

BIVARIATE LAGRANGE INTERPOLATION AT THE CHEBYSHEV NODES

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ABSTRACT. We discuss Lagrange interpolation on two sets of nodes in two dimensions where the coordinates of the nodes are Chebyshev points having either the same or opposite parity. We use a formula of Xu for Lagrange polynomials to obtain a general interpolation theorem for bivariate polynomials at either set of Chebyshev nodes. An extra term must be added to the interpolation formula to handle all polynomials with the same degree as the Lagrange polynomials. We express this term as a specifically determined linear combination of canonical polynomials that vanish on the set of Chebyshev nodes being considered.

As an application we deduce in an elementary way known minimal and near minimal cubature formulas applying to both the even and the odd Chebyshev nodes. Finally, we restrict to triangular subsets of the Chebyshev nodes to show unisolvence and deduce a Lagrange interpolation formula for bivariate symmetric and skew-symmetric polynomials. This result leads to another proof of the interpolation formula.

1. INTRODUCTION

Given a natural number m , let $T_m(t) = \cos(m \arccos(t))$ denote the Chebyshev polynomial of degree m . The corresponding Chebyshev points are given by

$$h_n = \cos\left(\frac{n\pi}{m}\right), \quad n = 0, \dots, m,$$

and satisfy $T_m(h_n) = (-1)^n$ for these n . In [5] we defined the sets of even and odd Chebyshev nodes as two dimensional generalizations of the Chebyshev points. These are important because they are the nodes of a Duffin-Schaeffer-type extension of Markov's theorem for derivatives of polynomials on the unit square of $\mathbb{R}^2$ (see [6]) and the nodes of a minimal (or near minimal) Gauss-Lobatto-type cubature formula in $\mathbb{R}^2$ (see [1]). More generally, these nodes arise naturally in bivariate polynomial inequalities on the unit square when the Chebyshev polynomials of a single variable are extremal. (See Lemma 6 of [5].)

The Padua points are a related set of nodes that have the advantage of being unisolvent so no additional terms are needed in the corresponding interpolation formula. In particular, cubature formulas and Lagrange polynomials at these nodes are developed in [2], [3] and [4].

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We recall from [5] the definition and properties of an explicit form of Lagrange polynomials for the Chebyshev nodes. Specifically, the set $\mathcal{N}_0$ of even (respectively, $\mathcal{N}_1$ of odd) Chebyshev nodes is the set of ordered pairs (h_n, h_q) , $0 \leq n, q \leq m$, where n and q are both even or both odd (respectively, n is even and q is odd or n is odd and q is even). Thus, if $k = 0$ or $k = 1$,

$$\begin{aligned}\mathcal{N}_k &= \{(h_n, h_q) : (n, q) \in Q_k\}, \quad \text{where} \\ Q_k &= \{(n, q) : 0 \leq n, q \leq m, n - q = k \bmod 2\}.\end{aligned}$$

As in [5], we define (by a single formula) a set of Lagrange polynomials for the even Chebyshev nodes and a set of Lagrange polynomials for the odd Chebyshev nodes. Let $c_j = 1$ for $j = 1, \dots, m-1$ and $c_j = 1/2$ when $j = 0$ or $j = m$. For $0 \leq n, q \leq m$, define

$$(1.1) \quad P_{n,q}(s, t) = \frac{2}{m^2} c_n c_q G_m(s, t, h_n, h_q),$$

where

$$(1.2) \quad \begin{aligned} G_m(s, t, u, v) &= 4 \sum_{i=0}^m \sum_{j=0}^m T_{i-j}(s) T_j(t) T_{i-j}(u) T_j(v) \\ &\quad - \frac{1}{2} [T_m(s) T_m(u) + T_m(t) T_m(v)]. \end{aligned}$$

Here the symbol $''$ in a sum indicates that the first and last terms of the sum are divided by 2. (When the sum has only one term, this term is divided by 2 only once.) Clearly each $P_{n,q}$ is a polynomial of degree m . Our formula has been obtained from [8].

The following Lagrange property of these polynomials is basic to our main results. An elementary proof is given in the Appendix of [6].

Lemma 1.1. *Suppose $0 \leq n, q \leq m$. Let $k = 0$ when $n - q$ is even and let $k = 1$ when $n - q$ is odd. Then $P_{n,q}(h_n, h_q) = 1$ and $P_{n,q}(x) = 0$ whenever $x \in \mathcal{N}_k$ and $x \neq (h_n, h_q)$.*

Given $k = 0$ or $k = 1$, define

$$(1.3) \quad V_i(s, t) = T_{m-i}(s) T_i(t) - (-1)^k T_i(s) T_{m-i}(t), \quad i = 0, \dots, m,$$

and note that V_i is a polynomial of degree m (except when $V_i \equiv 0$). An important property of these polynomials is that they vanish on the set $\mathcal{N}_k$ of Chebyshev nodes. This is easy to verify since $T_{m-i}(h_n) = (-1)^n T_i(h_n)$ for $i = 0, \dots, m$ and $n = 0, \dots, m$.

The following is an interpolation formula for the Chebyshev nodes that is shown in [5] to follow from Lemma 1.1 by basic linear algebra.

Lemma 1.2. *Let $k = 0$ or $k = 1$. If $p(s, t)$ is a polynomial of degree at most m , then*

$$p = \sum_{(n,q) \in Q_k} p(h_n, h_q) P_{n,q} + \bar{p}_k,$$

where $\bar{p}_k$ is a linear combination of the polynomials (1.3).

Our motivation for considering Lagrange interpolation in this context is to obtain multivariate polynomial inequalities similar to those that have been obtained with single-variable Lagrange interpolation. (See [6].)

2. LAGRANGE INTERPOLATION

The next theorem obtains an explicit determination of the linear combination mentioned in Lemma 1.2. In particular, the theorem shows that Lagrange interpolation holds on $\mathcal{N}_0$ and $\mathcal{N}_1$ for all bivariate polynomials of degree at most $m - 1$. This case has been considered previously in [1, Theorem 2.4] but the polynomials in the interpolation formula there are not the Lagrange polynomials for the points of interpolation.

Theorem 2.1. *Let $k = 0$ or $k = 1$. Suppose $p(s, t)$ is a polynomial of degree at most m and write*

$$(2.1) \quad p(s, t) = \sum_{i=0}^m a_i s^{m-i} t^i + p_{m-1}(s, t),$$

where p_{m-1} is a polynomial of degree at most $m - 1$. Then

$$(2.2) \quad p = \sum_{(n,q) \in Q_k} p(h_n, h_q) P_{n,q} + \frac{1}{2^{m-1}} \sum_{i=0}^m a_i V_i.$$

Corollary 2.2. *If p is given by (2.1), then*

$$(2.3) \quad p = \sum_{(n,q) \in Q_k} p(h_n, h_q) P_{n,q}$$

if and only if $a_{m-i} = (-1)^k a_i$ for $i = 0, \dots, m$.

Proof. To prove Theorem 2.1, put

$$(2.4) \quad \alpha_i = \frac{c_i a_i}{2^{m-2}}, \quad i = 0, \dots, m,$$

where c_i is defined before (1.1). Since each of the polynomials

$$a_i s^{m-i} t^i - \alpha_i T_{m-i}(s) T_i(t), \quad i = 0, \dots, m,$$

has degree at most $m - 1$, equation (2.1) may be written as

$$(2.5) \quad p(s, t) = \sum_{i=0}^m \alpha_i T_{m-i}(s) T_i(t) + p_{m-1}(s, t),$$

where p_{m-1} is a (different) polynomial of degree at most $m - 1$. By Lemma 1.2, we may write

$$(2.6) \quad p = \sum_{(n,q) \in Q_k} p(h_n, h_q) P_{n,q} + \bar{p}_k,$$

where $\bar{p}_k = \sum_{i=0}^m \bar{\alpha}_i V_i$ and $\bar{\alpha}_0, \dots, \bar{\alpha}_m$ are constants. Thus it suffices to show that $\bar{p}_k = \frac{1}{2} \sum_{i=0}^m \alpha_i V_i$. We do this by taking the inner product of each side of (2.6) with V_i .

Specifically, define an inner product on the space X of real polynomials in two variables of degree at most m by

$$(2.7) \quad (f, g) = \int_{-1}^1 \int_{-1}^1 f(s, t) g(s, t) w(s) w(t) ds dt, \quad f, g \in X,$$

where $w(t) = 1/(\pi\sqrt{1-t^2})$. Clearly $\{T_{i-j}(s)T_j(t) : 0 \leq j \leq i \leq m\}$ is an orthogonal basis for X . Put $\gamma_i = \|T_{m-i}(s)T_i(t)\|^2$ and note that $\gamma_{m-i} = \gamma_i$ for

$i = 0, \dots, m$. Given $0 \leq i \leq m$, one can verify by taking inner products term by term that

$$\begin{aligned}(p, V_i) &= [\alpha_i - (-1)^k \alpha_{m-i}] \gamma_i, \\ (\bar{p}_k, V_i) &= 2[\bar{\alpha}_i - (-1)^k \bar{\alpha}_{m-i}] \gamma_i.\end{aligned}$$

Also, $(P_{n,q}, V_i) = 0$ for all $(n, q) \in Q_k$ since $(G_m(\cdot, \cdot, u, v), V_i) = 2c_i^2 \gamma_i V_i(u, v)$. Hence by (2.6),

$$\alpha_i - (-1)^k \alpha_{m-i} = 2[\bar{\alpha}_i - (-1)^k \bar{\alpha}_{m-i}].$$

Thus, it follows from the identity $V_i = -(-1)^k V_{m-i}$ that

$$\bar{p}_k = \frac{1}{2} \left(\sum_{i=0}^m \bar{\alpha}_i V_i + \sum_{i=0}^m \bar{\alpha}_{m-i} V_{m-i} \right) = \frac{1}{2} \sum_{i=0}^m \alpha_i V_i = \frac{1}{2^{m-1}} \sum_{i=0}^m a_i V_i,$$

as asserted.

To prove Corollary 2.2, note that (2.3) holds if and only if $\sum_{i=0}^m a_i V_i = 0$. Hence this corollary follows from the identity

$$\sum_{i=0}^m a_i V_i = \sum_{i=0}^m [a_i - (-1)^k a_{m-i}] T_{m-i}(s) T_i(t).$$

□

As an application of Lemma 1.2, we deduce the following minimal and near minimal cubature formulas. (See [8], [1] and [5]).

Corollary 2.3. *Let $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$ and let $k = 0$ or $k = 1$. Then*

$$(2.8) \quad \int_{-1}^1 \int_{-1}^1 p(s, t) w(s) w(t) ds dt = \frac{2}{m^2} \sum_{(n,q) \in Q_k} c_n c_q p(h_n, h_q),$$

for all polynomials $p(s, t)$ of degree at most $2m - 1$.

Proof. Let X be the space of all real polynomials in two variables of degree at most $m - 1$ with inner product given by (2.7). The reproducing kernel for X is given by

$$K_{m-1}(s, t, u, v) = \sum_{i=0}^{m-1} \sum_{j=0}^i \frac{1}{\alpha_{i-j} \alpha_j} T_{i-j}(s) T_j(t) T_{i-j}(u) T_j(v),$$

where

$$\alpha_j = \int_{-1}^1 T_j(t)^2 w(t) dt, \quad j = 0, \dots, m-1.$$

Since $\alpha_j = 1$ when $j = 0$ and $\alpha_j = 1/2$ when $j > 0$, we may write

$$K_{m-1}(s, t, u, v) = 4 \sum_{i=0}^{m-1} \sum_{j=0}^i T_{i-j}(s) T_j(t) T_{i-j}(u) T_j(v),$$

where the symbol $'$ indicates that the first term of the sum is divided by 2. Hence, for fixed real numbers u and v , it follows that $G_m(s, t, u, v) - K_{m-1}(s, t, u, v)$ is a linear combination of the functions $\{T_{m-j}(s) T_j(t) : 0 \leq j \leq m\}$. Therefore, if p_1 is in X , then p_1 is orthogonal to these functions, so

$$(2.9) \quad p_1(u, v) = (p_1, K_{m-1}(\cdot, \cdot, u, v)) = (p_1, G_m(\cdot, \cdot, u, v)).$$

In a similar way, $(p_1, V_i) = 0$ for $i = 0, \dots, m$.

If p_2 is a polynomial in two variables of degree at most m , then by (2.6),

$$p_2(s, t) = \sum_{(n,q) \in Q_k} \beta_{n,q} G_m(s, t, h_n, h_q) + \sum_{i=0}^m \bar{\alpha}_i V_i(s, t),$$

where $\beta_{n,q} = \frac{2}{m^2} c_n c_q p_2(h_n, h_q)$ for $(n, q) \in Q_k$. Hence by (2.9),

$$\begin{aligned} (p_1, p_2) &= \sum_{(n,q) \in Q_k} \beta_{n,q} p_1(h_n, h_q) \\ &= \frac{2}{m^2} \sum_{(n,q) \in Q_k} c_n c_q p_1(h_n, h_q) p_2(h_n, h_q). \end{aligned}$$

Therefore (2.8) holds when $p = p_1 p_2$ and thus it holds when p is any polynomial in two variables of degree at most $2m - 1$ since p is a linear combination of monomials of the form $p_1 p_2$. $\square$

3. INTERPOLATION OF SYMMETRIC AND SKEW-SYMMETRIC POLYNOMIALS

In this section we show without appealing to Theorem 2.1 or Corollary 2.2 that the Lagrange interpolation formula holds without extra terms in the case of symmetric and skew symmetric polynomials. We then sketch an alternate proof of Theorem 2.1 using this fact and the cubature formula.

To state our results, suppose $k = 0$ or $k = 1$ and let $\mathcal{S}_k$ be the space of all polynomials p in two variables with degree at most m satisfying $p(t, s) = (-1)^k p(s, t)$ for all $(s, t) \in \mathbb{R}^2$. Hence $\mathcal{S}_0$ is a space of symmetric polynomials and $\mathcal{S}_1$ is a space of skew-symmetric polynomials. Since the value of such a polynomial on a node (h_q, h_n) is known when its value at (h_n, h_q) is known, we restrict to the case $q \leq n$ and denote this with the superscript Δ . Thus, we define

$$\begin{aligned} Q_k^\Delta &= \{(n, q) : 0 \leq q \leq n \leq m, n - q = k \bmod 2\} \quad \text{and} \\ \mathcal{N}_k^\Delta &= \{(h_n, h_q) : (n, q) \in Q_k^\Delta\}. \end{aligned}$$

Clearly,

$$\mathcal{N}_k^\Delta = \{(h_n, h_q) \in \mathcal{N}_k : q \leq n\}.$$

We refer to the above as a triangular set of nodes. An important property of these nodes is that they have the same cardinality as the dimension of the corresponding space of symmetric or skew-symmetric polynomials. The following theorem shows that the Lagrange interpolation problem is unisolvent with respect to the triangular even (respectively, odd) Chebyshev nodes for the space of symmetric (respectively, skew-symmetric) polynomials in two variables with degree at most m .

Theorem 3.1. *Let $k = 0$ or $k = 1$. For every real-valued function f on Q_k^Δ there exists exactly one polynomial p in $\mathcal{S}_k$ such that $p(h_n, h_q) = f(n, q)$ for all $(n, q) \in Q_k^\Delta$.*

Given $(n, q) \in Q_k^\Delta$, define

$$(3.1) \quad P_{n,q}^\Delta(s, t) = \begin{cases} P_{n,q}(s, t) + (-1)^k P_{n,q}(t, s) & \text{if } q < n \\ P_{n,n}(s, t) & \text{if } q = n \end{cases}.$$

Clearly $P_{n,q}^\Delta$ is in $\mathcal{S}_k$. By Lemma 1.1, $P_{n,q}^\Delta(h_n, h_q) = 1$ and $P_{n,q}^\Delta(x) = 0$ whenever $x \in \mathcal{N}_k^\Delta$ and $x \neq (h_n, h_q)$. Thus $\{P_{n,q}^\Delta : (n, q) \in Q_k^\Delta\}$ is a set of Lagrange polynomials for $\mathcal{N}_k^\Delta$.

Corollary 3.2. *If p is a polynomial in $\mathcal{S}_k$, then*

$$(3.2) \quad p = \sum_{(n,q) \in Q_k^\Delta} p(h_n, h_q) P_{n,q}^\Delta = \sum_{(n,q) \in Q_k} p(h_n, h_q) P_{n,q}.$$

Proof. We first observe that the number N_k of nodes in $\mathcal{N}_k^\Delta$ is the same as the dimension of $\mathcal{S}_k$. Indeed, given $(n, q) \in Q_k^\Delta$, define

$$\phi_{n,q}(s, t) = s^i t^j + (-1)^k s^j t^i, \quad \text{where}$$

$$i = \frac{n+q+k}{2}, \quad j = \frac{n-q-k}{2}.$$

Then $\{\phi_{n,q} : (n, q) \in Q_k^\Delta\}$ is a basis for $\mathcal{S}_k$. Clearly, $P_{n,q}^\Delta$ is a linear combination of the polynomials $\{\phi_{n,q} : (n, q) \in Q_k^\Delta\}$ since it is in $\mathcal{S}_k$. Since there is a linear combination p of the $P_{n,q}^\Delta$'s that satisfies the required equations, the $N_k \times N_k$ system of equations

$$(3.3) \quad \sum_{(n,q) \in Q_k^\Delta} \phi_{n,q}(h_{n'}, h_{q'}) x_{n,q} = f(n', q'), \quad (n', q') \in Q_k^\Delta,$$

has a solution $\{x_{n,q} : (n, q) \in Q_k^\Delta\}$ for each f . Thus these solutions are unique, which proves Theorem 3.1.

To prove the corollary, observe that the difference between the the sides of the first equality of (3.2) is a polynomial in $\mathcal{S}_k$ that vanishes on all points of $\mathcal{N}_k^\Delta$ and hence is identically zero by uniqueness. The second inequality of (3.2) follows directly from (3.1), the identity $P_{n,q}(t, s) = P_{q,n}(s, t)$ and the symmetry of Q_k . $\square$

Alternate proof of Theorem 2.1. We deduce Theorem 2.1 from the cubature formula given in Corollary 2.3 and from Corollary 3.2. Indeed, it follows from formula (2.9) and Corollary 2.3 that Theorem 2.1 holds whenever p is a polynomial of degree at most $m-1$. (Compare [1, p. 120].) Thus by (2.5), it suffices to show that

$$(3.4) \quad T_{m-i}(s)T_i(t) = \sum_{(n,q) \in Q_k} T_{m-i}(h_n)T_i(h_q)P_{n,q}(s, t) + \frac{1}{2}V_i(s, t)$$

for $i = 0, \dots, m$. Given such i , note that Corollary 3.2 applies to

$$S_i(s, t) = \frac{1}{2}[T_{m-i}(s)T_i(t) + (-1)^k T_i(s)T_{m-i}(t)]$$

since S_i is in $\mathcal{S}_k$. Hence (3.4) follows since $S_i(h_n, h_q) = T_{m-i}(h_n)T_i(h_q)$ for all $(n, q) \in Q_k$ and

$$T_{m-i}(s)T_i(t) = S_i(s, t) + \frac{1}{2}V_i(s, t).$$

$\square$

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