Interpolation and cubature for rectangular sets of nodes

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Abstract. This article obtains specific formulas for Lagrange polynomials for rectangular versions of the even and the odd product nodes in $\mathbb{R}^2$. These polynomials are applied to obtain an exact cubature formula for bivariate polynomials when the number of rows of the nodes exceeds the number of columns by one.

1. Bivariate interpolation nodes

Let $X = \{h_0, \ldots, h_m\}$ and $Y = \{\tilde{h}_0, \ldots, \tilde{h}_{m+\sigma}\}$ each be sets of distinct numbers arranged in decreasing order, where $m$ and $\sigma$ are integers with $m \geq 1$ and $\sigma \geq 0$, and consider the rectangular set

$$N_\sigma^m = X \times Y = \{(h_i, \tilde{h}_j) : 0 \leq i \leq m, 0 \leq j \leq m + \sigma\}$$

of nodes in $\mathbb{R}^2$. The parameter $\sigma$ is the number of rows of $N_\sigma^m$ beyond its number of columns. Clearly $n(N_\sigma^m) = (m+1)(m+\sigma+1)$, where $n(S)$ denotes the number of elements of a set $S$.

Let $P_n(\mathbb{R}^2)$ denote the set of all bivariate polynomials of degree at most $n$. There can never exist Lagrange polynomials for $N_\sigma^m$ in $P_n(\mathbb{R}^2)$ unless

$$(m+1)(m+\sigma+1) \leq \frac{(n+1)(n+2)}{2},$$

since if such polynomials exist, then by their linear independence the number of points in $N_\sigma^m$ is at most the dimension of $P_n(\mathbb{R}^2)$. In particular, there are no Lagrange polynomials of degree at most $m$ for $N_\sigma^m$. Because of this restriction, we consider instead a checkerboard arrangement of even and odd nodes given by

$$N_\sigma^m,0 = \{(h_i, \tilde{h}_j) : 0 \leq i \leq m, \ 0 \leq j \leq m + \sigma, \ i-j \text{ is even}\},$$
$$N_\sigma^m,1 = \{(h_i, \tilde{h}_j) : 0 \leq i \leq m, \ 0 \leq j \leq m + \sigma, \ i-j \text{ is odd}\},$$

respectively. Note that $i - j$ is even if and only if both $i$ and $j$ are even or both $i$ and $j$ are odd, and $i - j$ is odd if and only if one of $i$ and $j$ is even and the other is odd. For example, a plot of $N_\sigma^m,k$ is given in Figure 1 for the case $\sigma = 1$, $m = 6$ and $k = 0, 1$.

We refer to the sets $N_\sigma^m,0$ and $N_\sigma^m,1$ as the even-odd product nodes. Clearly the even and odd nodes are disjoint sets with $N_\sigma^m = N_\sigma^m,0 \cup N_\sigma^m,1$. It is easy to verify that

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The nodes $N_{m,k}$ arise naturally in connection with Lagrange polynomials and cubature formulas when $\sigma = 0$ and

\[(2) \quad h_i = \tilde{h}_i = \cos\left(\frac{i\pi}{m}\right), \quad i = 0, \ldots, m.\]

These are called Chebyshev points by Xu in [13]. Morrow and Patterson [12, p. 963] arrived at these nodes for $m$ even by applying ideal theory and by alternating the Gauss-Chebyshev and Lobatto-Chebyshev rules of degree $m - 1$. Bojanov and Petrova [2] arrive at them by splitting a known one-dimensional cubature formula of degree $2n - 1$ into even and odd weighted sums and applying the sums successively. They are also obtained in [8, Lemma 6] in the course of estimating linear functionals on polynomials in two variables that assume their norm at Chebyshev polynomials in each variable. All these nodes are special cases of the Geronimus nodes considered in [9] and are even-odd product nodes.

Padua points are self-intersections of Lissajous curves. The first and third families [3] are the odd and even nodes, respectively, with $\sigma = 1$ and

\[h_i = \cos\left(\frac{\pi i}{m}\right), \quad i = 0, \ldots, m,\]

\[\tilde{h}_j = \cos\left(\frac{j\pi}{m+1}\right), \quad j = 0, \ldots, m + 1.\]

A plot of these points is given for $m = 6$ in Figure 1. Padua points are a special case of the Morrow-Patterson nodes discussed in [9] and are extended here.
Erb [5] considers another extension of Padua points that arises from studying intersections of Lissajous curves where $\sigma = p$. The papers [12, 13, 3, 5] consider Lagrange polynomials and cubature where the coordinates of the nodes are restricted to cosines of equispaced angles. This is also true of Bojanov and Petrova [2] in the case of Lagrange polynomials but their approach to cubature is more general than in previous papers.

2. Construction of Lagrange polynomials for $\mathcal{N}_{m,k}^\sigma$

See [6] for a survey of polynomial interpolation in several variables. Most formulas for Lagrange polynomials apply to specific cases but notable exceptions are given by Kronecker (see [7, Section 2]) and Xu in [14, Section 4.3].

One can obtain Lagrange polynomials for $\mathcal{N}_{m}^\sigma$ of degree no more than $2m + \sigma$ in two variables simply by taking the product of the classical Lagrange polynomials of a single variable for each of the sets $X$ and $Y$. In particular, these polynomials are Lagrange polynomials for the subsets $\mathcal{N}_{m,k}^\sigma$ for $k = 0, 1$. Our effort to obtain Lagrange polynomials of low degree is motivated by a desire to obtain cubature formulas from Lemma 1 of [9] and its extensions.

Our construction is analogous to that given in Section 4 of [10], which considers the case $\sigma = 0$. Let $h_0 > h_1 > \cdots > h_m$ and $\tilde{h}_0 > \tilde{h}_1 > \cdots > \tilde{h}_{m+\sigma}$ be any given numbers. Suppose we have a polynomial function $G_m^\sigma(s, t, u, v)$ of $(s, t)$ and $(u, v)$ satisfying the following:

$\begin{align*}
(3) \quad (s - u)G_m^\sigma(s, t, u, v) &= 0, \quad (t - v)G_m^\sigma(s, t, u, v) = 0, \quad G_m^\sigma(u, v, u, v) > 0
\end{align*}$

for all $(s, t)$ and $(u, v)$ in $\mathcal{N}_{m,k}^\sigma$, where $k = 0$ and $k = 1$. Clearly, the first two equations of (3) imply that $G_m^\sigma(s, t, u, v) = 0$ whenever $(s, t)$ and $(u, v)$ are unequal nodes in $\mathcal{N}_{m,k}^\sigma$.

For each $(h_n, \tilde{h}_q) \in \mathcal{N}_{m,k}^\sigma$, let

$\begin{align*}
(4) \quad \lambda_{n,q} &= \frac{1}{G_m^\sigma(h_n, \tilde{h}_q, h_n, \tilde{h}_q)},
\end{align*}$

and define

$\begin{align*}
(5) \quad P_{n,q}^\sigma(s, t) &= \lambda_{n,q}G_m^\sigma(s, t, h_n, \tilde{h}_q).
\end{align*}$

Then $P_{n,q}^\sigma$ is the desired Lagrange polynomial for $(h_n, \tilde{h}_q)$, i.e.,

$\begin{align*}
P_{n,q}^\sigma(h_n', \tilde{h}_q') &= \delta_{n,n'}\delta_{q,q'} \quad \text{for all } (h_n', \tilde{h}_q') \in \mathcal{N}_{m,k}^\sigma.
\end{align*}$

To construct $G_m^\sigma$, first observe that by Theorem A.1 (or see Theorem 4 of [10]), there exist orthogonal polynomials $p_0, p_1, \ldots, p_m$ and orthogonal polynomials $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{m+\sigma}$ satisfying the alternation conditions

$\begin{align*}
(6) \quad p_{m-i}(h_n) &= (-1)^np_i(h_n), \quad i, n = 0, 1, \ldots, m, \\
(7) \quad \tilde{p}_{m+\sigma-j}(h_n) &= (-1)^n\tilde{p}_j(\tilde{h}_n), \quad j, n = 0, 1, \ldots, m + \sigma.
\end{align*}$

By orthogonality, there exists a three-term recurrence relation

$\begin{align*}
(8) \quad p_0(x) &= 1, \quad p_1(x) = a_0x + b_0, \\
(9) \quad p_{i+1}(x) &= (a_ix + b_i)p_i(x) - c_ip_{i-1}(x), \quad i = 1, \ldots, m - 1,
\end{align*}$

where $a_0 > 0$ and $a_i, c_i > 0$ for $i = 1, \ldots, m - 1$. Equations similar to (8) and (9) hold as well for $\tilde{p}_j$ with respect to constants $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j$ and with $m$ replaced by $m + \sigma$. Also, by Lemma A.2, the orthogonal polynomials can be chosen so that $c_i = 1$ for $i = 1, \ldots, m - 1$. Some of the papers that consider this approach, including [11, 12, 14], are listed in the references.
and $\tilde{c}_j = 1$ for $j = 1, \ldots, m + \sigma - 1$. The corresponding bivariate reproducing kernels are given by

$$K_n(s, t, u, v) = \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{p_j(s)\tilde{p}_{i,j}(t)p_j(u)\tilde{p}_{i-j}(v)}{H_j H_{i-j}}, \quad 0 \leq n \leq m,$$

where $H_i = a_0 H_0 / a_i$, for $i = 0, \ldots, m$ and $\tilde{H}_j = \tilde{a}_0 \tilde{H}_0 / \tilde{a}_j$ for $j = 0, \ldots, m + \sigma$. Here $H_0$ and $\tilde{H}_0$ are given positive numbers and $a_m$ and $\tilde{a}_{m+\sigma}$ are defined by these equations. By [10, Lemma 2],

$$H_{m-i} = H_i, \quad i = 1, \ldots, m - 1,$$

$$\tilde{H}_{m+\sigma-j} = \tilde{H}_j, \quad j = 1, \ldots, m + \sigma - 1.$$

In terms of the inner product $(\cdot, \cdot)$ of orthogonality, $H_j = (p_j, p_j)$ and similarly for $\tilde{H}_j$. In Section 4 we use the fact that

$$p_i(s)\tilde{p}_j(t) - (-1)^k p_{m-i}(s)\tilde{p}_{m+\sigma-j}(t) = 0, \quad i = 0, \ldots, m, \quad j = 0, \ldots, m + \sigma,$$

for all $(s, t)$ in $\mathcal{N}_{m,k}^\sigma$, which follows easily from (6) and (7).

Let

$$\Gamma^\ell_m(s, t, u, v) = \frac{p_m(s)\tilde{p}_n(t)p_m(u)\tilde{p}_n(v)}{H_0 \tilde{H}_0} + \sum_{i=0}^{m-1} \frac{p_i(s)\tilde{p}_{m+\ell-i}(t)p_i(u)\tilde{p}_{m+\ell-i}(v)}{H_i \tilde{H}_{m+\ell-i}}$$

for $0 \leq \ell \leq \sigma$ and define $G^\sigma_m$ recursively for $\sigma \geq 1$ by

$$G^\sigma_m = \begin{cases} K_{m-1} + \Gamma^0_m & \text{if } \sigma = 1, \\ G^\sigma_{m-1} + \frac{1}{2} \Gamma^{[\sigma/2]}_{m} & \text{if } \sigma \geq 2, \end{cases}$$

where $m \geq 1$. Then $G^\sigma_m$ is a polynomial of degree $m + [\sigma/2]$ in $(s, t)$ and in $(u, v)$, and

$$G^\sigma_m = \begin{cases} K_{m-1} + \sum_{\ell=0}^{(\sigma-1)/2} \Gamma^\ell_m & \text{if } \sigma \text{ is odd}, \\ K_{m-1} + \frac{1}{2} \Gamma^\sigma_m + \sum_{\ell=0}^{\sigma/2-1} \Gamma_m & \text{if } \sigma \text{ is even}. \end{cases}$$

Thus, in particular, the polynomials $P^\sigma_{n,q}(s, t)$ are a linear combination of the polynomials $p_i(s)\tilde{p}_{j-i}(t)$ for $0 \leq i \leq m$, $i \leq j$ and $0 \leq j \leq m + [\sigma/2]$.

To see that $G^\sigma_m(u, v, u, v) > 0$ holds, as required by (4), observe that all the terms of $K_{m-1}(u, v, u, v)$ and $\Gamma_m(u, v, u, v)$ are of the form

$$T_{i,j} = \frac{p_i(u)^2\tilde{p}_j(v)^2}{H_i H_j} \geq 0 \quad \text{or} \quad T_{m,j} = \frac{p_m(u)^2\tilde{p}_j(v)^2}{H_0 H_j} \geq 0,$$

and, in particular, the first term of $K_{m-1}(u, v, u, v)$ is $T_{0,0} > 0$.

**Theorem 1.** Let $\sigma = 1$. The polynomials $G^\sigma_m$ define Lagrange polynomials of degree $m$ for the even-odd product nodes $\mathcal{N}_{m,k}^\sigma$ when $k = 0, 1$. There are no other Lagrange polynomials for $\mathcal{N}_{m,k}^\sigma$ if all these polynomials have degree at most $m$. 
An analogue of the first part of this theorem has already been established for the case $\sigma = 0$ in [10, Theorem 5] and the case $\sigma = 1$ has been proved in [4] when the even-odd nodes are Morrow-Patterson nodes, i.e., the coefficients in (9) and in its analogue are constant. In this note, we prove Theorem 1 for arbitrary even-odd product nodes.

**Corollary 2.** The polynomial $G_{m+\sigma-1}^1$ defines Lagrange polynomials of degree $m+\sigma-1$ for the nodes $N_{m,k}^\sigma$ when $\sigma \geq 1$. Here $k = 0, 1$.

Corollary 2 follows easily from Theorem 1 and the observation that we may obtain the nodes of $N_{m+1,k}^{\sigma-1}$ by adding an extra column to $N_{m,k}^{\sigma}$. Hence $N_{m,k}^{\sigma} \subseteq N_{m+1,k}^{\sigma-1}$ and, by induction, $N_{m,k}^{\sigma} \subseteq N_{m+\sigma-1,k}^{1}$.

**Conjecture 3.** The polynomials $G_m^\sigma$ define Lagrange polynomials of degree $m + \lceil \sigma/2 \rceil$ for the nodes $N_{m,k}^\sigma$ when $\sigma \geq 1$ and $k = 0, 1$.

This conjecture has been verified by computer for $m, \sigma = 1, \ldots, 15$ and $k = 0, 1$, where \{h_i\} and \{h_j\} are randomly selected sets of rational numbers for each $m$ and $\sigma$ (with each set taken from the interval $-1 \times 10^6 \ldots 1 \times 10^6$ and arranged in decreasing order). Computations were done using Maple 2017 with the Random Tools package.

**Case $\sigma = 1$.** The Lagrange polynomials for $N_{m,k}^1$ are obtained from the function

$$G_m^1(s, t, u, v) = K_{m-1}(s, t, u, v) + \frac{p_m(s)p_m(u)}{H_0 H_0} + \sum_{j=0}^{m-1} \frac{p_j(s)\tilde{p}_{m-j}(t)p_j(u)\tilde{p}_{m-j}(v)}{H_j H_{m-j}}$$

(14)

as indicated by (5) and (13). They have the same degree as for the square case (i.e., $\sigma = 0$) although an extra row of nodes has been added. This makes the number of nodes of $N_{m,k}^1$ equal to the dimension of the space $P_m(\mathbb{R}^2)$ for $k = 0$ and $k = 1$ by (1). Since any set of Lagrange polynomials for $N_{m,k}^1$ in $P_m(\mathbb{R}^2)$ is linearly independent, these polynomials are a basis for $P_m(\mathbb{R}^2)$. A consequence is that any given real-valued function $f$ on $N_{m,k}^1$ agrees with some polynomial $p$ in $P_m(\mathbb{R}^2)$ and $p$ is unique.

To see this, note that it follows from the first part of Theorem 1 that for each $x \in N_{m,k}^1$ there exists a Lagrange polynomial $L_x$ in $P_m(\mathbb{R}^2)$ with $L_x(x) = 1$ and $L_x(y) = 0$ for all $y \in N_{m,k}^1$ with $y \neq x$. The desired polynomial is then

$$p = \sum f(x)L_x,$$

where the sum is taken over all $x \in N_{m,k}^1$. To show that $p$ is unique, it suffices to show that $p = 0$ when $p$ is in $P_m(\mathbb{R}^2)$ and $p(x) = 0$ for all $x \in N_{m,k}^1$. Since $p$ is a linear combination of the basis elements $\{L_x : x \in N_{m,k}^1\}$, it follows by evaluation at $x$ that each of the coefficients of the linear combination is 0 and hence $p = 0$.

Thus, by definition, $N_{m,k}^1$ is unisolvent for $P_m(\mathbb{R}^2)$. This phenomenon has been observed for Padua points in [3, p. 44].

Another consequence is that the Lagrange polynomials of Theorem 1 have minimal degree. It is natural to wonder whether the same is true for the Lagrange polynomials of Conjecture 3.
According to the conjecture, Lagrange polynomials for the nodes $N_{m,k}^\sigma$ are obtained from the functions $G_m^\sigma$ given below.

**Case $\sigma = 2$.**

$$G_m^2(s, t, u, v) = G_m^1(s, t, u, v) + \frac{p_m(s)\tilde{p}_1(t)p_m(u)\tilde{p}_1(v)}{2H_0H_1} + \sum_{j=0}^{m-1} \frac{p_j(s)\tilde{p}_{m+1-j}(t)p_j(u)\tilde{p}_{m+1-j}(v)}{2H_jH_{m+1-j}}$$

**Case $\sigma = 3$.**

$$G_m^3(s, t, u, v) = G_m^1(s, t, u, v) + \frac{p_m(s)\tilde{p}_1(t)p_m(u)\tilde{p}_1(v)}{H_0\tilde{H}_1} + \sum_{j=0}^{m-1} \frac{p_j(s)\tilde{p}_{m+1-j}(t)p_j(u)\tilde{p}_{m+1-j}(v)}{H_j\tilde{H}_{m+1-j}}$$

**Case $\sigma = 4$.**

$$G_m^4(s, t, u, v) = G_m^1(s, t, u, v) + \frac{p_m(s)\tilde{p}_1(t)p_m(u)\tilde{p}_1(v)}{2H_0H_2} + \sum_{j=0}^{m-1} \frac{p_j(s)\tilde{p}_{m+1-j}(t)p_j(u)\tilde{p}_{m+1-j}(v)}{2H_jH_{m+2-j}}$$

**Case $\sigma = 5$.**

$$G_m^5(s, t, u, v) = G_m^1(s, t, u, v) + \frac{p_m(s)\tilde{p}_1(t)p_m(u)\tilde{p}_1(v)}{H_0\tilde{H}_1} + \sum_{j=0}^{m-1} \frac{p_j(s)\tilde{p}_{m+1-j}(t)p_j(u)\tilde{p}_{m+1-j}(v)}{H_j\tilde{H}_{m+1-j}} + \frac{p_m(s)\tilde{p}_2(t)p_m(u)\tilde{p}_2(v)}{H_0H_2} + \sum_{j=0}^{m-1} \frac{p_j(s)\tilde{p}_{m+2-j}(t)p_j(u)\tilde{p}_{m+2-j}(v)}{H_j\tilde{H}_{m+2-j}}$$

The above conjecture can also be applied to the case where $-m < \sigma < 0$, i.e., the number of rows of the nodes $N_{m,k}^\sigma$ is less than the number of columns. In this case, $N_{m,k}^\sigma$ is obtained from $N_{m+\sigma,k}^\sigma$ by an interchange of coordinates. The corresponding Lagrange polynomial for a point $(h_n, \tilde{h}_q)$ in $N_{m,k}^\sigma$ is given by

$$P_n,q(s, t) = \frac{G_{m+\sigma}^{-\sigma}(s, \tilde{h}_q, h_n)}{G_{m+\sigma}^{-\sigma}(\tilde{h}_q, h_n, \tilde{h}_q, h_n)}.$$

### 3. Cubature for $\sigma = 1$

We next show that the algebraic structure of our Lagrange polynomials implies a cubature formula for both the even and odd product nodes. Let $k = 0, 1$.

**Lemma 4.** Let $\{x_i\}$ be the points of $N_{m,k}^1$ and let $\{P_i\}$ be the Lagrange polynomials of Theorem 1 with $P_i(x_i) = 1$. Let $\mu$ and $\tilde{\mu}$ be measures in which the polynomials of (6) and (7)
are orthogonal, respectively. Then for each \( i \) there exists a polynomial \( \lambda_i \) of degree \( m \) and a real number \( \lambda_i > 0 \) given by (4) such that
\[
P_i(x) = \lambda_i K_{m-1}(x, x_i) + S_i(x), \quad x \in \mathbb{R}^2,
\]
where \( K_{m-1} \) is the bivariate reproducing Kernel given by (10) and
\[
\int_{\mathbb{R}^2} p(s, t) S_i(s, t) \, d\mu(s) \, d\bar{\mu}(t) = 0
\]
for all \( p \in \mathcal{P}_{m-1}(\mathbb{R}^2) \).

**Theorem 5.** If \( p(s, t) \) is any polynomial of degree at most \( 2m - 1 \), then
\[
\int_{\mathbb{R}^2} p(s, t) \, d\mu(s) \, d\bar{\mu}(t) = \sum_{n,q} \lambda_{n,q} p(h_n, \bar{h}_q),
\]
where the sum is taken over all nodes \((h_n, \bar{h}_q)\) in \( \mathcal{N}^1_{m,k} \). Here, \( \lambda_{n,q} \) is given by
\[
\frac{1}{\lambda_{n,q}} = \sum_{j=0}^{m} \sum_{i=0}^{m-j} p_i(h_n) \tilde{p}_j(\bar{h}_q)^2 \frac{1}{H_i H_j} + \frac{1}{H_0} \left( \frac{1}{H_0} - \frac{1}{H_m} \right) p_m(h_n)^2,
\]
where \( H_0 = \mu(\mathbb{R}), \tilde{H}_0 = \bar{\mu}(\mathbb{R}) \).

Special cases of this theorem have been given previously in [9, Theorem 5] and [4, Theorem 4]. Other expressions can be obtained for \( \lambda_{n,q} \) from the discussion in [10, p. 49].

Of particular interest is the choice \( h_i = \cos \theta_i, \quad 0 \leq i \leq m \), where \( \theta_i \) is one of the expressions
\[
\theta_i = i \frac{\pi}{m}, \quad \frac{m}{m + 2}, \quad \frac{i \pi}{m + 1}, \quad \frac{(i + 1) \pi}{m + 1}.
\]
It can be verified that corresponding orthogonal polynomials satisfying (6) are the Chebyshev polynomials of kinds I-IV, i.e., \( p_i(x) = T_i(x), U_i(x), V_i(x), W_i(x) \), respectively, with their associated measures. (See [11].) There always exists discrete measures for \( \mu \) by Theorem A.1 of Section 5. The same discussion holds for \( \tilde{h}_i \) when \( m \) is replaced by \( m + 1 \).

**Proof.** To prove Lemma 4, let \( x_i = (h_i, \bar{h}_q) \) and let \( S_i(s, t) = \lambda_{n,q} \Gamma_m^0(s, t, h_n, \bar{h}_q) \). Then (15) holds by our definitions. To prove (16), it suffices to prove that if \( \ell_1 + \ell_2 \leq m - 1 \) and \( \ell_1, \ell_2 \geq 0 \), then
\[
\int_{\mathbb{R}^2} s^{\ell_1} t^{\ell_2} p_j(s) \tilde{p}_{m-j}(t) \, d\mu(s) \, d\bar{\mu}(t) = 0 \quad \text{for all } 0 \leq j \leq m - 1.
\]
Now if \( i_1 + j_1 = m - 1 \) and \( i_1, j_1 \geq 0 \) then \( i_1 < j_1 < m - j_1 \) (otherwise, \( i_1 + j_1 \geq m \)). Therefore,
\[
\int_{\mathbb{R}^2} p_{i_1}(s) \tilde{p}_{j_1}(t) p_j(s) \tilde{p}_{m-j}(t) \, d\mu(s) \, d\bar{\mu}(t) = \int_{\mathbb{R}} p_{i_1}(s) p_j(s) \, d\mu(s) \int_{\mathbb{R}} \tilde{p}_{j_1}(t) \tilde{p}_{m-j}(t) \, d\bar{\mu}(t) = 0.
\]
Hence (17) follows from the fact that \( s^{\ell_1} \) and \( t^{\ell_2} \) are in the span of the polynomials \( p_0, p_1, \ldots, p_{\ell_1} \) and \( \tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{\ell_2} \), respectively. This proves Lemma 4.

By Lemma 1 of [9], Lemma 4 together with the existence of a Lagrange interpolation formula for \( \sigma = 1 \) is equivalent to the cubature formula given in Theorem 5. Also, by (4) and (14),
\[
\frac{1}{\lambda_{n,q}} = K_m(h_n, \bar{h}_q, h_n, \bar{h}_q) + \frac{1}{H_0} \left( \frac{1}{H_0} - \frac{1}{H_m} \right) p_m(h_n)^2.
\]
The sum in Theorem 5 is obtained by reversing the order of summation in \( K_m \).

\[ \square \]

4. Proof of Theorem 1

Fix \( k = 0 \) or \( k = 1 \) and let \( \epsilon = (-1)^k \). Define

\[
Y_i(s,t) = \begin{cases} 
  p_1(s)p_m(s) - p_{m-1}(s) & \text{if } i = 0, \\
  p_{m+1-i}(s)\tilde{p}_i(t) - \epsilon p_{i-1}(s)\tilde{p}_{m+1-i}(t) & \text{if } 1 \leq i \leq m+1.
\end{cases}
\]

It follows from (11) that \( Y_i(s,t) \) vanishes on each of the nodes of \( \mathcal{N}^1_{m,k} \). By our discussion in Section 2, it suffices to verify (3) and this follows from (14) and the identities

\[
a_0 H_0(s - u)G^1_m(s,t,u,v) = \sum_{i=0}^{m} Y_i(s,t)p_{m-i}(u)\tilde{p}_i(v) - Y_i(u,v)p_{m-i}(s)\tilde{p}_i(t),
\]

\[
\tilde{a}_0 \tilde{H}_0(t - v)G^1_m(s,t,u,v) = \sum_{i=0}^{m} Y_{i+1}(s,t)p_{m-i}(u)\tilde{p}_i(v) - Y_{i+1}(u,v)p_{m-i}(s)\tilde{p}_i(t).
\]

Here ‘ in the sum of (19) means that the term for \( i = 0 \) is multiplied by \( H_m/H_0 \). Identities of this kind were given previously in [4] but there the coefficients in the recursion equation (9) were required to be constant.

To prove (18), let

\[
A_i = \frac{p_i(s)p_i(u)}{H_i}, \quad \tilde{A}_j = \frac{\tilde{p}_j(t)\tilde{p}_j(v)}{\tilde{H}_j}.
\]

The reproducing kernel can be written as

\[
K_m(s,t,u,v) = \sum_{i=0}^{m} \sum_{j=0}^{i} \tilde{A}_{i-j}A_j = \sum_{i=0}^{m} \tilde{A}_i \sum_{j=0}^{m-i} A_j
\]

\[
= \tilde{A}_0 A_m + \sum_{j=0}^{m-1} A_j + \sum_{i=1}^{m} \tilde{A}_i \sum_{j=0}^{m-i} A_j.
\]

and

\[
G^1_m(s,t,u,v) = \frac{1}{H_0} \left( \frac{H_m}{H_0} - 1 \right) A_m + K_m(s,t,u,v)
\]

by (14). It follows from the Christoffel-Darboux formula that

\[
a_0 H_0(s - u) \sum_{j=0}^{m-i} A_j = p_{m+1-i}(s)p_{m-i}(u) - p_{m+1-i}(u)p_{m-i}(s), \quad 1 \leq i \leq m,
\]
since \( a_n H_n = a_0 H_0 \) for \( 0 \leq n < m \). Hence

\[
(21) \quad a_0 H_0 (s - u) G_m^1 (s, t, u, v) = \frac{a_0 (s - u) H_m A_m}{H_0} + \frac{p_m (s) p_{m-1} (u) - p_{m-1} (s) p_m (u)}{H_0} + \sum_{i=1}^{m} \tilde{A}_i [p_{m+1-i} (s) p_{m-i} (u) - p_{m+1-i} (u) p_{m-i} (s)].
\]

Thus we may write

\[
(22) \quad a_0 H_0 (s - u) G_m^1 (s, t, u, v) = X_m (s, t, u, v) - X_m (u, v, s, t),
\]

where

\[
X_m (s, t, u, v) = \frac{[p_1 (s) p_m (s) - p_{m-1} (s)] p_m (u)}{H_0} + \sum_{i=1}^{m} \tilde{A}_i p_{m+1-i} (s) p_{m-i} (u).
\]

Now define

\[
E_m = \sum_{i=1}^{m} \frac{e_i}{H_i}, \quad e_i (s, t, u, v) = p_{i-1} (s) \tilde{p}_{m+1-i} (t) p_{m-i} (u) \tilde{p}_i (v).
\]

Since

\[
Y_i (s, t) p_{m-i} (u) \tilde{p}_i (v) = \tilde{H}_i \tilde{A}_i p_{m+1-i} (s) p_{m-i} (u) - \epsilon e_i (s, t, u, v), \quad 1 \leq i \leq m,
\]

and the first term of \( X_m \) is \( Y_0 (s, t) p_m (u) / \tilde{H}_0 \), we obtain

\[
(23) \quad \sum_{i=0}^{m} \frac{Y_i (s, t) p_{m-i} (u) \tilde{p}_i (v)}{H_i} = (X_m - \epsilon E_m) (s, t, u, v).
\]

Let \( T \) be the operator that interchanges \((s, t)\) and \((u, v)\). Then \( e_{m+1-i}^T = e_i \) and \( \tilde{H}_{m+1-i} = \tilde{H}_i \) for \( 1 \leq i \leq m \). Hence (18) follows from (22) and (23).

Identity (19) is proved by a similar technique. We begin with the observation that the reproducing kernel can be written as

\[
K_m (s, t, u, v) = \sum_{i=0}^{m} A_i \sum_{j=0}^{m-i} \tilde{A}_j = \sum_{i=0}^{m} A_{m-i} \sum_{j=0}^{i} \tilde{A}_j,
\]

and that the analogous Christoffel-Darboux formula (with \( p \) replaced by \( \tilde{p} \) and \( A_j \) by \( \tilde{A}_j \)) applies to show that

\[
\tilde{a}_0 \tilde{H}_0 (t - v) G_m^1 (s, t, u, v) = \frac{\tilde{a}_0 H_m A_m}{H_0} (t - v) + \sum_{i=1}^{m} A_{m-i} [\tilde{p}_{i+1} (t) \tilde{p}_i (v) - \tilde{p}_{i+1} (v) \tilde{p}_i (t)].
\]

Hence,

\[
(24) \quad \tilde{a}_0 \tilde{H}_0 (t - v) G_m^1 (s, t, u, v) = \tilde{X}_m (s, t, u, v) - \tilde{X}_m (u, v, s, t),
\]

where

\[
\tilde{X}_m (s, t, u, v) = \frac{H_m A_m}{H_0} \tilde{p}_1 (t) + \sum_{i=1}^{m} A_{m-i} \tilde{p}_{i+1} (t) \tilde{p}_i (v).
\]
Now define
\[ \tilde{E}_m = \frac{e_0}{H_0} + \sum_{i=1}^{m} \frac{\tilde{e}_i}{H_{m-i}}, \]
\[ \tilde{e}_i(s, t, u, v) = p_i(s)\tilde{p}_{m-i}(t)p_{m-i}(u)\tilde{p}_i(v). \]

Since
\[ Y_{i+1}(s, t)p_{m-i}(u)\tilde{p}_i(v) = H_{m-i}A_{m-i}\tilde{p}_{i+1}(t)\tilde{p}_i(v) - \epsilon \tilde{e}_i(s, t, u, v), \quad 0 \leq i \leq m, \]
we obtain
\[ \frac{Y_1(s, t)p_m(u)}{H_0} + \sum_{i=1}^{m} \frac{Y_{i+1}(s, t)p_{m-i}(u)\tilde{p}_i(v)}{H_{m-i}} = (\tilde{X}_m - \epsilon \tilde{E}_m)(s, t, u, v). \] (25)

Then \( \tilde{E}_m^T = \tilde{E}_m \) since \( \tilde{e}_{m-i}^T = \tilde{e}_i \) for \( 0 \leq i \leq m \) and \( H_{m-i} = H_i \) for \( 1 \leq i \leq m - 1 \). Hence (19) follows from (24) and (25).

5. Appendix

The purpose of this appendix is to give an alternate proof of the existence of orthogonal polynomials that have given decreasing numbers as alternation points and to provide a simpler algorithm for computation of these polynomials than given in [10]. We also show that when the polynomials are rescaled, the coefficient of the polynomial of lowest degree in the recursion relation for \( p_n \) may be taken to be 1. This condition simplifies formulas we use in the proof of Theorem 1.

Given a decreasing finite sequence \( h_0 > h_1 > \cdots > h_m \), define a semi-inner product for the polynomials of a single variable by
\[ (p, q) = \frac{1}{C_m} \sum_{n=0}^{m} w_n p(h_n)q(h_n), \] (A.1)
where
\[ w_n = \frac{(-1)^n}{\prod_{j \neq n}(h_n - h_j)} > 0 \quad \text{and} \quad C_m = \sum_{n=0}^{m} w_n. \]

Note that \((\cdot, \cdot)\) is an inner product for the vector space \( \mathcal{P}_m(\mathbb{R}) \) of real polynomials of degree at most \( m \) since \((p, p) = 0 \) implies \( p = 0 \) when \( p \in \mathcal{P}_m(\mathbb{R}) \). As usual, \( \|p\| = \sqrt{(p, p)} \) for all polynomials \( p \).

**Theorem A.1.** Let \( p_0, p_1, \ldots, p_m \) be the orthonormal polynomials obtained by applying the Gram-Schmidt process to the polynomials \( 1, x, \ldots, x^m \) with respect to the inner product (A.1). Then
\[ p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad j, n = 0, 1, \ldots, m. \] (A.2)

**Proof.** Note that each \( p_n \) in the Gram-Schmidt construction has positive leading coefficient. Thus \( p_n(h_0) > 0 \) since all the zeros of \( p_n \) are in the interval \( (h_m, h_0) \) by [1, p. 8].

We first establish a useful identity for the semi-inner product. Let \( Q(x) = \prod_{j=0}^{m}(x - h_j) \) and let \( p(x) \) be any polynomial of degree at most \( m - 1 \). Then \( Q'(h_n) = \prod_{j \neq n}(h_n - h_j) \). By the method of partial fractions,
\[ \frac{p(x)}{Q(x)} = \sum_{n=0}^{m} \frac{A_n}{x - h_n}, \quad \text{where} \quad A_n = \frac{p(h_n)}{Q'(h_n)}. \]
and hence
\[ \sum_{n=0}^{m} A_n = \lim_{x \to \infty} \frac{xp(x)}{Q(x)} = 0. \]

By Lagrange interpolation and the intermediate value theorem, there exists a polynomial \( q_m \) of degree \( m \) satisfying \( q_m(h_n) = (-1)^n \) for \( 0 \leq n \leq m \). Therefore,

\[ (A.3) \quad (pq_m, 1) = \frac{1}{C_m} \sum_{n=0}^{m} \frac{p(h_n)}{Q'(h_n)} = 0. \]

To show that \( p_m = q_m \), observe that \( (q_m, p_k) = 0 \) for \( 0 \leq k \leq m - 1 \) by \( (A.1) \) and \( \| q_m \| = 1 \) by \( (A.3) \). Hence, by the orthogonal decomposition of \( q_m \) with respect to \( \{p_k\} \), there is a constant \( \alpha \) such that \( q_m = \alpha p_m \) and \( |\alpha| = 1 \). Then \( \alpha = 1 \) since both \( p_m(h_0) \) and \( q_m(h_0) \) are positive.

Define \( p \equiv q \) to mean that \( p(h_n) = q(h_n) \) whenever \( 0 \leq n \leq m \). To prove the theorem, it suffices to show by induction on \( n \) that \( p_j p_m \equiv p_{m-j} \) for \( 0 \leq j \leq m \). This is true when \( j = 0 \) since \( p_0 = \alpha \) is a positive constant and \( |\alpha| = \| p_0 \| = 1 \). Hence \( p_0 = 1 \).

Proceeding to the inductive step, let \( 0 \leq s < m \) and suppose \( p_j p_m \equiv p_{m-j} \) for \( j = 0, 1, \ldots, s \). It follows from Lagrange interpolation that there exists a unique polynomial \( \phi \) of degree at most \( m \) with \( \phi \equiv p_{s+1}p_m \). It suffices to show that \( \phi = p_{m-(s+1)} \) since then \( p_{s+1}p_m \equiv p_{m-(s+1)} \). If \( 0 \leq m - j \leq s \) then

\[ (\phi, p_j) = (p_{s+1}, p_{m}p_j) = (p_{s+1}, p_{m-j}) = 0, \]

by the induction hypothesis and the orthogonality of the polynomials \( \{p_n\} \). Also, if \( 0 \leq j \leq m - s - 2 \) then it follows from \( (A.3) \) that \( (\phi, p_j) = 0 \). Therefore, by the orthogonal decomposition of \( \phi \) with respect to \( \{p_n\} \), there exists a number \( \alpha \) with \( \phi = \alpha p_{m-s-1} \). Further, \( |\alpha| = 1 \) since

\[ \| \phi \| = \| p_{s+1}p_m \| = \| p_{s+1} \| = 1. \]

Hence \( \alpha = 1 \) since both \( p_{s+1}(h_0)p_m(h_0) \) and \( p_m(h_0) \) are positive. Thus \( \phi = p_{m-(s+1)} \), as we wished to show.

Conversely, it follows from [10, Theorem 8] that if \( p_0, p_1, \ldots, p_m \) are orthogonal polynomials satisfying \( (A.2) \) then the polynomials are orthogonal with respect to the inner product \( (A.1) \).

If there exist orthogonal polynomials \( p_0, p_1, \ldots, p_m \) and a decreasing finite sequence \( h_0, h_1, \ldots, h_m \) satisfying \( (A.2) \), then the finite sequence is called a set of alternation points for the orthogonal polynomials. (See [10]). The following lemma shows that by multiplying each term of a finite sequence \( \{p_n\} \) of orthogonal polynomials by an appropriate positive constant, we may assume that \( p_{n+1}(x) = (a_n x + b_n)p_n(x) - p_{n-1}(x) \) for \( n \geq 1 \). Moreover, this can be done so that alternation points are preserved.

**Lemma A.2.** Let \( \{p_n\}_0^m \) be a finite sequence of polynomials such that \( p_0(x) = 1, p_1(x) = a_0 x + b_0, \) and

\[ (A.4) \quad p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x), \quad 1 \leq n < m, \]

where \( \{a_n\}_0^{m-1}, \{b_n\}_0^{m-1} \) and \( \{c_n\}_0^{m-1} \) are finite sequences of real numbers with \( c_n > 0 \) for \( 1 \leq n < m \). Then there exists positive numbers \( \{\gamma_n\}_0^m \) such that the polynomials \( P_n = \gamma_n p_n \) satisfy \( P_0(x) = 1, \) \( P_1(x) = A_0 x + B_0, \) and

\[ (A.5) \quad P_{n+1}(x) = (A_n x + B_n)P_n(x) - P_{n-1}(x), \quad 1 \leq n < m, \]
where \( A_n = \frac{\gamma_{n+1} a_n}{\gamma_n} \) and \( B_n = \frac{\gamma_{n+1} b_n}{\gamma_n} \) for \( 0 \leq n < m \). Also, alternation points for \( \{p_n\} \), if they exist, are still alternation points for \( \{P_n\} \).

**Proof.** We assume that \( m \geq 2 \) since the lemma is obvious for \( m = 1 \) with \( \gamma_0 = \gamma_1 = 1 \). By multiplying (A.4) by \( \gamma_{n+1} \), we obtain

\[
P_{n+1}(x) = (A_n x + B_n) P_n(x) - c_n \frac{\gamma_{n+1}}{\gamma_n} P_{n-1}(x), \quad 1 \leq n < m.
\]

Let \( \gamma_0 = 1 \) and \( \gamma_1 > 0 \). Then \( P_0 = 1 \) and the solution to the equations (A.6)

\[
\gamma_{n+1} = \frac{\gamma_{n-1}}{c_n}, \quad 1 \leq n < m,
\]

is

\[
\gamma_n = \begin{cases} 
\frac{1}{c_1 c_3 \cdots c_{n-1}} & \text{for } n \text{ even} \\
\frac{c_2 c_4 \cdots c_{n-1}}{c_1 c_3 \cdots c_{n-1}} & \text{for } n \text{ odd}
\end{cases}
\]

when \( 1 < n \leq m \). Thus (A.5) holds with these values of \( \gamma_0, \ldots, \gamma_m \).

To show that \( \{P_n\}_0^m \) has the same alternation points as \( \{p_n\}_0^m \), it suffices to show that \( R_n = 1 \) for \( 0 \leq n < m/2 \), where

\[
R_n = \frac{\gamma_{m-n}}{\gamma_n}.
\]

If \( \{p_n\}_0^m \) has alternation points then \( c_{m-n} c_n = 1 \) for \( 0 < n < m \) by [10, Lemma 2]. Hence if \( m \) is even, successive applications of this equality leads to

\[
\frac{1}{R_n} = c_{n+1} c_{n+3} \cdots c_{m-n-3} c_{m-n-1} = 1, \quad 0 \leq n < \frac{m}{2},
\]

where the indices are in increasing order. If \( m \) is odd, choose \( \gamma_1 = \gamma_{m-1} \). Then \( R_1 = 1 \) by definition and \( R_0 = 1 \) since

\[
\frac{1}{R_0} = \frac{c_2 c_4 \cdots c_{m-1}}{\gamma_1} = c_1 c_2 \cdots c_{m-1} = 1.
\]

It follows from (A.6) that

\[
R_{n+2} = c_{n+1} c_{m-n-1} R_n = R_n, \quad 0 \leq n \leq m - 2.
\]

Hence by induction, \( R_n = 1 \) whenever \( 1 \leq n \leq m \). \( \square \)

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