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapmann & Hall/CRC, Boca Raton, 2003.
9. Yuan Xu, *Common Zeros of Polynomials in Several Variables and Higher-dimensional Quadrature*, Pitman Research Notes in Mathematics, Longman, Essex 1994.
10. Yuan Xu, *Lagrange interpolation on Chebyshev points of two variables*, J. Approx. Theory **87** (1996), 220–238.

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