Lagrange Polynomials, Reproducing Kernels and Cubature in Two Dimensions☆

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Abstract

We obtain by elementary methods necessary and sufficient conditions for a k-dimensional cubature formula to hold for all polynomials of degree up to $2m-1$ when the nodes of the formula have Lagrange polynomials of degree at most m. The main condition is that the Lagrange polynomial at each node is a scalar multiple of the reproducing kernel of degree $m-1$ evaluated at the node plus an orthogonal polynomial of degree m. Stronger conditions are given for the case where the cubature formula holds for all polynomials of degree up to $2m$.

This result is applied in one dimension to obtain a quadrature formula where the nodes are the roots of a quasi-orthogonal polynomial of order 2. In two dimensions the result is applied to obtain constructive proofs of cubature formulas of degree $2m-1$ for the Geronimus and the Morrow-Patterson classes of nodes. A cubature formula of degree $2m$ is obtained for a subclass of Morrow-Patterson nodes. Our discussion gives new proofs of previous theorems for the Chebyshev points and the Padua points, which are special cases.

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1. Introduction

This paper focuses on the nodes for quadrature and bivariate cubature that are derived from the Chebyshev polynomials. These usually can be

☆Dedicated to Dick Askey on his 80th birthday.

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computed explicitly and allow constructive, uncomplicated proofs. Our arguments are based on two elementary lemmas that characterize the existence of cubature formulas in terms of properties of the Lagrange polynomials at each of the nodes. We need to consider to what extent the Lagrange interpolation formula holds and how close the Lagrange polynomial for a point is to a multiple of the reproducing kernel evaluated at the point.

Most of the orthogonal polynomials we consider are Geronimus polynomials. These are a normalization of the polynomials whose three-term recurrence relation has constant coefficients. Thus the Geronimus polynomials include all four kinds of the the Chebyshev polynomials as well as a number of subtle examples that are not as amenable to computation.

An important property of the Geronimus polynomials $\{p_n\}$ is that there exist roots of an associated quasi-orthogonal polynomial π_m of order 2 that are alternation points of p_m and satisfy a compatibility condition with respect to the previous terms. The Chebyshev points for T_n are a representative example. We give a table of the alternation points for each of the four kinds of Chebyshev polynomials.

The two classes of nodes we consider for bivariate cubature are the Geronimus nodes and the Morrow-Patterson nodes. They are extensions of examples given in the classic paper of Morrow-Patterson [24]. In both cases, the even nodes are the pairs of alternation points whose indices have the same parity and the odd nodes are the pairs of alternation points whose indices have opposite parity. What differentiates the classes is that the Geronimus nodes use the same alternation points in both coordinates but the Morrow-Patterson nodes use alternation points of the next degree in the second coordinate.

The Geronimus nodes include what are sometimes called the Chebyshev points [28] or the Xu points [3] and were introduced in [20]. The Morrow-Patterson nodes contain the Padua points [7] as well as the Morrow-Patterson points as defined in [6]. An essential difference is that we consider the nodes generated by any Geronimus polynomial rather than by just a single one of the four kinds of Chebyshev polynomials. (References to examples of nodes considered before 2001 can be found in Section 7 of [11]. Many of them belong to the classes we consider.)

We prove cubature formulas of degree $2m-1$ for both classes of nodes. Previous statements of our formula for the Geronimus nodes [24, 28, 2] considered only the case of the Chebyshev polynomials T_n. We show that a cubature formula of degree $2m$ holds for the Morrow-Patterson nodes if and only if the coefficients of x are the same in the recurrence equations for the Geronimus polynomials generating the nodes. There is no cubature formula of degree $2m$ for the Geronimus nodes.

During the proofs we obtain explicit formulas for the Lagrange polynomials for the Geronimus and Morrow-Patterson nodes. They are constructed using Christoffel-Darboux formulas with polynomial coefficients having common zeros at the nodes. We also indicate another approach to our theorems depending on arguments of Bojanov and Petrova in [2]. An important additional fact is that the Christoffel numbers in our bivariate cubature formulas are twice the product of the Christoffel numbers in the associated quadrature formulas.

See [10] for a readable survey of the extensive field of cubature and see [11] for a summary of the connections with orthogonal polynomials. References to early work can be found in [25].

2. A condition for cubature

Given integers k and m with $k \geq 1$ and $m \geq 0$, let $\mathcal{P}_m(\mathbb{R}^k)$ denote the space of all real-valued polynomials in k variables with degree at most m. Let μ be a positive measure on $\mathbb{R}^k$ with finite multivariate moments and suppose that the only $p \in \mathcal{P}_m(\mathbb{R}^k)$ satisfying

$$\int_{\mathbb{R}^k} p(x)^2 \, d\mu(x) = 0$$

is $p = 0$. Then

$$(p, q) = \int_{\mathbb{R}^k} p(x)q(x) \, d\mu(x)$$

defines a complete inner product on $\mathcal{P}_m(\mathbb{R}^k)$. For example, μ may be given by $d\mu(x) = w(x)\, dx$, where w is a weight function and dx is Lebesgue measure on $\mathbb{R}^k$. Let $\mathcal{S}_m$ be the space of all orthogonal polynomials on $\mathbb{R}^k$ of degree m, i.e.,

$$\mathcal{S}_m = \{p \in \mathcal{P}_m(\mathbb{R}^k) : \ (q, p) = 0 \text{ for all } q \in \mathcal{P}_{m-1}(\mathbb{R}^k)\}.$$

It is well known [12] that there exists a reproducing kernel $K_m : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ with

$$p(y) = (p, K_m(\cdot, y)) = \int_{\mathbb{R}^k} p(x) K_m(x, y) \, d\mu(x)$$

for all $p \in \mathcal{P}_m(\mathbb{R}^k)$ and $y \in \mathbb{R}^k$. In particular, for each $y \in \mathbb{R}^k$, $q(x) = K_m(x,y)$ is in $\mathcal{P}_m(\mathbb{R}^k)$, $K_m(x,y) = K_m(y,x)$ and $K_m(x,x) > 0$ for all $x \in \mathbb{R}^k$.

Let $\{x_i\}_{i=1}^n$ be n distinct points of $\mathbb{R}^k$ and suppose $\{P_i\}_{i=1}^n$ is a corresponding set of Lagrange polynomials in $\mathcal{P}_m(\mathbb{R}^k)$, i.e.,

$$P_i(x_j) = \delta_{i,j}, \quad 1 \le i,j \le n.$$

It is often a difficult problem to determine whether such Lagrange polynomials exist. (A survey of results up to 2000 is given in [13].) The following elementary lemmas show an equivalence between a cubature formula and a formula for the Lagrange polynomials for the nodes.

Lemma 1. *Suppose $\{\lambda_i\}_{i=1}^n$ are real numbers and $m \ge 1$. Conditions* (a) *and* (b) *below are equivalent.*

a) *If $p \in \mathcal{P}_m(\mathbb{R}^k)$ then there is an $S \in \mathcal{S}_m$ with*

$$p = \sum_{i=1}^n p(x_i) P_i + S. \tag{1}$$

Also, for each $i = 1, \ldots, n$, there is an $S_i \in \mathcal{S}_m$ with

$$P_i(x) = \lambda_i K_{m-1}(x, x_i) + S_i(x), \quad x \in \mathbb{R}^k. \tag{2}$$

b)

$$\int_{\mathbb{R}^k} p(x)\, d\mu(x) = \sum_{i=1}^n \lambda_i p(x_i) \tag{3}$$

for all $p \in \mathcal{P}_{2m-1}(\mathbb{R}^k)$.

Lemma 2. *Suppose $\{\lambda_i\}_{i=1}^n$ are real numbers and $m \ge 1$. Conditions* (a′) *and* (b′) *below are equivalent.*

a′) *If $p \in \mathcal{P}_m(\mathbb{R}^k)$ then $p = \sum_{i=1}^n p(x_i) P_i$ and $P_i(x) = \lambda_i K_m(x, x_i)$ for all $x \in \mathbb{R}^k$ and $i = 1, \ldots, n$.*

b′) *Equation (3) holds for all $p \in \mathcal{P}_{2m}(\mathbb{R}^k)$.*

Proof. We first prove Lemma 1.

(a)⇒(b). Let $q_1 \in \mathcal{P}_{m-1}(\mathbb{R}^k)$. For each $i = 1, \ldots, n$, by the reproducing property,

$$\lambda_i q_1(x_i) = (q_1, \lambda_i K_{m-1}(\cdot, x_i)) = (q_1, P_i - S_i) = (q_1, P_i)$$

since $(q_1, S_i) = 0$. Now if $q_2 \in \mathcal{P}_m(\mathbb{R}^k)$, then it follows from (1) that

$$(q_1, q_2) = \sum_{i=1}^{n} q_2(x_i)(q_1, P_i) + (q_1, S) = \sum_{i=1}^{n} \lambda_i q_1(x_i) q_2(x_i).$$

Therefore (3) holds when $p = q_1 q_2$ and thus it holds when $p \in \mathcal{P}_{2m-1}(\mathbb{R}^k)$ since p is a linear combination of monomials of the form $q_1 q_2$.

(b)$\Rightarrow$(a). Let $p \in \mathcal{P}_m(\mathbb{R}^k)$ and take $q = p - \sum_{i=1}^{n} p(x_i) P_i$. Clearly $q(x_i) = 0$ whenever $i = 1, \ldots, n$. If $q_1 \in \mathcal{P}_{m-1}(\mathbb{R}^k)$ then $q_1 q \in \mathcal{P}_{2m-1}(\mathbb{R}^k)$ and so $(q_1, q) = 0$ by (3). Hence $q \in \mathcal{S}_m$, which is (1).

Now, given $i = 1, \ldots, n$, take $q(x) = P_i(x) - \lambda_i K_{m-1}(x, x_i)$. If $q_1 \in \mathcal{P}_{m-1}(\mathbb{R}^k)$, then $(q_1, P_i) = \lambda_i q_1(x_i)$ by (3) and $(q_1, K_{m-1}(\cdot, x_i)) = q_1(x_i)$ by the reproducing property. Hence $(q_1, q) = 0$ for all $q_1 \in \mathcal{P}_{m-1}(\mathbb{R}^k)$ so $q \in \mathcal{S}_m$, as required.

The proof of Lemma 2 is simpler since the stronger hypotheses allow us to take $q_1 \in \mathcal{P}_m(\mathbb{R}^k)$ in the above proof. Lemma 2 is immediate from [10, Th. 7.3], which also proves the existence of the Lagrange polynomials for nodes satisfying (b$'$).

3. The classical case

Our condition for cubature is useful even in the classical case of one variable. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of real-valued orthogonal polynomials on $\mathbb{R}$ with respect to a positive measure μ as described in Section 2. As usual, the degree of p_n is n and the highest coefficient of p_n is denoted by k_n. We shall assume that $k_n > 0$ for all $n \geq 0$. Set

$$H_n = \int_{\mathbb{R}} p_n(x)^2 \, d\mu(x)$$

and note that $H_n > 0$ for all $n \geq 0$ by our hypothesis.

Theorem 3. *For $m \geq 1$, let $\pi_m(x) = (ax + b)p_m(x) - p_{m-1}(x)$, where a and b are real constants with $a > 0$. Then the roots $x_0, \ldots, x_m$ of π_m are distinct and*

$$\int_{\mathbb{R}} p(x) \, d\mu(x) = \sum_{i=0}^{m} \lambda_i p(x_i) \tag{4}$$

for all $p \in \mathcal{P}_{2m-1}(\mathbb{R})$, *where* $p_m(x_i)\pi'_m(x_i) \neq 0$ *and*

$$\lambda_i = \frac{k_m H_{m-1}}{k_{m-1} p_m(x_i)\pi'_m(x_i)} > 0$$

for $i = 0, \ldots, m$.

As is well known [1, p. 244], orthogonal polynomials satisfy a three-term recurrence relation

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \quad n \geq 0,$$

$$\text{where } A_n = \frac{k_{n+1}}{k_n}, \; C_{n+1} = \frac{A_{n+1}H_{n+1}}{A_n H_n} > 0, \text{ and } p_{-1} = 0.$$

If $a = A_m/C_m$ and $b = B_m/C_m$, then $C_m \pi_m = p_{m+1}$ so (4) is Gaussian quadrature and holds for all $p \in \mathcal{P}_{2m+1}(\mathbb{R})$.

Proof. Suppose we modify the three-term recurrence relation for $n \geq m$ by taking $A_n = a$, $B_n = b$ and $C_n = 1$. By Favard's theorem, the new set of polynomials is orthogonal with respect to some positive measure and hence the roots of π_m are real and distinct. (See [8].)

Let $i = 0, \ldots, m$ and put $c_i = \pi'_m(x_i)$. By Lemma 1, it suffices to show that $p_m(x_i)c_i > 0$ and

$$P_i(x) = \lambda_i K_{m-1}(x, x_i) + \beta_i p_m(x), \tag{5}$$

where $\beta_i = a/c_i$. Clearly $\pi_m(x) = c_i(x - x_i)P_i(x)$, where P_i is the Lagrange polynomial for x_i. By the classical Christoffel-Darboux formula [1, p. 246] and the equation $\pi_m(x_i) = 0$, we have

$$\begin{aligned}
\frac{k_m H_{m-1}}{k_{m-1}}(x - x_i)K_{m-1}(x, x_i) &= p_m(x)p_{m-1}(x_i) - p_m(x_i)p_{m-1}(x) \\
&= p_m(x_i)[(ax_i + b)p_m(x) - p_{m-1}(x)] \\
&= p_m(x_i)[\pi_m(x) - a(x - x_i)p_m(x)] \\
&= p_m(x_i)(x - x_i)[c_i P_i(x) - a p_m(x)].
\end{aligned}$$

In particular, dividing by $x - x_i$ and taking the limit as $x \to x_i$, we obtain

$$p_m(x_i)c_i = \frac{k_m H_{m-1}}{k_{m-1}} K_{m-1}(x_i, x_i) + a p_m(x_i)^2 > 0.$$

Thus one can solve for $P_i(x)$ to obtain (5).

Remark 1. Theorem 3 applies to the roots of most quasi-orthogonal polynomials of order 2. By definition, a polynomial p of degree $m+1$ is quasi-orthogonal of order 2 if there exist real constants α_1 and α_2 such that

$$p = p_{m+1} + \alpha_1 p_m + \alpha_2 p_{m-1}.$$

Suppose $\alpha_2 < C_m$ and take $\pi_m = p/(C_m - \alpha_2)$. Then Theorem 3 applies with

$$a = \frac{A_m}{C_m - \alpha_2}, \quad b = \frac{B_m + \alpha_1}{C_m - \alpha_2}.$$

In particular, if $\alpha_2 = 0$ then $a > 0$ and

$$\beta_i = \frac{A_m}{C_m c_i} = \frac{A_{m-1} H_{m-1}}{H_m c_i} = \frac{k_m H_{m-1}}{k_{m-1} H_m c_i} = \frac{\lambda_i p_m(x_i)}{H_m},$$

so

$$P_i(x) = \lambda_i K_m(x, x_i), \quad i = 0, \ldots, m,$$

by (5). Hence (4) holds for all $p \in \mathcal{P}_{2m}(\mathbb{R})$ by Lemma 2. Of course, if $\alpha_1 = \alpha_2 = 0$, then (4) is Gaussian quadrature. See [14, p. 21] and [29] for references.

4. The Geronimus polynomials

Let α, β, γ, and δ be real constants with $\alpha > 0$ and $\gamma > 0$. The Geronimus polynomials are the sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials defined by the recursive equations

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = \alpha x + \beta, \\ p_{n+1}(x) &= (\gamma x + \delta) p_n(x) - p_{n-1}(x), \quad n \geq 1. \end{aligned} \tag{6}$$

These were first considered by Geronimus in [16] for the case $\gamma = 2$ and $\delta = 0$. It follows from Favard's theorem [8, Theorem 4.4] that $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to a compactly supported positive Borel measure μ with $H_0 = \gamma/\alpha$ and $H_n = 1$ for all $n \geq 1$. (The measure can have up to two atoms. See [9] and [26].) Let $\pi_m(x)$ be defined as in Theorem 3. It is shown in [20] that if $a = \alpha$, $b = \beta$ and $x_0, \ldots, x_m$ are the roots of π_m in decreasing order then

$$p_{m-j}(x_i) = (-1)^i p_j(x_i), \quad i, j = 0, \ldots, m. \tag{7}$$

In particular, $p_m(x_i) = (-1)^i$ for $i = 0, \ldots, m$. Theorem 3 for this case has been given in [20, Theorem 2.2]. It is easy to verify that

$$\pi_{m+1}(x) = (\gamma x + \delta)\pi_m(x) - \pi_{m-1}(x), \quad m \geq 2.$$

The main examples of the Geronimus polynomials are the four kinds of Chebyshev polynomials, which correspond to four choices of the pair (α, β) with $\gamma = 2$ and $\delta = 0$. See Table 1. The associated measure μ is given by $d\mu(x) = w_i(x)\,dx$ where the weight function w_i is taken to be 0 outside the interval $(-1, 1)$. Many Bernstein-Szegö polynomials derived from these weights are also Geronimus polynomials. (See [15] and the examples given in [8, p. 204–206].)

Table 1: The four kinds of Chebyshev polynomials

Kind	(α, β)	Definition for $x = \cos\theta$	weight	$x_i = \cos\theta_i$
1st	$(1, 0)$	$T_n(x) = \cos n\theta$	$w_1(x) = \dfrac{2}{\pi\sqrt{1-x^2}}$	$\theta_i = \dfrac{i\pi}{m}$
2nd	$(2, 0)$	$U_n(x) = \dfrac{\sin(n+1)\theta}{\sin\theta}$	$w_2(x) = \frac{2}{\pi}\sqrt{1-x^2}$	$\theta_i = \dfrac{(i+1)\pi}{m+2}$
3rd	$(2, -1)$	$V_n(x) = \dfrac{\cos(n+1/2)\theta}{\cos(\theta/2)}$	$w_3(x) = \frac{1}{\pi}\sqrt{\dfrac{1+x}{1-x}}$	$\theta_i = \dfrac{i\pi}{m+1}$
4th	$(2, 1)$	$W_n(x) = \dfrac{\sin(n+1/2)\theta}{\sin(\theta/2)}$	$w_4(x) = \frac{1}{\pi}\sqrt{\dfrac{1-x}{1+x}}$	$\theta_i = \dfrac{(i+1)\pi}{m+1}$

It follows easily by induction that

$$p_n(x) = (\alpha x + \beta)U_{n-1}\left(\frac{\gamma x + \delta}{2}\right) - U_{n-2}\left(\frac{\gamma x + \delta}{2}\right), \quad n \geq 1.$$

Hence, for example, the conclusions of Theorem 3 hold with $d\mu(x) = w_2(x)\,dx$ when $x_0, \ldots, x_m$ are the roots of any Geronimus polynomial p_{m+1} with $\gamma = 2$ and $\delta = 0$ since in this case we may take $\pi_m = p_{m+1}$.

If $\{p_n\}_{n=0}^{\infty}$ is one of the kinds of Chebyshev polynomials, then it is easy to verify from (6) that

$$\begin{array}{ll} \pi_m = \frac{1}{2}(p_{m+1} - p_{m-1}) & \text{when } a = 1, b = 0, \\ \pi_m = p_{m+1} & \text{when } a = 2, b = 0, \\ \pi_m = p_{m+1} - p_m & \text{when } a = 2, b = -1, \\ \pi_m = p_{m+1} + p_m & \text{when } a = 2, b = 1. \end{array}$$

In each case, the roots of π_m and the values in Theorem 3 can be computed easily with elementary trigonometry. The results are given in Table 2. (See

[21, Chapter 8] and compare [22].) The last column of Table 2 gives the value(s) of i where one obtains the required weight by dividing the formula for λ_i by 2. Thus each row of the table specifies a quadrature formula. For example, the quadrature formula corresponding to the first row is the Gauss-Lobatto formula [14, p. 26]

$$\int_{-1}^{1} p(x)\frac{2}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{m}p(1)+\sum_{i=1}^{m-1}\frac{2}{m}p\left(\cos\frac{i\pi}{m}\right)+\frac{1}{m}p(-1), \quad p \in \mathcal{P}_{2m-1}(\mathbb{R}).$$

The quadrature formula corresponding to the second row for each weight function is Gaussian quadrature and thus is valid for all $p \in \mathcal{P}_{2m+1}(\mathbb{R})$. The quadrature formulas corresponding to the third and fourth rows for each weight function are valid for all $p \in \mathcal{P}_{2m}(\mathbb{R})$ by Remark 1.

In the next sections we use the Geronimus polynomials to specify classes of nodes in $\mathbb{R}^2$ for cubature.

Table 2: Quadrature for the Chebyshev weights

weight	(a,b)	$\pi_m(x)$	$x_i = \cos\theta_i$	λ_i	$\lambda_i/2$
w_1	$(1,0)$	$(x^2-1)U_{m-1}(x)$	$\theta_i = \frac{i\pi}{m}$	$\frac{2}{m}$	$i=0,m$
w_1	$(2,0)$	$T_{m+1}(x)$	$\theta_i = \frac{(2i+1)\pi}{2(m+1)}$	$\frac{2}{m+1}$	
w_1	$(2,-1)$	$(x-1)W_m(x)$	$\theta_i = \frac{2i\pi}{2m+1}$	$\frac{4}{2m+1}$	$i=0$
w_1	$(2,1)$	$(x+1)V_m(x)$	$\theta_i = \frac{(2i+1)\pi}{2m+1}$	$\frac{4}{2m+1}$	$i=m$
w_2	$(1,0)$	$T_{m+1}(x)$	$\theta_i = \frac{(2i+1)\pi}{2(m+1)}$	$\frac{2(1-x_i^2)}{m+1}$	
w_2	$(2,0)$	$U_{m+1}(x)$	$\theta_i = \frac{(i+1)\pi}{m+2}$	$\frac{2(1-x_i^2)}{m+2}$	
w_2	$(2,-1)$	$V_{m+1}(x)$	$\theta_i = \frac{(2i+1)\pi}{2m+3}$	$\frac{4(1-x_i^2)}{2m+3}$	
w_2	$(2,1)$	$W_{m+1}(x)$	$\theta_i = \frac{2(i+1)\pi}{2m+3}$	$\frac{4(1-x_i^2)}{2m+3}$	
w_3	$(1,0)$	$(x-1)W_m(x)$	$\theta_i = \frac{2i\pi}{2m+1}$	$\frac{2(1+x_i)}{2m+1}$	$i=0$
w_3	$(2,0)$	$V_{m+1}(x)$	$\theta_i = \frac{(2i+1)\pi}{2m+3}$	$\frac{2(1+x_i)}{2m+3}$	
w_3	$(2,-1)$	$2(x-1)U_m(x)$	$\theta_i = \frac{i\pi}{m+1}$	$\frac{1+x_i}{m+1}$	$i=0$
w_3	$(2,1)$	$2T_{m+1}(x)$	$\theta_i = \frac{(2i+1)\pi}{2(m+1)}$	$\frac{1+x_i}{m+1}$	
w_4	$(1,0)$	$(x+1)V_m(x)$	$\theta_i = \frac{(2i+1)\pi}{2m+1}$	$\frac{2(1-x_i)}{2m+1}$	$i=m$
w_4	$(2,0)$	$W_{m+1}(x)$	$\theta_i = \frac{2(i+1)\pi}{2m+3}$	$\frac{2(1-x_i)}{2m+3}$	
w_4	$(2,-1)$	$2T_{m+1}(x)$	$\theta_i = \frac{(2i+1)\pi}{2(m+1)}$	$\frac{1-x_i}{m+1}$	
w_4	$(2,1)$	$2(x+1)U_m(x)$	$\theta_i = \frac{(i+1)\pi}{m+1}$	$\frac{1-x_i}{m+1}$	$i=m$

5. The Geronimus nodes

In this section we apply Lemma 1 to obtain a two dimensional analogue of Theorem 3. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of Geronimus polynomials given by (6) and let μ be the corresponding measure. Let π_m be defined as in Theorem 3 with $a=\alpha$ and $b=\beta$, i.e., $\pi_m = p_1 p_m - p_{m-1}$, and let $x_0, \ldots, x_m$ be the roots of π_m in decreasing order.

We consider two sets of nodes where we evaluate functions of two variables. Define the even Geronimus nodes $\mathcal{N}_0$ to be the set of ordered pairs

(x_n, x_q) with $0 \le n, q \le m$, where n and q are both even or both odd and define the odd Geronimus nodes $\mathcal{N}_1$ to be the set of ordered pairs (x_n, x_q) with $0 \le n, q \le m$, where n is even and q is odd or n is odd and q is even. Thus if $k = 0$ or $k = 1$, then the Geronimus nodes are given by

$$\mathcal{N}_k = \{(x_n, x_q) : \ (n, q) \in Q_k\},$$

where

$$Q_k = \{(n, q) : \ 0 \le n, q \le m, \ n - q = k \bmod 2\}.$$

Theorem 4. *Let $m \ge 1$ and let $k = 0$ or $k = 1$. Then there exist positive real numbers $\lambda_{n,q}$ satisfying*

$$\int_{\mathbb{R}^2} p(x, y)\, d(\mu \times \mu)(x, y) = \sum_{(n,q)\in Q_k} \lambda_{n,q}\, p(x_n, x_q) \tag{8}$$

for all $p \in \mathcal{P}_{2m-1}(\mathbb{R}^2)$.

It is easy to verify that the number of nodes satisfies

$$\begin{aligned} n(\mathcal{N}_0) = n(\mathcal{N}_1) &= \frac{(m+1)^2}{2} \quad \text{for } m \text{ odd},\\ n(\mathcal{N}_0) - 1 = n(\mathcal{N}_1) &= \frac{m(m+2)}{2} \quad \text{for } m \text{ even}. \end{aligned}$$

By a theorem of Möller [23], if μ is a measure induced by a centrally symmetric weight function, the number of nodes in any bivariate cubature formula of degree $2m - 1$ is at least

$$\binom{m+1}{2} + \left[\frac{m}{2}\right].$$

Thus for these measures, the number of nodes in Theorem 4 is at most one more than the minimal number and is minimal when m is even and $k = 1$.

The coordinates of the nodes and their weights are given explicitly for each of the four kinds of the Chebyshev polynomials in Table 2 and in this case the nodes lie in the unit square $[-1, 1] \times [-1, 1]$. The Chebyshev nodes are the special case of the Geronimus nodes where $p_n(x) = T_n(x)$ so that $\pi_m(x) = (x^2 - 1)U_{m-1}(x)$. Proofs of Theorem 4 have been given only for the Chebyshev nodes. (See [24, 28, 2, 18].) We point out that the Chebyshev nodes also arise naturally in the extension of V. Markov's polynomial inequality to higher dimensions [20].

We deduce Theorem 4 from Lemma 1. (Compare [28].) It is easy to check that the set of polynomials

$$\{p_{i-j}(x)p_j(y):\ 0 \le j \le i,\ i = 0, 1, \ldots\}$$

forms an orthogonal system with respect to the product measure $\mu \times \mu$ and hence the reproducing kernel for $\mathcal{P}_m(\mathbb{R}^2)$ is given by

$$\begin{aligned} K_m(x, y, u, v) &= \sum_{i=0}^{m} \sum_{j=0}^{i} \frac{p_{i-j}(x)p_j(y)p_{i-j}(u)p_j(v)}{H_{i-j}H_j} \\ &= {\sum_{i=0}^{m}}{}' {\sum_{j=0}^{i}}{}'' p_{i-j}(x)p_j(y)p_{i-j}(u)p_j(v). \end{aligned} \tag{9}$$

Here $'$ means that the first term of the sum is multiplied by α/γ and $''$ means that the first and last terms of the sum are multiplied by α/γ (but only once if these terms are the same). Define polynomials

$$\begin{aligned} X_i(x, y) &= p_{m-i}(x)p_i(y) - \epsilon\, p_i(x)p_{m-i}(y), \quad i = 0, \ldots, m, \\ Y_0(x, y) &= (\alpha x + \beta)p_m(x) - p_{m-1}(x) = \pi_m(x), \\ Y_i(x, y) &= p_{m-i+1}(x)p_i(y) - \epsilon\, p_{i-1}(x)p_{m-i}(y), \quad i = 1, \ldots, m, \end{aligned}$$

where ϵ is a constant. All of these polynomials vanish at the nodes $\mathcal{N}_k$ when $\epsilon = (-1)^k$ and $k = 0, 1$ by (7). It can be shown as in [19, p. 380] that the polynomial

$$\begin{aligned} P_m(x, y, u, v) &= {\sum_{i=0}^{m-1}}{}' [X_i(x, y)p_{m-i-1}(u)p_i(v) - X_i(u, v)p_{m-i-1}(x)p_i(y)] \\ &\quad + {\sum_{i=0}^{m}}{}'' [Y_i(x, y)p_{m-i}(u)p_i(v) - Y_i(u, v)p_{m-i}(x)p_i(y)] \end{aligned} \tag{10}$$

is independent of ϵ. When $\epsilon = 0$ it is evident that $P_m(x, y, u, v) = 0$ when $x = u$ so we can write

$$P_m(x, y, u, v) = 2\gamma(x - u)G_m(x, y, u, v),$$

where G_m is a polynomial. The previous identity can be regarded as a Christoffel-Darboux formula since it can be verified as in [19] that

$$\begin{aligned} G_m(x,y,u,v) &= \frac{1}{2}[K_{m-1}(x,y,u,v) + K_m(x,y,u,v)] \\ &\quad + \frac{\alpha(\alpha-\gamma)}{2\gamma^2}[p_m(x)p_m(u) + p_m(y)p_m(v)]. \end{aligned}$$

Clearly G_m is a polynomial of degree at most m in its first two and last two variables and $G_m(y,x,v,u) = G_m(x,y,u,v)$. By separating off the last term in the sum for K_m, we obtain

$$G_m = K_{m-1} + \frac{S_m}{2}, \tag{11}$$

where

$$S_m(x,y,u,v) = \sum_{i=1}^{m-1} p_{m-i}(x)p_i(y)p_{m-i}(u)p_i(v) + \frac{\alpha^2}{\gamma^2}[p_m(x)p_m(u) + p_m(y)p_m(v)].$$

Given $(n,q) \in Q_k$, define

$$P_{n,q}(x,y) = \lambda_{n,q} G_m(x,y,x_n,x_q), \quad \lambda_{n,q} = \frac{1}{G_m(x_n,x_q,x_n,x_q)},$$

and note that $\lambda_{n,q}$ is well defined and positive by (11). Hence $P_{n,q}$ is a Lagrange polynomial of degree m for (x_n,x_q) in $\mathcal{N}_k$ by (10) with $\epsilon = (-1)^k$. Moreover,

$$P_{n,q}(x,y) = \lambda_{n,q} K_{m-1}(x,y,x_n,x_q) + S_{n,q}(x,y), \tag{12}$$

where

$$S_{n,q}(x,y) = \frac{\lambda_{n,q}}{2} S_m(x,y,x_n,x_q)$$

and $S_{n,q}$ is an orthogonal polynomial of degree m. This establishes condition (2) of Lemma 1 and condition (1) follows from an elementary dimension argument given in [17, Theorem 5]. Note that the equality $\lambda_{n,q} = 2\lambda_n\lambda_q$ is not evident from this approach although it is true by Theorem 7 in the last section.

We point out that none of the Geronimus nodes have a cubature formula of degree $2m$, i.e., satisfying (b$'$) in Lemma 2. Indeed, the first equality of (a$'$) in Lemma 2 does not hold since there is at least one polynomial X_i with $\epsilon = (-1)^k$ and degree m such that X_i vanishes on a given set $\mathcal{N}_k$ of Geronimus nodes.

6. The Morrow-Patterson nodes

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of Geronimus polynomials given by (6) and let μ be the corresponding measure. Let $a = \alpha$ and $b = \beta$ in both π_m and π_{m+1} and let $x_0, \dots, x_m$ and $y_0, \dots, y_{m+1}$ be the corresponding roots in decreasing order. As in the previous section, we define two sets of nodes, $\mathcal{N}_0$ and $\mathcal{N}_1$, which we call the Morrow-Patterson nodes. Given $k = 0$ or $k = 1$, define

$$\mathcal{N}_k = \{(x_n, y_q) : \ (n, q) \in Q_k\},$$

where

$$Q_k = \{(n, q) : \ 0 \le n \le m,\ 0 \le q \le m + 1,\ n - q = k \bmod 2\}.$$

Note that the number $n(\mathcal{N}_k)$ of Morrow-Patterson nodes is given by

$$n(\mathcal{N}_k) = \frac{(m + 2)(m + 1)}{2}, \quad k = 0, 1,$$

which is also the dimension of $\mathcal{P}_m(\mathbb{R}^2)$. The Morrow-Patterson nodes for each of the four kinds of Chebyshev polynomials lie in the unit square and satisfy $\mathcal{N}_1 = -\mathcal{N}_0$, which can be deduced from the identity $\cos(\pi - \theta) = -\cos(\theta)$. The classical Morrow-Patterson nodes were given in [24, p. 960] and are the nodes for the case where $p_n = U_n$, $k = 1$ and m is even.

We will deduce the following theorems from Lemmas 1 and 2. Let

$$C(n, q) = {\sum_{j=0}^{m}}' {\sum_{i=0}^{m-j}}' p_i(x_n)^2 p_j(y_q)^2, \quad (n, q) \in Q_k.$$

Theorem 5. *Let $m \ge 1$ and $k = 0$ or $k = 1$. Then there exist positive real numbers $\lambda_{n,q}$ satisfying*

$$\int_{\mathbb{R}^2} p(x, y)\, d(\mu \times \mu)(x, y) = \sum_{(n,q) \in Q_k} \lambda_{n,q} p(x_n, y_q) \tag{13}$$

for all $p \in \mathcal{P}_{2m-1}(\mathbb{R}^2)$ and

$$\frac{1}{\lambda_{n,q}} = C(n, q) + \frac{\alpha}{\gamma^2}(\alpha - \gamma), \quad (n, q) \in Q_k. \tag{14}$$

Theorem 6. *Let $m \geq 1$ and $k = 0$ or $k = 1$. Then there exist real numbers $\lambda_{n,q}$ satisfying (13) for all $p \in \mathcal{P}_{2m}(\mathbb{R}^2)$ if and only if $\alpha = \gamma$ for the Geronimus polynomials generating the nodes. In that case, $\lambda_{n,q}$ is positive and $\lambda_{n,q} = 1/C(n,q)$ for all $(n,q) \in Q_k$.*

Note that the Chebyshev polynomials of the second, third and fourth kind all satisfy the equation $\alpha = \gamma$ but the Chebyshev polynomials of the first kind do not. The cubature formula of Theorem 6 does not hold for any choice of a fewer number of nodes since the number of nodes in a bivariate cubature formula of degree $2m$ is at least $(m+2)(m+1)/2$ by a result of A. H. Stroud [27, p. 118].

To apply Lemma 1, we first construct Lagrange polynomials for the nodes $\mathcal{N}_k$ as in the previous section. Define polynomials of degree $m+1$ by

$$\begin{aligned} Y_0(x,y) &= (\alpha x + \beta)p_m(x) - p_{m-1}(x) = \pi_m(x), \\ Y_i(x,y) &= p_{m-i+1}(x)p_i(y) - (-1)^k p_{i-1}(x)p_{m-i+1}(y), \quad i = 1, \ldots, m+1. \end{aligned}$$

Then $\mathcal{N}_k$ is a set of common zeros for these polynomials by (7). (Again we could introduce a parameter ϵ but choose the more direct approach.) Define

$$G_m(x,y,u,v) = K_m(x,y,u,v) + \frac{\alpha(\alpha-\gamma)}{\gamma^2} p_m(x)p_m(u).$$

Then

$$G_m = K_{m-1} + S_m, \tag{15}$$

where

$$S_m(x,y,u,v) = \sum_{i=1}^{m-1} p_{m-i}(x)p_i(y)p_{m-i}(u)p_i(v) + \frac{\alpha^2}{\gamma^2} p_m(x)p_m(u) + \frac{\alpha}{\gamma} p_m(y)p_m(v).$$

The Christoffel-Darboux formulas

$$\begin{aligned} \gamma(x-u)G_m(x,y,u,v) &= \sum_{i=0}^{m}{}' \left[Y_i(x,y)p_{m-i}(u)p_i(v) - Y_i(u,v)p_{m-i}(x)p_i(y)\right], \\ \gamma(y-v)G_m(x,y,u,v) &= \sum_{i=0}^{m}{}'' \left[Y_{i+1}(x,y)p_{m-i}(u)p_i(v) - Y_{i+1}(u,v)p_{m-i}(x)p_i(y)\right] \end{aligned}$$

can be verified as in [19, p. 380]. Given $(n,q) \in Q_k$, define

$$P_{n,q}(x,y) = \lambda_{n,q} G_m(x,y,x_n,y_q), \quad \lambda_{n,q} = \frac{1}{G_m(x_n,y_q,x_n,y_q)}, \tag{16}$$

and note that $\lambda_{n,q}$ is well defined and positive by (15). Hence $P_{n,q}$ is a Lagrange polynomial of degree m for the node (x_n, y_q) in $\mathcal{N}_k$ by the Christoffel-Darboux formulas. Moreover,

$$P_{n,q}(x,y) = \lambda_{n,q} K_{m-1}(x,y,x_n,y_q) + S_{n,q}(x,y),$$

where

$$S_{n,q}(x,y) = \lambda_{n,q} S_m(x,y,x_n,y_q),$$

and $S_{n,q}$ is an orthogonal polynomial of degree m.

Note that $\{P_{n,q} : (n,q) \in Q_k\}$ is a basis for $\mathcal{P}_m(\mathbb{R}^2)$ since it is a set of N linearly independent polynomials in $\mathcal{P}_m(\mathbb{R}^2)$ where N is the dimension of $\mathcal{P}_m(\mathbb{R}^2)$. Hence, if $p \in \mathcal{P}_m(\mathbb{R}^2)$ then

$$p = \sum_{(n,q)\in Q_k} p(x_n, y_q) P_{n,q}$$

since one can compute the coordinates of p in this basis by evaluating at the nodes of $\mathcal{N}_k$. Thus the nodes $\mathcal{N}_k$ satisfy hypothesis (a) of Lemma 1 and hence Theorem 5 holds with weights $\lambda_{p,q}$ given by (16). One can obtain the expression for $C(n,q)$ by reversing the sums in (9). Our arguments also show that $\mathcal{N}_k$ is unisolvent. (See [4].)

If (n_0, q_0) and (n,q) are different elements of Q_k, then

$$\gamma^2 K_m(x_n, y_q, x_{n_0}, y_{q_0}) = (-1)^{n+n_0}\alpha(\gamma - \alpha) \tag{17}$$

by the definition of G_m and (16). Theorem 6 follows from this and Lemma 2.

Remark 2. Since 2005, many papers have appeared discussing one or more of the four families of Padua points defined in [7]. It is not difficult to see that the first and third families are the Morrow-Patterson nodes for $p_n = T_n$ with $k = 1$ and $k = 0$, respectively. The second and fourth families are the first and third families with the coordinates interchanged. In particular, the paper [5] considers the first family of Padua points and hence our results include Theorem 3.2, Proposition 3.4 and Corollary 2.4 given there.

7. An alternate approach

Let μ and ν be positive measures as described in the beginning of Section 2 and let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be sequences of real-valued orthogonal polynomials on $\mathbb{R}$ for μ and ν, respectively. The following product theorem follows directly from the extension of Theorem 2.1 of [2] to two measures and the arguments given in the proof of Theorem 3.4 of [20].

Theorem 7. *Suppose that quadrature formulas*

$$\begin{aligned} \int_{\mathbb{R}} p(x)\, d\mu(x) &= \sum_{i=0}^{I} \lambda_i p(x_i), \quad p \in \mathcal{P}_{2I-1}(\mathbb{R}), \\ \int_{\mathbb{R}} q(y)\, d\nu(y) &= \sum_{j=0}^{J} \gamma_j q(y_j), \quad q \in \mathcal{P}_{2J-1}(\mathbb{R}) \end{aligned}$$

hold with given weights and nodes. If $p_I(x_i) = (-1)^i$ *and* $q_J(y_j) = (-1)^j$ *for* $0 \le i \le I$ *and* $0 \le j \le J$*, then*

$$\int_{\mathbb{R}^2} p(x,y)\, d(\mu \times \nu)(x,y) = 2 \sum_{(i,j)\in Q_k} \lambda_i \gamma_j\, p(x_i, y_j)$$

for all $p \in \mathcal{P}_{2m-1}(\mathbb{R}^2)$*, where* $m = \min\{I, J\}$*,*

$$Q_k = \{(i,j) :\ 0 \le i \le I,\, 0 \le j \le J,\ i - j = k \bmod 2\}$$

and $k = 0$ *or* $k = 1$*.*

Theorems 4 and 5 follow immediately from Theorem 3, equation (7) and Theorem 7. Moreover, $\lambda_{n,q} = 2\lambda_n\lambda_q$ in both of the theorems.

Suppose the measures, orthogonal polynomials and quadrature formulas are the same in each variable. The next two examples show that the conclusions of Theorem 7 can fail dramatically when the hypothesis that $p_m(x_i) = (-1)^i$ is omitted.

Example 1. Let the given sequence of orthogonal polynomials be the Chebyshev polynomials of the first kind. Take $a = 2$, $b = -1$, $m = 2$ and $k = 0$. Then by the third row of Table 2, we have that the roots of π_2 are $x_0 = 1$, $x_1 = (\sqrt{5} - 1)/4$, and $x_2 = -(\sqrt{5} + 1)/4$. Thus

$$\mathcal{N}_0 = \{(x_0, x_0), (x_0, x_2), (x_1, x_1), (x_2, x_0), (x_2, x_2)\}.$$

One can obtain quadratic Lagrange polynomials for each of these nodes by solving a consistent system of 5 linear equations in 6 unknowns. However, for each node in $\mathcal{N}_0$, the system of equations obtained from (2) by replacement of x by each of the nodes in $\mathcal{N}_0$ is a system of 5 linear equations in 4 unknowns that has no solution. Thus none of the Lagrange polynomials for $\mathcal{N}_0$ have the form of (2). In particular, there is no cubature formula of the form (3) for the nodes of $\mathcal{N}_0$.

Example 2. Let the given sequence of orthogonal polynomials and a, m and k be as in the previous example but let $b = 0$. Then $\pi_2 = T_3$ so the roots of π_2 are $x_0 = \sqrt{3}/2$, $x_1 = 0$, and $x_2 = -\sqrt{3}/2$. Again, the Lagrange polynomials exist for the corresponding set $\mathcal{N}_0$ and one can show that $\lambda_i = 2/3$ in Theorem 3 while an equation of the form (12) holds with $\lambda_{n,q} = 2/3$ except that $\lambda_{1,1} = 4/3$. Thus the equality $\lambda_{n,q} = 2\lambda_n\lambda_q$ never holds in this case.

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