# Interpolation and cubature at the Morrow-Patterson nodes generated by different Geronimus polynomials

Stephen Deterding and Lawrence A. Harris

Abstract. We extend the definition of Morrow-Patterson nodes to include pairs of real numbers where each coordinate consists of the alternation points of a possibly different Geronimus polynomial. We give an explicit formula for the Lagrange polynomials for these nodes that involves the reproducing kernel for the product polynomials and deduce cubature formulas for polynomials in two variables with respect to a product measure.

## 1. Introduction

Geronimus polynomials are a normalization of the polynomials that can be defined by a three-term recurrence relation having constant coefficients. These include all four kinds of the Chebyshev polynomials and many other more subtle examples. Each Geronimus polynomial has alternation points, which reduce to the Chebshev points in the case of Chebyshev polynomials of the first kind.

Our object is to extend the discussion of Morrow-Patterson nodes in [**8**] to allow pairs of alternation points for two different Geronimus polynomials with degree differing by one. This extends the cubature formulas in [**8**, Section 6] to integrals with respect to the product of the measures for each of the Geronimus polynomials. See [**9**] for an analogous discussion of the Geronimus nodes and see [**4**] for an extension in a different direction of the case of Chebyshev polynomials of the first kind.

## 2. Geronimus polynomials

Let $a, b, c$ and $d$ be real constants with $a > 0$ and $c > 0$. The corresponding Geronimus polynomials are the terms of a sequence $\{p_n\}$ of polynomials defined recursively by

$$p_0(x) = 1, \quad p_1(x) = ax + b, \tag{1}$$
$$p_{n+1}(x) = (cx + d)p_n(x) - p_{n-1}(x), \quad n \geq 1.$$

These were first considered by Geronimus [**6**] in the case $c = 2$ and $d = 0$. An important property of the Geronimus polynomials given in [**7**] is that for each positive integer $m$ there exists unique alternation points, i.e., numbers

$$h_0 > h_1 > \cdots > h_m \text{ such that}$$

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$$p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad n, j = 0, 1, \ldots, m. \tag{2}$$

In particular, $p_m(h_n) = (-1)^n$ for $n = 0, 1, \ldots, m$. The proof in [**7**] shows that $\{h_n\}_0^m$ is the set of roots of the polynomial $\pi_m = p_1 p_m - p_{m-1}$.

The main examples of the Geronimus polynomials are the four kinds of Chebyshev polynomials $\{T_n\}$, $\{U_n\}$, $\{V_n\}$ and $\{W_n\}$, which correspond to the cases where $(a, b, c, d)$ is $(1, 0, 2, 0)$, $(2, 0, 2, 0)$, $(2, -1, 2, 0)$ and $(2, 1, 2, 0)$, respectively. (See Table 1, [**10**] and [**7**].)

**Table 1: The four kinds of Chebyshev polynomials**

| Kind | Definition | Weight | $h_n = \cos\theta_n$ |
|---|---|---|---|
| 1st | $T_n(\cos\theta) = \cos n\theta$ | $w_1(x) = \dfrac{2}{\pi\sqrt{1-x^2}}$ | $\theta_n = \dfrac{n\pi}{m}$ |
| 2nd | $U_n(\cos\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$ | $w_2(x) = \frac{2}{\pi}\sqrt{1-x^2}$ | $\theta_n = \dfrac{(n+1)\pi}{m+2}$ |
| 3rd | $V_n(\cos\theta) = \dfrac{\cos(n+1/2)\theta}{\cos(\theta/2)}$ | $w_3(x) = \frac{1}{\pi}\sqrt{\dfrac{1+x}{1-x}}$ | $\theta_n = \dfrac{n\pi}{m+1}$ |
| 4th | $W_n(\cos\theta) = \dfrac{\sin(n+1/2)\theta}{\sin(\theta/2)}$ | $w_4(x) = \frac{1}{\pi}\sqrt{\dfrac{1-x}{1+x}}$ | $\theta_n = \dfrac{(n+1)\pi}{m+1}$ |

Many Bernstein-Szegö polynomials are Geronimus polynomials. (See [**3**, p. 204-206] and [**5**] .) By Favard's theorem [**3**, p. 21], the Geronimus polynomials are orthogonal polynomials with respect to a moment functional $\ell$ satisfying

$$\ell(1) = \frac{c}{a}, \quad \ell(p_n^2) = 1, \quad n \geq 1.$$

Define $H_n = \ell(p_n^2)$ for all nonnegative integers $n$. Then $\{p_n/\sqrt{H_n}\}$ is an orthonormal sequence. Clearly $H_n = 1$ for all positive integers $n$ and $H_0 = c/a$. For example, the moment functional for the Chebyshev polynomials of the $i$th kind is given by

$$\ell(p) = \int_{-1}^{1} p(x) w_i(x)\, dx,$$

where $w_i(x)$ is as given in Table 1.

## 3. Morrow-Patterson nodes

Let $\{p_n\}$ and $\{\tilde{p}_n\}$ be the Geronimus polynomials determined by coefficients $(a, b, c, d)$ and $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, respectively. Let $m$ be a given nonnegative integer and let $\{h_n\}_0^m$ and $\{\tilde{h}_n\}_0^{m+1}$ be the corresponding decreasing sequences of alternation points. These are the roots of associated polynomials $\pi_m$ and $\tilde{\pi}_{m+1}$, respectively. We define the even Morrow-Patterson nodes $\mathcal{N}_0$ to be the set of ordered pairs $(h_n, \tilde{h}_q)$, with $0 \leq n \leq m$ and $0 \leq q \leq m+1$, where $n$ and $q$ are both even or both odd and the odd Morrow-Patterson nodes $\mathcal{N}_1$ to be the set of ordered pairs $(h_n, \tilde{h}_q)$, with $0 \leq n \leq m$ and $0 \leq q \leq m+1$, where $n$ is even and $q$ is odd or $n$ is odd and $q$ is even. Thus if $k = 0$ or $k = 1$, then the Morrow-Patterson nodes are given by

$$\mathcal{N}_k = \{(h_n, \tilde{h}_q) : \ (n, q) \in Q_k\},$$

where

$$Q_k = \{(n, q) : \ 0 \leq n \leq m, \ 0 \leq q \leq m+1, \ n - q = k \bmod 2\}.$$

As in [**8**], our Morrow-Patterson nodes are unisolvent for the set $\mathcal{P}_m(\mathbb{R}^2)$ of polynomials of degree at most $m$ in two variables. This follows from Theorem 1 below since the number of Morrow-Patterson nodes for $k = 0$ or $k = 1$ is the same as the dimension of $\mathcal{P}_m(\mathbb{R}^2)$, which is $(m + 2)(m + 1)/2$. The Morrow-Patterson nodes of [**8**] are the case where both sequences of Geronimus polynomials are the same. The classical Morrow-Patterson nodes were introduced in [**11**, p. 960] and are the case where $p_n = \tilde{p}_n = U_n$, $k = 1$ and $n$ is even. (See also [**1**, p. 269].) The first and third families of the Padua points as defined in [**2**] are the case where $p_n = \tilde{p}_n = T_n$ with $k = 1$ and $k = 0$, respectively.

Note that there are 16 classes of even Morrow-Patterson nodes $\mathcal{N}_0$ corresponding to the four different kinds of Chebyshev polynomials $p_n$ and $\tilde{p}_n$. Let $\mathcal{N}_1$ be the set of odd Morrow-Patterson nodes with corresponding Chebyshev polynomials $q_n$ and $\tilde{q}_n$. Define $q_n = p_n$ when $p_n = T_n$ or $U_n$, $q_n = W_n$ when $p_n = V_n$, and $q_n = V_n$ when $p_n = W_n$. Also, define $\tilde{q}_n$ similarly in terms of $\tilde{p}_n$. Then $\mathcal{N}_1 = -\mathcal{N}_0$ by the identity $\cos(\pi - \theta) = -\cos(\theta)$. Thus it suffices to consider only the even (or odd) Morrow-Patterson nodes when the generating Geronimus polynomials are one of the kinds of Chebyshev polynomials.

Since $\{p_n\}$ and $\{\tilde{p}_n\}$ are both orthogonal polynomials with respect to inner products induced by their respective moment functionals, the set

$$\{p_{i-j}(s)\tilde{p}_j(t) : 0 \le j \le i,\ i = 0, 1, \ldots\}$$

is a set of orthogonal polynomials in two real variables. Thus the reproducing kernel for the Hilbert space $\mathcal{P}_n(\mathbb{R}^2)$ is given by

$$K_n(s,t,u,v) = \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{p_{i-j}(s)\tilde{p}_j(t)p_{i-j}(u)\tilde{p}_j(v)}{H_{i-j}\tilde{H}_j}, \tag{3}$$

where $H_n$ is as given in Section 2 for $\{p_n\}$ and $\tilde{H}_n$ is defined similarly for $\{\tilde{p}_n\}$. By considering the first term in the above sum, we see that $K_n(s, t, s, t) > 0$.

## 4. Construction of Lagrange polynomials

Our construction follows a plan given in [**12**, §4.2] and [**9**]. Let $m$ be a given positive integer, fixed throughout, and define

$$G_m(s,t,u,v) = K_m(s,t,u,v) + \frac{\tilde{a}(a-c)}{c\tilde{c}} p_m(s)p_m(u).$$

Clearly $G_m(s,t,u,v)$ is a polynomial of degree at most $m$ in $(s,t)$ and in $(u,v)$. Also, we may write

$$G_m(s,t,u,v) = K_{m-1}(s,t,u,v) + S_m(s,t,u,v),$$

where

$$S_m(s,t,u,v) = \sum_{i=1}^{m-1} p_{m-i}(s)\tilde{p}_i(t)p_{m-i}(u)\tilde{p}_i(v) + \frac{a\tilde{a}}{c\tilde{c}} p_m(s)p_m(u) + \frac{a}{c}\tilde{p}_m(t)\tilde{p}_m(v).$$

Hence $G_m(s, t, s, t) > 0$ for all $(s, t) \in \mathbb{R}^2$. Given $(n, q) \in Q_k$, define

$$P_{n,q}(s,t) = \lambda_{n,q} G_m(s,t,h_n,\tilde{h}_q), \text{ where } \lambda_{n,q} = \frac{1}{G_m(h_n,\tilde{h}_q,h_n,\tilde{h}_q)}.$$

THEOREM 1. *Let $k = 0$ or $k = 1$ and let $(n, q) \in Q_k$. Then $P_{n,q}$ is a polynomial of degree $m$ satisfying $P_{n,q}(h_n, \tilde{h}_q) = 1$ and $P_{n,q}(x) = 0$ for all $x \in \mathcal{N}_k$ with $x \neq (h_n, \tilde{h}_q)$.*

Our proof of Theorem 1 follows easily from two Christoffel-Darboux identities for bivariate polynomials. (Compare [**12**, Theorem 4.2.1] and [**9**].) To state the identities, first define polynomials

$$\begin{aligned} Y_0(s,t) &= (as+b)p_m(s) - p_{m-1}(s) = \pi_m(s), \\ Y_i(s,t) &= p_{m-i+1}(s)\tilde{p}_i(t) - \epsilon\, p_{i-1}(s)\tilde{p}_{m-i+1}(t), \quad i = 1, 2, \ldots, m+1, \end{aligned}$$

where $\epsilon$ is a constant. It follows from (2) that all of these polynomials vanish at the nodes $\mathcal{N}_k$ when $\epsilon = (-1)^k$ and $k = 0, 1$.

PROPOSITION 2.

$$c(s-u)G_m(s,t,u,v) = \sum_{i=0}^{m}{}' \, [Y_i(s,t)p_{m-i}(u)\tilde{p}_i(v) - Y_i(u,v)p_{m-i}(s)\tilde{p}_i(t)].$$

*Here $'$ means that the term for $i = 0$ is multiplied by $\tilde{a}/\tilde{c}$.*

PROPOSITION 3.

$$\tilde{c}(t-v)G_m(s,t,u,v) = \sum_{i=0}^{m}{}'' \, [Y_{i+1}(s,t)p_{m-i}(u)\tilde{p}_i(v) - Y_{i+1}(u,v)p_{m-i}(s)\tilde{p}_i(t)].$$

*Here $''$ means that the terms for $i = 0$ and $i = m$ are multiplied by $a/c$.*

To deduce Theorem 1, note that $P_{n,q}(h_n, \tilde{h}_q) = 1$ by definition. If $x = (h_{n'}, \tilde{h}_{q'})$ is a node in $\mathcal{N}_k$, then

$$\begin{aligned} (h_{n'} - h_n)P_{n,q}(x) &= 0, \\ (\tilde{h}_{q'} - \tilde{h}_q)P_{n,q}(x) &= 0, \end{aligned}$$

by Propositions 2 and 3. Hence, if $x \neq (h_n, \tilde{h}_q)$ then $P_{n,q}(x) = 0$, as required.

The same arguments that established Theorems 5 and 6 of [**8**] apply in this case to obtain cubature theorems of degree $2m - 1$ and $2m$ for our more general Morrow-Patterson nodes. The weights are positive numbers $\lambda_{n,q}$ given by

$$\frac{1}{\lambda_{n,q}} = \sum_{j=0}^{m} \sum_{i=0}^{m-j} \frac{p_i(h_n)^2 \tilde{p}_j(\tilde{h}_q)^2}{H_i \tilde{H}_j} + \frac{\tilde{a}(a-c)}{c\tilde{c}}, \quad (n,q) \in Q_0 \cup Q_1. \tag{4}$$

THEOREM 4. *Let $\{p_n\}$ and $\{\tilde{p}_n\}$ be two sequences of Geronimus polynomials and suppose their moment functionals are given by weight functions $w$ and $\tilde{w}$. Let $m$ be a positive integer and let $\{h_n\}_0^m$ and $\{\tilde{h}_q\}_0^{m+1}$ be corresponding alternation points. Then*

$$\iint_{\mathbb{R}^2} p(s,t)w(s)\tilde{w}(t)\, ds\, dt = \sum_{(n,q) \in Q_k} \lambda_{n,q}\, p(h_n, \tilde{h}_q) \tag{5}$$

*for all bivariate polynomials $p$ of degree at most $2m - 1$ and for $k = 0, 1$.*

THEOREM 5. *There exist real numbers $\lambda_{n,q}$ such that the cubature formula (5) holds for all $p \in \mathcal{P}_{2m}(\mathbb{R}^2)$ if and only if $a = c$ for the Geronimus polynomials generating the first coordinates of the nodes. In that case, $\lambda_{n,q}$ is given by (4) and the number of nodes is a minimum.*

It follows from [**8**, Theorem 7] and [**7**, Theorem 2.2] that the weights $\lambda_{n,q}$ in (4) can be given more simply by

$$\frac{1}{\lambda_{n,q}} = \frac{(-1)^{n+q}}{2c\tilde{c}} {\pi_m}'(h_n)\, \tilde{\pi}'_{m+1}(\tilde{h}_q), \quad (n,q) \in Q_0 \cup Q_1.$$

## 5. Proof of Propositions 2 and 3

As in [**9**], we begin with the following identity for the reproducing kernel:

(6)

$$c(s-u)K_m(s,t,u,v) = \sum_{j=0}^{m} \frac{\tilde{p}_j(t)\tilde{p}_j(v)}{\tilde{H}_j} \left[p_{m-j+1}(s)p_{m-j}(u) - p_{m-j}(s)p_{m-j+1}(u)\right].$$

Define

$$\begin{aligned} f_j(s,u) &= p_{m-j+1}(s)p_{m-j}(u) - p_{m-j}(s)p_{m-j+1}(u), & j = 0,\ldots,m, \\ g_j(s,t,u,v) &= p_{j-1}(s)\tilde{p}_{m-j+1}(t)p_{m-j}(u)\tilde{p}_j(v), & j = 1,\ldots,m. \end{aligned}$$

Then

$$c(s-u)K_m(s,t,u,v) = \frac{\tilde{a}}{\tilde{c}} f_0(s,u) + \sum_{j=1}^{m} f_j(s,u)\tilde{p}_j(t)\tilde{p}_j(v).$$

By a computation,

$$\begin{aligned} &f_j(s,u)\tilde{p}_j(t)\tilde{p}_j(v) + \epsilon[g_j(s,t,u,v) - g_{m-j+1}(s,t,u,v)] \\ &\quad = Y_j(s,t)p_{m-j}(u)\tilde{p}_j(v) - Y_j(u,v)p_{m-j}(s)\tilde{p}_j(t) \end{aligned}$$

for $j = 1,2,\ldots,m$. Since

$$\sum_{j=1}^{m}(g_j - g_{m-j+1}) = 0,$$

it follows that

$$\begin{aligned} (7) \qquad & c(s-u)K_m(s,t,u,v) \\ &= \frac{\tilde{a}}{\tilde{c}} f_0(s,u) + \sum_{j=1}^{m}[Y_j(s,t)p_{m-j}(u)\tilde{p}_j(v) - Y_j(u,v)p_{m-j}(s)\tilde{p}_j(t)]. \end{aligned}$$

Let

$$h(s,t,u,v) = Y_0(s,t)p_m(u) - Y_0(u,v)p_m(s) - f_0(s,u).$$

Then the right-hand side of the identity in Proposition 2 is just the sum of the right-hand side of (7) and

$$\frac{\tilde{a}}{\tilde{c}} h(s,t,u,v). \tag{8}$$

Applying (1) and the definitions of each of the terms of $h(s,t,u,v)$, we obtain

$$h(s,t,u,v) = (a-c)(s-u)p_m(s)p_m(u).$$

Thus the left-hand side of the identity in Proposition 2 is also just the sum of the left-hand side of (7) and (8), which verifies Proposition 2.

We will deduce Proposition 3 from (6) in a similar way. By interchanging $p$ and $\tilde{p}$, $s$ and $t$, and $u$ and $v$ in (6) and applying the symmetry in the definition of $K_m$, we obtain

$$\tilde{c}(t-v)K_m(s,t,u,v) = \sum_{j=0}^{m} \tilde{f}_{m-j}(t,v)\frac{p_j(s)p_j(u)}{H_j},$$

where $\tilde{f}_j(t,v) = \tilde{p}_{j+1}(t)\tilde{p}_j(v) - \tilde{p}_j(t)\tilde{p}_{j+1}(v)$. Then

$$\begin{aligned}\tilde{c}(t-v)K_m(s,t,u,v) &= A_0 + {\sum_{j=0}^{m}}'' \tilde{f}_{m-j}(t,v)p_j(s)p_j(u),\\ &= A_0 + {\sum_{j=0}^{m}}'' \tilde{f}_j(t,v)p_{m-j}(s)p_{m-j}(u),\end{aligned}$$

where

$$\begin{aligned}A_0 &= \left(1-\frac{a}{c}\right) p_m(s)p_m(u)[\tilde{p}_1(t)-\tilde{p}_1(v)],\\ &= \tilde{a}\left(1-\frac{a}{c}\right)(t-v)p_m(s)p_m(u).\end{aligned}$$

Hence

$$\tilde{c}(t-v)G_m(s,t,u,v) = {\sum_{j=0}^{m}}'' \tilde{f}_j(t,v)p_{m-j}(s)p_{m-j}(u). \tag{9}$$

By a computation,

$$\begin{aligned}&\tilde{f}_j(t,v)p_{m-j}(s)p_{m-j}(u) + \epsilon[\tilde{g}_j(s,t,u,v) - \tilde{g}_{m-j}(s,t,u,v)]\\ &\quad = Y_{j+1}(s,t)p_{m-j}(u)\tilde{p}_j(v) - Y_{j+1}(u,v)p_{m-j}(s)\tilde{p}_j(t),\end{aligned} \tag{10}$$

where $\tilde{g}_j(s,t,u,v) = p_{m-j}(s)p_j(u)\tilde{p}_j(t)\tilde{p}_{m-j}(v)$ and $j = 0,1,\ldots,m$. Since

$${\sum_{j=0}^{m}}'' (\tilde{g}_j - \tilde{g}_{m-j}) = 0,$$

Proposition 3 follows from (9) and (10).

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
*E-mail address*: sde243@g.uky.edu

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
*E-mail address*: larry@ms.uky.edu